Stable Secondary Cohomology Operations

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Introduction

The purpose of this paper is to investigate for each positive integer n the stable secondary cohomology operations which are defined on every mod 2 cohomology class of dimension n. Such operations correspond to relations in the mod 2 Steenrod algebra A of the form $0=\Sigma a_i b_i$ with excess b_i greater than n. The set of all such operations is a left A-module. Thus we derive a basis for the module of operations. We shall call operations in a basis *basic operations* and their corresponding relations *basic relations.*

Let $B(n)$ denote the left A-ideal of the Steenrod algebra which annihilates all mod 2 cohomology of dimension n or less. In [9] it is shown that the set of admissible monomials of excess greater than n is a basis for $B(n)$ as a Z_2-module. In his paper [1], J. F. Adams uses homological algebra to find relations in A. For our problem the generators of $\mathrm{Ext}_A^{s,t}(B(n), Z_2)$ for $s=0, 1$ as a Z_2-module are in one to one correspondence with a minimal set of A-generators and basic relations respectively for $B(n)$. Wall formalizes the connection between generator and relations, and homological algebra [10].

Our main results are the following.

THEOREM A. $\mathrm{Ext}_A^{0,t}(B(n)\, Z_2)\cong Z_2$ *for pairs* (n, t) *such that either*

(a) $t=2^i$ *and* $0\leqslant n<t$, *or*

(b) $t\equiv 2^i(2^{i+1})$, $t>2^i$, *and* $n=t-r$ *for* $0<r<2^{i+1}$.

Otherwise the group is 0. *The corresponding generator of* $B(n)$ *can be chosen as* Sq^t.

THEOREM B. *For* $t\leqslant 3n+4$, $\mathrm{Ext}_A^{1,t}(B(n), Z_2)\cong Z_2$ *for pairs* (n, t) *satisfying all of the following:*

(a) *Given* t *determine all non-negative integers* i, j *such that* $t=m+3\cdot 2^j$, $m\equiv 2^i(2^{i+1})$, *m may be negative.*

(b) $n=m+2^j-r \quad 0\leqslant r<2^{i+1}-1$

(c) $n\not\equiv 2^j(2^{j+1})$

For any element $\theta\in A$ we define $H(\theta)\in A\otimes A$ as follows. Let $\psi: A\to A\otimes A$ be the coproduct. Since A is co-commutative, $\psi(\theta)=\Sigma\theta_i'\otimes\theta_i''+\eta\otimes\eta+\Sigma\theta_i''\otimes\theta_i'$. Let $H(\theta)= =\Sigma\theta_i'\otimes\theta_i''$. If x is a cohomology class, we define $H(\theta)x=\Sigma\theta_i' x\cup\theta_i'' x$.

THEOREM C. *Let* $0=a\mathrm{Sq}^{n+1}+\Sigma a_i\mathrm{Sq}^i(i>n+1)$ *be a relation in* $B(n)$. *There exists* *a stable secondary operation Φ such that on the fundumental class $\iota_n \in H^n(Z_2, n; Z_2)$ we have $(H(a)\,\iota_n) \subset \Phi(\iota_n)$ and if* $\dim x < n$, $(0) \subset \Phi(\iota_n)$.

The first two theorems are proved in Section 2. Theorem C is proved in Section 3. There some relations are listed.

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Section 1. Algebraic Preliminaries

In this section we obtain some results on the structure of $B(n)$ and related A-modules. Let A^* denote the dual of A. We use the Cartan basis of admissible monomials for A and the Milnor basis of monomials in ξ_i for A^* [9]. We employ the conventious of writing $\mathrm{Sq}(I)$, $\xi(I)$ and $\xi_1(i)\,\xi_2(j)\ldots\xi_p(k)$, where $I=(i,j,\ldots k)$ is a finite sequence of non-negative integers, to denote $\mathrm{Sq}^i\mathrm{Sq}^j\ldots\mathrm{Sq}^k$ and $\xi_1^i\xi_2^j\ldots\xi_p^k$ in A and A^* respectively. We first summarize those results of Milnor [8] which we require.

THEOREM 1.1 (a) *As an algebra A^* is a graded polynomial algebra over Z_2 on generators ξ_i of grade 2^i-1 $i\geqslant 1$.*

(b) *The coproduct $\varphi^*: A^* \to A^* \otimes A^*$ is a homomorphism of algebras given by $\varphi^*(\xi_k) = \Sigma_{i+j=k}\xi_i(2^j) \otimes \xi_j$.*

(c) *The evaluation $\langle \mathrm{Sq}^k, \xi_1^k \rangle = 1$ and $\langle \mathrm{Sq}^k, \alpha \rangle = 0$ for α any other monomial in the ξ_i.*

(d) *Let $\mathrm{Sq}(I)$ with $I=(i_1, i_2, \ldots, i_k)$ be an admissible monomial in A. Form the sequence $I'=(i'_1, i'_2, \ldots, i'_k)$ where $i'_k = i_k$ and $i'_j = i_j - 2i_{j+1}$ for $1 \leqslant j \leqslant k-1$. Then grade $\mathrm{Sq}(I) =$ grade $\xi(I')$ and $\langle \mathrm{Sq}(I), \xi(I') \rangle = 1$. We call $\xi(I')$ the monomial associated with $\mathrm{Sq}(I)$.*

We consider $B(n)$ as a graded A-module with the grading and module action that inherited as a submodule. We shall need dual information for $B(n)^*$.

DEFINITION. *Let $\xi(I)$ for $I=(i_1, i_2, \ldots, i_k)$ be a monomial in A^*. The multiplicity of $\xi(I)$, written either as $m\xi(I)$ or $m(I)$, is defined to be Σi_j.*

PROPOSITION 1.2 *$B(n)^*$ is the quotient of A^* by $B(n)^\dagger$, the annihilator of $B(n)$, spanned by all monomials of multiplicity less than or equal to n. $B(n)^\dagger$ is a sub A^*-comodule of A^* and $B(n)^*$ has the induced comodule structure.*

Proof. Since the coproduct φ^* in A^* has the property that if $\varphi^*(\alpha) = \Sigma\alpha' \otimes \alpha''$, then $m(\alpha'') \leqslant m(\alpha)$, the only non-trivial statement is the description of $B(n)^\dagger$. Let α be a monomial in A^* with $m(\alpha) > n$. Let $\mathrm{Sq}(J)$ be the admissible monomial in A such that

α is the associated monomial of $\mathrm{Sq}(J)$. Then $\langle \mathrm{Sq}(J), \alpha\rangle = 1$ and excess $\mathrm{Sq}(J) = m(\alpha)$. Hence α is not in $B(n)^{\dagger}$. Now let the monomial α have $m(\alpha) \leqslant n$. Let X be an n-fold product of RP^{∞}. Let $u \in H^n(X; Z_2)$ be the element such that $p_i^*(x_i) = u$ where $p_i: X \to RP^{\infty}$ is the i-th projection and x_i generates $H^1(RP^{\infty}; Z_2)$. Consider the formula [9]

$$\theta(u) = \textstyle\sum_{m(I)\leqslant n} < \theta, \xi(I) > x(I),$$

where $\theta \in A$, $\xi(I) \in A^*$ and $x(I)$ are linearly independent elements of $H^*(X; Z_2)$. We can assume θ is admissible. If $\theta \in B(n)$ then $\theta(u) = 0$. Hence by the linear independence of the $X(I)$, $\langle \theta, \xi(I)\rangle = 0$ for all I with $m(I) \leqslant n$. Thus $B(n)^{\dagger}$ is as described.

We use the basis for $B(n)^*$ consisting of all classes $[\alpha]$ of monomials in the Milnor generators for which $m(\alpha) \geqslant n+1$. The next few lemmas concern primitives of the co-action $\varphi^*: B(n)^* \to A^* \otimes B(n)^*$. Primitives are those elements for which $\varphi^*[\alpha] = 1 \otimes [\alpha]$. In 1.3–1.6 β always denotes a homogeneous (in grade) sum of monomials in A^*, $\beta = \Sigma \alpha_i$.

LEMMA 1.3. *Suppose $m(\alpha_i) > m$ for all i and $[\beta]$ is primitive in $B(n)^*$ for some $n < m$. Then $[\beta]$ is primitive in $B(k)^*$ for all k such that $n \leqslant k < p$, $p = \max_i(m(\alpha_i))$.*

Proof. The hypothesis says that in A^*, $\varphi^*(\beta) = 1 \otimes \beta + \Sigma \alpha_i' \otimes \alpha_1''$ with $m(\alpha_i'') \leqslant n$, from which the lemma is obvious.

LEMMA 1.4. *Suppose $m(\alpha_i) = m$ for all i and $\beta \neq \xi_1^m$. Then $[\beta]$ is not primitive in $B(m-1)^*$.*

Proof. Write each $\alpha_i = \xi(n_{i,1}, n_{i,2}, \ldots, n_{ik})$ with k large enough to be independent of i (some $n_{i,k}$ may be zero). Define a lexicographic type ordering $\alpha_i > \alpha_j$ provided there is an integer J (depending on the pair) such that $n_{i,J} > n_{j,J}$ and $n_{i,p} = n_{j,p}$ for all $p > J$. The ordering is well defined because it is transitive. Since the α_i are all of the same grade and multiplicity, no two of them can differ only in the ξ_1 and ξ_2 factors. Thus any J involved in the determination of order is greater than 2. Now let J be the integer which determines $\alpha_1 > \alpha_2$, α_1 being the first ordered element. The coproduct $\varphi^*(\xi_J(n_{1,J}))$ contains the summand $\xi_{J-1}(2n_{1,J}) \otimes \xi_1(n_{1,J})$. Since we are interested in those summands $\alpha_1' \otimes \alpha_i'' \subset \varphi^*(\alpha_i)$ for which $\alpha_i'' \neq 1$ and $m(\alpha_i'') = m$, the factor $\xi_p(n_{i,p}) \otimes 1$ cannot be involved in forming $\alpha_i' \otimes \alpha_i''$. Since J is greater than 2, the appearance of ξ_{J-1} on the left of $\otimes$ in the coproduct of a monomial means that monomial must have a factor ξ_p with $p \geqslant J$. Now $\varphi^*(\alpha_1)$ contains the summand

$$\xi_{J-1}(2n_{1,J}) \otimes \xi(n_{1,1} + n_{1,J}, n_{1,2}, \ldots, n_{1,J-1}, 0, n_{1,J+1}, \ldots, n_{1,k}). \tag{1}$$

This term is not cancelled in $\varphi^*(\alpha_1)$. Since α_1 is ordered first, its exponents after the 0 are all respectively greater than or equal to the corresponding exponents of any other α_i. There are two cases. Either $n_{1,J} > n_{i,J}$ or $n_{1,p} > n_{i,p}$ for some $p > J$. In the latter case such α_i cannot have (1) in their coproducts since they cannot produce the term on the right of $\otimes$. For the former case the remarks about the appearence of ξ_{J-1} on the left of $\otimes$ indicate that α_i has a factor $\xi_p(n_{i,p})$ for $p \geqslant J$. But $n_{1,p} = n_{i,p}$ for $p \geqslant J+1$ means only the term $1 \otimes \xi_p(n_{i,p})$ can be used if $\varphi^*(\alpha_i)$ can possible contain (1). Thus $p = J$. But $n_{1,J} > n_{i,J}$. In passing to $B(m-1)^*$ we note that (1) is a non-zero element of $A^* \otimes B(m-1)^*$.

The argument is illustrated by $\xi(27, 3, 6, 3, 1) > \xi(15, 21, 0, 3, 1) > \xi(21, 6, 12, 0, 1)$.

LEMMA 1.5. *Let grade $\beta = t$ and $[\beta]$ primitive in $B(n)^*$. Then one of the $\alpha_i = \xi_1^t$ and $[\beta]$ is the only primitive in $B(n)^*$ in this grade.*

Proof. By 1.4 not all α_i have the same multiplicity. Let $\beta' = \Sigma_{m(\alpha_i) > n+1} \alpha_i$. Then by 1.3 $[\beta']$ (non-zero) is a primitive in $B(n+1)^*$ and has fewer summands. Inductively we obtain a monomial primitive which is $[\xi_1^t]$. If there was another primitive $[\gamma]$ of the same grade, then $[\beta + \gamma]$ would be a primitive in $B(n)^*$ not having the summand ξ_1^t (over Z_2).

LEMMA 1.6. *Let $m = \min_i(m(\alpha_i))$. Suppose for some α_i with $m(\alpha_i) = m$, $\alpha_i = \xi(n_1, n_2, \ldots, n_k)$ with some n_j odd. Then $[\beta]$ is not primitive in $B(m-2)^*$.*

Proof. In A^* the coproduct $\varphi^*(\alpha_i)$ contains the summand $\xi_j \otimes \xi(n_1, \ldots, n_j - 1, \ldots, n_k)$ which is not in the coproduct of any other α in A^*.

Some other modules and algebras we shall employ are the following. Let $C(n+1) = B(n)/B(n+1)$. Let $a^*: A^* \to A^*$ be the squaring map $a^*(\alpha) = \alpha^2$. Let $A_e^* = \operatorname{Im} a^*$, $B(n)_e^\dagger = B(n)^\dagger \cap A_e^*$, $B(n)_e^* = A_e^*/B(n)_e^\dagger$. Let N be the dual of A^*/A_e^*, P the dual of $A^*//A_e^* = A^*/A_e^*.\, I(A^*)$. If we let $a: A \to A$ be the dual of a^*, then it is well known that $P \to A$ is an inclusion of Hopf algebras, P is normal in A and $\ker a = A \cdot I(P)$, [3]. It is easy to see that $\operatorname{Ext}_P^{*,*}(Z_2, Z_2) = Z_2[q_0, q_1, \ldots]$ where q_i are all generators of bidegree $(1, 2^{i+1} - 1)$. We shall call this polynomial algebra W. Recall that an A-module is *cyclic* if it is generated by a single element. The next proposition summarizes the information we shall need about the above structures.

PROPOSITION 1.7. (a) *$C(n)^*$ is spanned by all classes of monomials $[\alpha]$ for which $m(\alpha) = n$.*

(b) *$C(n)$ is isomorphic as an A-module to a cyclic module $A/R(n)$ on a single generator (n) of grade n. The generator corresponds to* Sq^n.

(c) *N is a submodule of $R(n)$ and $R(n)/N \cong B(2n)_e$.*

(d) *Since $B(2n)_e^* = 0$ in grades $t \leqslant 2n+1$, $C(n)^* = A_e^* \cdot (n)$ as A^* comodules in this* *range. This isomorphism is given by* $[\alpha]\to\alpha'(n)$ *where* $\varphi^*(\alpha)$ *has the summand* $\alpha'\otimes\xi_1^n$.

(e) $t\leqslant 3n+1$, $\mathrm{Ext}_A^{s,t}(C(n), Z_2)$ *is a free* W *module over* Z_2 *on a single generator* (n) *of bidegree* $(0, n)$.

Proof. (a) is obvious. To show (b) let $f: A\to C(n)$ be the A-map defined by $f(1)=\mathrm{Sq}^n$. Since f^* is adjoint to multiplication by Sq^n, an application of 1.1 (c) shows that if $\alpha=\xi(n_1, n_2,\ldots, n_k)$ with $\Sigma n_i=n$ then $f^*([\alpha])=(2n_2, 2n_3,\ldots, 2n_k)$. It is immediate that f^* is monic. We let $R(n)=\ker f$. We obtain (c) via dualizing. $R(n)^*=\mathrm{coker}\, f^*=A^*/B(2n)_e^\dagger$ as A^*-comodules. Since $B(2n)_e^*=A_e^*/B(2n)_e^*$ we have $R(n)^*/B(2n)_e^*\cong$ $\cong A^*/A_e^*$. This also gives (d). We obtain (e) for $t\leqslant 2n+1+n=3n+1$ from the sequence of isomorphisms,

$$\begin{aligned}\mathrm{Ext}_A^{s,t}(C(n), Z_2) &\cong \mathrm{Ext}_A^{s-1,t-n}(N, Z_2)\\ &\cong \mathrm{Ext}_A^{s-1,t-n}(\ker a, Z_2) = \mathrm{Ext}_A^{s,t-n}(A/A\cdot I(P), Z_2)\\ &\cong \mathrm{Ext}_P^{s,t-n}(Z_2, Z_2)\cong W(n).\end{aligned}$$

The penultimate isomorphism is an application of Cor 1.5 of [3].

We shall use the following maps.

DEFINITION. *The map* $s_0: B(n)^*\to B(n+1)^*$ *is given as a map of* Z_2*-modules by* $s_0[\alpha]=[\xi_1\alpha]$. *The map* s_i *is defined as* s_0^{2i}. *In particular* $s_i[\alpha]=[\xi_1(2^i)\alpha]$. *The codomain of* s_i *is* $B(n+2^i)^*$.

The next lemma is an easy exercise with the co-action and its proof is omitted. It accounts for the periodicity in Theorems A and B.

LEMMA 1.8. *The map* $s_0: B(n)_t^*\to B(n)_{t+1}^*$ *is an isomorphism of* Z_2*-modules in a range of grades* $t\leqslant 3n+4$. *The map* $s_i: B(n)_t^*\to B(n+2^i)_{t+2^i}^*$ *is an* A^**-comodule isomorphism in a range of grades* $t\leqslant\min(n+2^i, 3n+4)$.

Section 2. Computations

The main idea of this section is to investigate the spectral sequence obtained from the exact couple $\langle D, E\rangle$ where D, E are triply graded Z_2-modules;

$$\begin{aligned}E_{p,q,t} &= \mathrm{Ext}_A^{p+q,t}(C(p), Z_2)\\ D_{p,q,t} &= \mathrm{Ext}_A^{p+q,t}(B(p-1), Z_2),\\ &\quad p\geqslant 0,\ p+q\geqslant 0.\end{aligned}$$

With maps induced from

$$0\to B(p)\xrightarrow{i} B(p-1)\xrightarrow{j} C(p)\to 0.$$

We observe that if $x \in E_{p,q,t}$, then x is a non-bounding r-1 cycle if and only if x pulls back via $r-1$ iterates of i^* to a non-zero class in $\mathrm{Ext}_A^{p+q,t}(B(p-r), Z_2)$.

The computations are facilitated by employing 1.7 to introduce products in the exact (compare [7]) in a certain range of t. We show that the various differentials are derivations. We shall obtain a product P,

$$\begin{aligned} P\colon \mathrm{Ext}_A^{s,t}(C(p), Z_2) \otimes \mathrm{Ext}_A^{u,v}(C(q), Z_2) \to \\ \mathrm{Ext}_A^{s+u,t+v}(C(p+q), Z_2) \quad \text{for} \quad t+v \leqslant 3(p+q)+1\,. \end{aligned}$$

We employ the method of Mac Lane [6 p. 220]. First let X and Y be A free resolutions of $C(p)$ and $C(q)$ respectively. Using the Hom-$\otimes$ interchange we have an external cohomology product p which commutes with connecting homomorphisms and in this case is an isomorphism

$$\begin{tikzcd} H^s(\mathrm{Hom}_A(X, Z_2)) \otimes H^u(\mathrm{Hom}_A(Y, Z_2)) \arrow[r] \arrow[dr, "p"'] & H^{s+u}(\mathrm{Hom}_A(X, Z_2) \otimes \mathrm{Hom}_A(Y, Z_2)) \arrow[d] \\ & H^{s+u}(\mathrm{Hom}_{A\otimes A}(X \otimes Y, Z_2))\,. \end{tikzcd}$$

Let $\psi\colon A \to A\otimes A$ be the coproduct in A. ψ induces a change of rings $\psi^\#$,

$$\psi^\#\colon H^{s+u}(\mathrm{Hom}_{A\otimes A}(X \otimes Y, Z_2)) \to H^{s+u}(\mathrm{Hom}_A(X \otimes Y, Z_2))\,.$$

Using 1.7 we obtain an A-map $\Delta\colon C(p+q) \to C(p)\otimes C(q)$ for grades $t \leqslant 3(p+q)+1$ from $\Delta^*\colon A_e^*(p)\otimes A_e^*(q) \to A_e^*(p+q)$ by multiplication in A_e^*, $\Delta^*(\alpha(p)\,\beta(q)) = \alpha\beta(p+q)$. Let Δ^* also denote the induced map in Ext,

$$\Delta^*\colon \mathrm{Ext}_A^{s+u}(C(p)\otimes C(q), Z_2) \to \mathrm{Ext}_A^{s+u}(C(p+q), Z_2)\,.$$

Then the product P is defined to be $P = \Delta^*\psi^\# p$. P commutes with connecting homomorphisms because all the factors do.

We next compute P in terms of the information of 1.7(e). Since p is an isomorphism, we identify $\mathrm{Ext}_A^s(C(p), Z_2)\otimes\mathrm{Ext}_A^u(C(q), Z_2)$ with $\mathrm{Ext}_{A\otimes A}^{s+u}(C(p)\otimes C(q), Z_2)$. We analyze the change of rings directly using the method of [6 p. 91]. We first compute $\mathrm{Ext}_A(C(p)\otimes C(q), Z_2)$. By 1.7 we have $N = \ker a$ in the range we are considering. Since $\ker a = A\cdot I(C)$ we have A_e a Hopf algebra obtained from A as a quotent of A by a Hopf ideal [9]. Let ${}_D(A\otimes A)$ and ${}_L(A\otimes A)$ denote $A\otimes A$ considered as an A-module via ψ and left action alone respectively. Then there exists a map $h\colon {}_D(A\otimes A) \to {}_L(A\otimes A)$ which is an isomorphism of A-modules. h is the composite $(1\otimes\varphi)\circ(\psi\otimes 1)$, [5]. The remarks about the construction of A_e show that h projects to an isomorphism of ${}_D(A_e\otimes A_e)$ with ${}_L(A_e\otimes A_e)$. We thus obtain

$$\mathrm{Ext}_A(C(p)\otimes C(q), Z_2) = \mathrm{Ext}_A(C(p+q), Z_2) \otimes \mathrm{Ext}_A({}_L(A\otimes A_e), Z_2)\,. \tag{2}$$

From our computations of Ext we can obtain a resolution X of $C(p)\otimes C(q)$ as an $A\otimes A$ module. Pulling back along ψ makes ${}_\psi X$ an A free resolution of ${}_\psi(C(p)\otimes C(q))$.

Let X' be a resolution for ${}_D(C(p)\otimes C(q))$ as an A-module obtained from (2). The following commutative diagram

$$\begin{tikzcd} {}_\psi(C(p)\otimes C(q)) \arrow[d, "\mathrm{id}"] & {}_\psi X \arrow[l] \arrow[d, "f"] \\ {}_D(C(p)\otimes C(q))/ & X' \arrow[l] \end{tikzcd}$$

where f is a lifting of the identity, shows that a class in $W(p)\otimes W(q)$ maps to the obvious product class in $W(p+q)\otimes 1$ under the change of rings.

Finally lifting Δ to a map of resolutions we obtain P as

$$P\colon W(p)\otimes W(q)\to W(p+q);\qquad \alpha(p)\otimes\beta(q)\to\alpha\beta(p+q).$$

We next study how the differentials in the spectral sequence behave in the algebra under P.

PROPOSITION 2.1. *The differential d^r coincides with the connecting homomorphism associated with the short exact sequence*

$$0\to B(p-r)/B(p)\to B(p-r-1)/B(p)\to C(p-r)\to 0,$$

when d^r is defined.

Proof. For $r=1$ consider,

$$\begin{tikzcd} 0 \arrow[r] & B(p-1) \arrow[r] \arrow[d, "j"] & B(p-2) \arrow[r] \arrow[d, "j"] & C(p-1) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & C(p) \arrow[r] & B(p-2)/B(p) \arrow[r] & C(p-1) \arrow[r] & 0. \end{tikzcd}$$

The rows are exact and the squares commute, thus we obtain

$$\begin{tikzcd} \mathrm{Ext}^s_A(C(p)) \arrow[r, "\delta'"] \arrow[d, "j^*"] & \mathrm{Ext}^{s+1}_A(C(p-1)) \arrow[d, equal] \\ \mathrm{Ext}^s_A(B(p-1)) \arrow[r, "\delta"] & \mathrm{Ext}^{s+1}_A(C(p-1)). \end{tikzcd}$$

Thus $\delta'=\delta j^*=d^1$. The proof is completed inductively by giving the same argument on

$$\begin{tikzcd} 0 \arrow[r] & B(p-r) \arrow[r] \arrow[d] & B(p-r-1) \arrow[r] \arrow[d] & C(p-r) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B(p-r)/B(p) \arrow[r] & B(p-r-1)/B(p) \arrow[r] & C(p-r) \arrow[r] & 0. \end{tikzcd}$$

Using 2.1 we obtain that each d^r is a derivation in the range where P is defined because P commutes with connecting homomorphism and maps i^*, j^*.

We now carry out the calculations of Theorems A and B. We obtain Theorem A by a direct approach. This information gives us $d^r(n)$ which, when used with the algebra structure, gives us Theorem B.

We first interpret 1.5 in the setting of the exact couple. Let ξ_1^p represent (p) in $\mathrm{Ext}_A^{0,p}(B(p-1), Z_2)$. The proposition says that if $[\beta]$ represents a non-zero element in $\mathrm{Ext}_A^{0,p}(B(p-r), Z_2)$ then (p) pulls back via $r-1$ iterates of $(i^*)^{-1}$ to $[\beta]$. We also remark that 1.5 along with 1.1(c) implies we can choose Sq^p as the representative generator in $B(p-r)$.

PROPOSITION 2.2. $(3\cdot 2^i)$ *pulls back exactly to* $\mathrm{Ext}_A^{0,3\cdot 2^i}(B(2^i+1), Z_2)$.

Proof. We show that there are elements (not monomials in general) $p_i \in A^*$ of grade $3\cdot 2^i (i \geqslant 0)$ such that $[p_i]$ is primitive in $B(2^i+1)^*$ (and hence in $B(n)^*$ for $2^i+1 \leqslant n < 3\cdot 2^i$) but not primitive in $B(2^i)^*$. Prop 1.5 then gives the result. We obtain the p_i inductively, $p_0 = \xi_1^3$ and $p_1 = \xi_1^6 + \xi_1^3\xi_2$. By inspection these elements satisfy the proposition. The inductive hypothesis for $i \geqslant 2$ is that (a) p_{i-1} is constructed, (b) $[p_{i-1}]$ is primitive in $B(2^{i-1}+1)^*$ but not primitive in $B(2^{i-1})^*$, (c) one of the summands of p_{i-1} is $\xi_1(2^{i-1}+1)\,\xi_i$. Now by squaring, $[p_{i-1}^2]$ is primitive in $B(2^i+3)^*$. Consider the exact sequence,

$$\mathrm{Ext}_A^{0,t}(B(n), Z_2) \xrightarrow{i^*} \mathrm{Ext}_A^{0,t}(B(n+1), Z_2) \xrightarrow{\delta} \mathrm{Ext}_A^{1,t}(C(n+1), Z_2).$$

By 1.7 the third term is non-zero for $t \leqslant 3n+4$ only for $t = n+2^j$. Let $n = 2^i+2$. Since the grading requirements are met and $3\cdot 2^i \neq 2^i+2+2^j$, we have $[p_{i-1}^2]$ in the image of i^*. Now let $n = 2^i+1$. Again $3.2^i \neq 2^i+1+2^j$ so $[p_{i-1}^2]$ is in the image of $i^* \circ i^*$. Let p_i be a representative in A^* such that $i^* \circ i^*([p_i]) = [p_{i-1}^2]$. By 1.5 p_i must contain p_{i-1} as a summand, and *a fortiori* $\xi_i(2^i+2)\,\xi_i(2) = S$ as a summand. The coproduct in $B(2^i+1)^*$ of S contains the summand $\xi_i(2) \otimes [\xi_1(2^i+2)]$. This must be cancelled by some other summand of p_i in order for $[p_i]$ to be primitive in $B(2^i+1)^*$. The only other monomial in A^* which can do this is $\xi_1(2^i+1)\,\xi_{i+1}$. Thus the induction is completed by invoking 1.6 to obtain the whole statement (b). This completes the proof.

2.3. *Proof of Theorem A.* We show

(a) $d^r(2^i) = 0$ all r and i

(b) $d^r(n) = q_i(n-r)$ for $r = 2^{i+1}-1$ $n \equiv 2^i(2^{i+1})$, $n > 2^i$.

(a) is immediate since $\xi_1(2^i)$ is a primitive in $I(A^*) = B(0)^*$. The previous result establishes that $d^r(n) \neq 0$ for the lowest value of n in each residue class. But the maps s_j of 1.8 are comodule isomorphisms in the grades involved necessary to assert $d^r(n) \neq 0$ for all n in the residue class. The values given in (b) are the only ones possible in view of 1.7 (e).

2.4. *Proof of Theorem B.* In the gradings we are considering, E_1 of the exact couple consists of all $q_j(n)$ such that $n+2^{j+1}-1\leqslant 3n+1$ or $2^j\leqslant n+1$. Those which are eventually boundaries are by 2.3 all $q_j(n-r)$ with $r\equiv 2^j(2^{j+1})$, $n>2^j$ and $r=2^{j+1}-1$. These can be rewritten using P as $q_j(2^j+1)(k)$ with $k\equiv 0(2^{j+1})$. If we write $q_j(n)=q_j(2^j+1)(m)$ with $n=2^j+1+m$, $m\not\equiv 0(2^{j+1})$, $m\geqslant 0$ we obtain the non-boundaries except those classes which would formally correspond to $m=-2, -1$. But these exceptional cases are such that any class to which they pull back lies outside the gradings of Theorem B. They sit in the line $t=3n+4$ in table I. Hence in the proof of Theorem B the representation is adequate. In the gradings where we have the product, Theorem A implies the following,

$$\begin{aligned} d^r q_j(2^j+1)(m) &= q_j(2^j+1)\,d^r(m) \\ &= q_j q_i(2^j+1)(m-r), \quad m\equiv 2^i(2^{i+1}) \quad r=2^{i+1}-1 \end{aligned}$$

or

$$= 0 \quad \text{if} \quad r<2^{i+1}-1\,.$$

For values of m small with respect to 2^j (near $t=3n+4$ in Table I) the computation is invalid because either the differentials land outside the range where we have P or are not defined in the exact couple. However we can use the isomorphisms of 1.8 to obtain the results for these values of m. The statement of Theorem B is just a restatement in terms of (n, t) of the above.

For convenience, Table I is a graph of Theorem B. It also includes some further information obtained in [2] for $t>3n+4$.

A dot or *x* in position (n, t) of Table I represents an additive generator of $\mathrm{Ext}^{1,t}(B(n), Z_2)$. For $t\leqslant 3n+4$ the information is complete. The classes denoted by x represent generators determined by the rest of E_1 of the exact couple. They come from $B(2n)_e$ as indicated by Prop. 1.7. However, in this range many differentials are unknown. First, are there classes in the region $t>3n+4$ which are pull backs of classes in the region $t\leqslant 3n+4$? The answer is "no" for $t\leqslant 16$ by direct computation. Second, what is the action of differentials where a "?" is placed? In low grades direct computation gives the result indicated.

Section 3. Evaluation of the Operations

We use the method of universal examples as developed in [4] to evaluate the operations. We can assume that a typical relation in $B(n)$ is $0=a_1\mathrm{Sq}^{n+1}+\Sigma a_i\mathrm{Sq}^i$ with $i>n+1$. For each positive integer m let (E_m, B_m, F_m, p_m) be the fibre space over $B_m=K(Z_2, m)$ with fiber F_m a cartesian product $K(Z_2, m+n)\,xX_iK(Z_2, m+i-1)$ $i>n+1$, and k-invariants Sq^{n+1} and Sq^i from the relation. We let ι_m and η_{m+n}, η_i denote the fundumental classes of the base and factors in the fibre respectively. Let

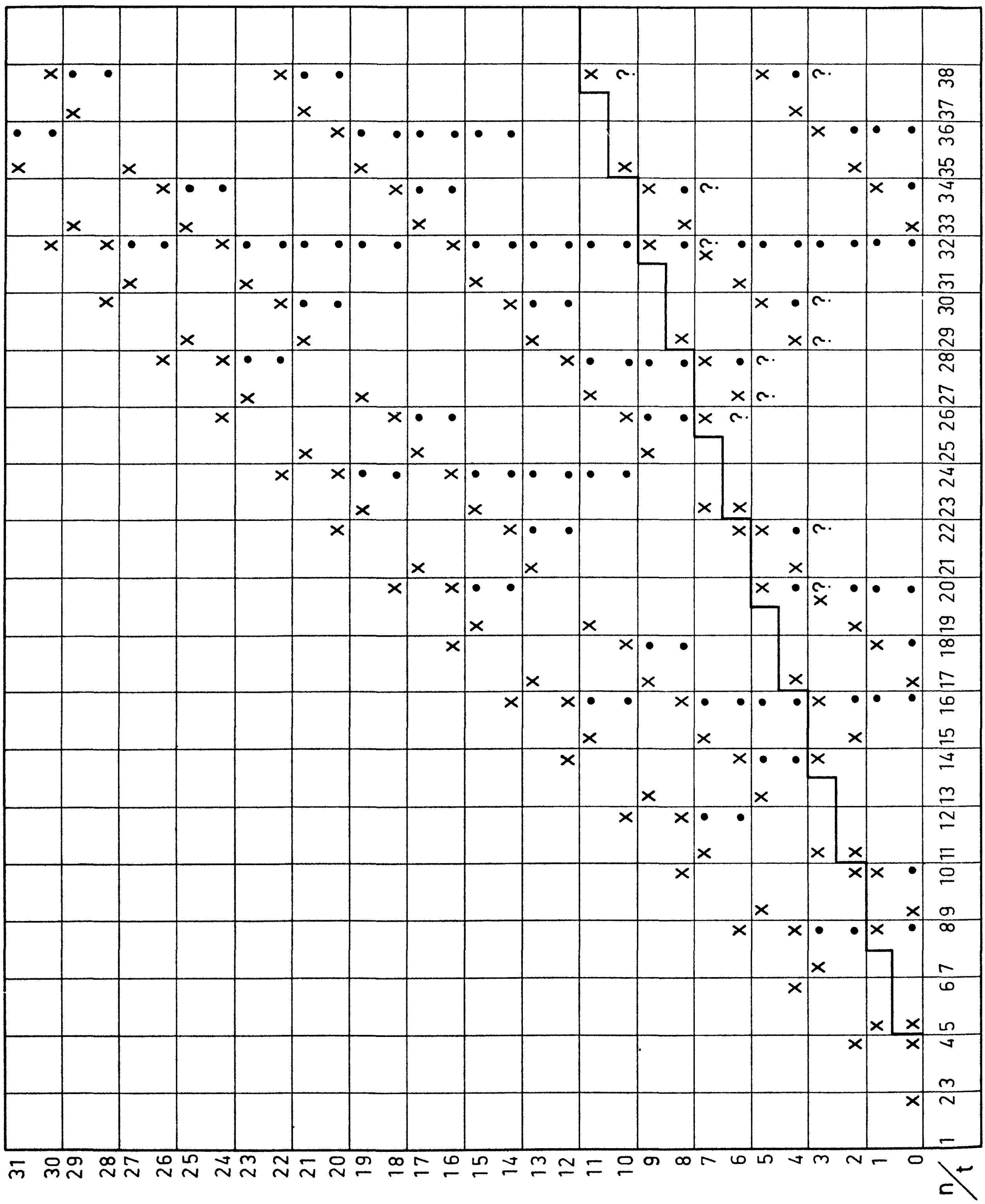

Table I

Graph of Theorem B in low dimensions, $\times$ *represents class in* E_2, ? *refers to unknown differentials on* $\times$*-classes for* $t > 3n + 4$.

$\kappa_m = p_m^*(\iota_m)$. Since F_m is $(m+n-1)$-connected, the Serre sequence (coefficients Z_2)

$$H^j(B_m) \xrightarrow{p^*} H^j(E_m) \xrightarrow{i^*} H^j(F_m) \xrightarrow{\tau} H^{j+1}(B_m)$$

is exact for $j < 2m+n-1$. Let m be large, then

$$\tau(a_1\eta_{n+m} + \Sigma a_i\eta_i) = a_1 \mathrm{Sq}^{n+1}\iota_m + \Sigma a_i \mathrm{Sq}^i \iota_m = 0.$$

By exactness, there exists a class $e_m \in H^j(E_m)$, $j = m+k-1$, where k is the grade of the (homogeneous) relation, such that $i^*(e_m) = a_1\eta_{n+m} + \Sigma a_i\eta_i$. If we perform the construction of the fibre spaces in such a way that $E_m = \Omega E_{m+1}$, we can assume e_m is primitive in the Hopf algebra $H^*(E_m)$. For small values of m, we can obtain a class e_m with the same properties by applying Ω. There is an indeterminacy in the choice of e_m. In [1] it is shown that this indeterminacy is the subgroup $\Sigma a_i H^{b_i}(E_m)$ $(i \geqslant n+1)$ where deg $a_i + b_i = k-1$. We shall denote the indeterminacy by $Q(r, m)$, r denoting the relation involved. We let Φ_m denote the secondary operation defined on κ_m with $\Phi(\kappa_m)$ equal to the coset of e_m modulo $Q(r, m)$. The collection $\{\Phi_m\}$ is the stable secondary operation associated with r.

In [1], Adams formalizes the procedure for directly connecting a relation with a universal example. The following proposition is an easy consequence of Theorem 3.7.2 [1], and its proof is omitted. We need it because there may be some choice in a relation representing a class in $\mathrm{Ext}_A^{1,*}(B(n), Z_2)$.

PROPOSITION 3.1. *Let $(C, \varepsilon\, d)$ and (C, ε', d') be two resolutions of $B(n)$. Let f be a chain equivalence of C with C'. Let r and r' be relations representing the same class in* Ext, *i.e. $f(r) = r'$. Then the secondary operations satisfy*

$$\Phi_m \bmod Q(r, m) = \Phi'_m \bmod Q(r', m) + c$$

where c is a primary operation.

3.2. *Proof of Theorem C.* All coefficients are Z_2. We use the universal examples $(E_m, B_m, F_m, \kappa_m, e_m)$ with $m = n, n+1$. The spaces are displayed in the commutative diagram below,

$$\begin{array}{ccccc}
F_n & \to & PF_{n+1} & \xrightarrow{f} & F_{n+1} \\
\downarrow & & \downarrow & & \downarrow \\
E_n & \to & PE_n & \xrightarrow{g} & E_{n+1} \\
\downarrow{\scriptstyle p_n} & & \downarrow & & \downarrow{\scriptstyle p_{n+1}} \\
B_n & \to & PB_{n+1} & \xrightarrow{h} & B_{n+1},
\end{array}$$

in which the rows and columns are fibrations, all spaces are H-spaces, all maps H-maps, the Serre cohomology spectral sequences are sequences of Hopf algebras, all differentials are Hopf algebra maps and the vertical maps induce maps of spectral sequences which are maps of differential Hopf algebras.

For $m=n$, all k-invariants are 0 and E_n is homotopically equivalent to $B_n x F_n$, however as an H-space it does not split this way. To find the H-space structure of E_n it is enough to determine the coproducts of the fundumental classes in the factors of F_n, η_{2n} and η_i. For $i>n+1$ it is easy to see that η_i is in the image of the suspension σ associated with the fibration g. This is because the k-invariants for $i>n+1$ are still zero in the fibration p_{n+1}. Thus these η_i are primitive. In the spectral sequence of g, we have $d_n(\kappa_n)=\kappa_{n+1}$. Thus $d_n(\kappa_n \otimes \kappa_{n+1})=\kappa_{n+1}^2$. But $\kappa_{n+1}^2=p^*(\iota_{n+1}^2)$ and $\iota_{n+1}^2= =\mathrm{Sq}^{n+1}\iota_{n+1}$ is the lowest dimensional k-invariant for p_{n+1}. Therefore $\kappa_{n+1}^2=0$. Since PE_{n+1} is acyclic we have $\kappa_n \otimes \kappa_{n+1}$ in the image of d_n. But the elements η_{2n} and κ_n^2 generate $H^{2n}(E_n)$. Since $d_n(\kappa_n^2)=0$ the only remaining possibility is $d_n(\eta_{2n})=\kappa_n \otimes \kappa_{n+1}$. Both κ_n and κ_{n+1} are primitive so $\kappa_n \otimes \kappa_{n+1}$ is not primitive. Since d_n is an H-map, η_{2n} is not primitive. By dimensionality the coproduct of η_{2n} is

$$1 \otimes \eta_{2n} + \kappa_n \otimes \kappa_n + \eta_{2n} \otimes 1 .$$

Since $i^*(e_n)=\eta_{2n}+\Sigma a_i \eta_i$ and e_n is primitive, e_n must have the summand $H(a_1)\,\kappa_n$. Projecting into B_n gives the result.

We conclude by listing some low dimensional ($t \leqslant n+8$) relations representing classes of Theorem B. They were obtain from minimal resolutions. We use $i=(i^*)^{-1}$ to represent pull backs.

Table II

Class in Ext	*Congruence class of n*	*Relation*
$q_0(n+1)$	$0(2)$	$\mathrm{Sq}^1\mathrm{Sq}^{n+1}$
$q_1(n+1)$	$3(4)$	$\mathrm{Sq}^3\mathrm{Sq}^{n+1}+\mathrm{Sq}^1\mathrm{Sq}^{n+3}$
$q_1(n+1)$	$0(4)\quad n>0$	$\mathrm{Sq}^3\mathrm{Sq}^{n+1}+\mathrm{Sq}^2\mathrm{Sq}^{n+2}$
$q_1(n+1)$	$1(4)$	$\mathrm{Sq}^3\mathrm{Sq}^{n+1}$
$iq_1(n+1)$	$3(4)$	$\mathrm{Sq}^3\mathrm{Sq}^1\mathrm{Sq}^{n+1}+\mathrm{Sq}^2\mathrm{Sq}^{n+3}$
$i^2q_1(n+1)$	$2(4)$	$\mathrm{Sq}^5\mathrm{Sq}^{n+1}$
$q_2(n+1)$	$5(8)$	$\mathrm{Sq}^7\mathrm{Sq}^{n+1}+\mathrm{Sq}^5\mathrm{Sq}^{n+3}+\mathrm{Sq}^1\mathrm{Sq}^{n+7}$
$q_2(n+1)$	$6(8)$	$\mathrm{Sq}^7\mathrm{Sq}^{n+1}+(\mathrm{Sq}^6+\mathrm{Sq}^5\mathrm{Sq}^1)\,\mathrm{Sq}^{n+2}+\mathrm{Sq}^2\mathrm{Sq}^{n+6}$
$q_2(n+1)$	$7(8)$	$\mathrm{Sq}^7\mathrm{Sq}^{n+1}+\mathrm{Sq}^3\mathrm{Sq}^{n+5}$
$q_2(n+1)$	$0(8)\quad n>0$	$\mathrm{Sq}^7\mathrm{Sq}^{n+1}+\mathrm{Sq}^6\mathrm{Sq}^{n+2}+\mathrm{Sq}^4\mathrm{Sq}^{n+4}$
$q_2(n+1)$	$1(8)\quad n>1$	$\mathrm{Sq}^7\mathrm{Sq}^{n+1}+\mathrm{Sq}^5\mathrm{Sq}^{n+3}$
$q_2(n+1)$	$2(8)$	$\mathrm{Sq}^7\mathrm{Sq}^{n+1}+\mathrm{Sq}^6\mathrm{Sq}^{n+2}$
$q_2(n+1)$	$3(8)$	$\mathrm{Sq}^7\mathrm{Sq}^{n+1}$

BIBLIOGRAPHY

[1] J. F. Adams, *On the non-existence of elements of Hopf invariant one*, Ann. Math. *72* (1960), 20–104.

[2] J. R. HARPER, *Stable secondary cohomology operations*, University of Chicago Disseration, Chicago, Illinois (1967).
[3] A. L. LIULEVICIUS, *Notes on homotopy of Thom spectra*, Amer. J. Math. *86* (1964), 1–16.
[4] ——, *The factorization of cyclic reduced powers by secondary cohomology operations*, Memoirs of the Amer. Math. Soc. *42* (1962).
[5] ——, *The cohomology of Massey–Peterson algebras*, to appear.
[6] S. MACLANE, *Homology* (Academic Press, New York 1963).
[7] W. S. MASSEY, *Products in exact couples*, Ann. Math. *59* (1954), 558–569.
[8] J. W. MILNOR, *The Steenrod algebra and its dual*, Ann. Math. *67* (1958), 150–171.
[9] N. E. STEENROD and D. B. A. EPSTEIN, *Cohomology Operations* (Princeton University Press, 1962).
[10] C. T. C. WALL, *Generators and relations for the Steenrod algebra*, Ann. Math. *72* (1960), 429–444.

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