A Note on a Paper of J.T. Marti.

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Recently, J. T. MARTI showed [1] that, under a certain commutativity condition on the operators involved, it was possible to obtain an operational calculus for pairs of closed linear operators. The purpose of this note is to observe that MARTI's conclusions are valid under more general conditions which, moreover, permit extension of the theory to n-tuples of operators.

Let X be a Banach space and T_i closed linear operators with domains D_i and ranges in X, $i=1, 2$. We shall also write D_{12} for the domain of $T_1 T_2$ i.e. $D_{12}=\{x\in D_2: T_2 x\in D_1\}$. Similarly we define D_{21}. Following [1], we define

$$\mathscr{C}(X)=\{(T_1, T_2): D_1\subseteq D_2;\ T_2 D_1\subseteq D_1;\ T_1 T_2 x=T_2 T_1 x \quad \text{for all} \quad x\in D_{21}\}.$$

Then for $(T_1, T_2)\in\mathscr{C}(X)$, suppose $\sigma_e(T_i)\subseteq V_i$, $i=1, 2$, where V_i is a Cauchy domain and let $H(V)$ denote the complex algebra of functions $\mathbf{C}^2\to\mathbf{C}$, locally holomorphic in $V=V_1\times V_2$, vanishing outside V. The main result of [1], (Theorem 2), asserts the existence of a proper homomorphism Φ of $H(V)$ into $B(X)$, mapping $(\lambda-\lambda_i)^{-1} \mid V$ onto $(\lambda-T_i)^{-1}$ for each $\lambda\notin V_i$ such that Φ is given by a Cauchy integral. The essential step is to show that the assumption $(T_1, T_2)\in\mathscr{C}(X)$ implies that the resolvent operators $R_i=(\lambda_i-T_i)^{-1}$ commute, $\lambda_i\in\varrho(T_i)$, $i=1, 2$.

PROPOSITION. *The resolvent operators R_1 and R_2 commute if and only if*

(i) $D_1\cap D_{12}=D_2\cap D_{21}$

and (ii) $T_1 T_2 x=T_2 T_1 x$ for $x\in D_{12}\cap D_{21}$

Proof. Suppose R_1 and R_2 commute. Then it is easy to verify that, for any x, $R_1 R_2 x\in D_{21}$. For D_i is the range of R_i so that $R_1 R_2 x\in D_1$ and hence $(\lambda_1-T_1) R_1 R_2 x$ is defined and equal to $R_2 x$ which lies in D_2. Thus $R_1 R_2 x\in D[T_2(\lambda_1-T_1)]=D_2\cap D_{21}$. Similarly $R_2 R_1 x\in D_{12}$ so that since $R_1 R_2 x=R_2 R_1 x$, we have $R_1 R_2 x\in D_{12}\cap D_{21}$.

Consider now the equation

$$0=(R_1 R_2-R_2 R_1)x=R_1 R_2[(\lambda_1-T_1)(\lambda_2-T_2)-(\lambda_2-T_2)(\lambda_1-T_1)] R_2 R_1 x. \quad (1)$$

Since $R_2 R_1 x\in D_{12}\cap D_{21}$, (1) is valid for all x and can be rewritten

$$0=R_1 R_2[T_2 T_1-T_1 T_2] R_2 R_1 x.$$

This implies

$$[T_2 T_1-T_1 T_2] R_2 R_1 x=0 \quad \text{for all} \quad x\in X.$$

But every $y\in D_{12}\cap D_{21}$ can be expressed as $R_2 R_1 x$. For let $x=(\lambda_1-T_1)(\lambda_2-T_2)y$. Hence condition (ii) is satisfied.

To prove (i), let $x \in D_1 \cap D_{12}$ and define $x' = (\lambda_1 - T_1)(\lambda_2 - T_2) x$. Clearly x' is well defined since $D[\lambda_1 - T_1)(\lambda_2 - T_2)] = D_1 \cap D_{12}$. Hence $x = R_2 R_1 x' = R_1 R_2 x'$. But $R_1 R_2 x' \in D_2 \cap D_{21}$. Hence $D_1 \cap D_{12} \subseteq D_2 \cap D_{21}$. The reverse inclusion is obtained similarly.

Now suppose that conditions (i) and (ii) hold. Let $x \in X$. Then there exists $x_2 \in D_2$ such that $(\lambda_2 - T_2) x_2 = x$ and $x_1 \in D_1$ such that $(\lambda_1 - T_1) x_1 = x_2$. Hence $x = (\lambda_2 - T_2) \times (\lambda_1 - T_1) x_1$ and evidently $x_1 \in D_{21} \cap D_2$. By assumption (i), $(\lambda_1 - T_1)(\lambda_2 - T_2) x_1$ is defined and by assumption (ii), it must equal x. Hence $x_1 = R_1 R_2 x = R_2 R_1 x$.

Corollary 1. *Theorem* 2 *of* [1] *is valid whenever* T_1 *and* T_2 *satisfy conditions* (i) *and* (ii).

Proof. An examination of the proof of Theorem 2 shows that the commutativity assumption involved in the definition of $\mathscr{C}(X)$ was used only to establish the commutativity of R_1 and R_2.

Corollary 2. *If* $(T_1, T_2) \in \mathscr{C}(X)$, *then* $T_1 D_1 \subseteq D_2$.

Proof. By [1] Lemma 1, R_1 and R_2 commute. Hence, in particular, by the above proposition, $D_1 \cap D_{12} = D_2 \cap D_{21}$. But $D_1 \cap D_{12} = \{x \in D_1 \cap D_2 : T_2 x \in D_1\} = D_1 \cap D_2 = D_1$. Hence $D_1 = D_2 \cap D_{21}$ and since $D_{21} \subseteq D_1 \subseteq D_2$, we must have $D_{21} = D_1$ i.e. $D_1 = \{x \in D_1 : T_1 x \in D_2\}$ i.e. $T_1 D_1 \subseteq D_2$.

Corollary 3. *Theorem* 2 *of* [1] *can now be extended in the obvious way to an operational calculus of n-tuples* $(T_1, T_2, \ldots, T_n)$ *of closed operators such that pairs* (T_i, T_j) *satisfy conditions* (i) *and* (ii). *The other results of* [1] *also have obvious extensions.*

REFERENCE

[1] J. T. Marti, *Operational Calculus for Two Commuting Closed Operators*, Comment. Math. Helv. *43* (1968), 87–97.

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