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# Foliations on Open Manifolds, II

by ANTHONY PHILLIPS

## 1. Introduction

Consider an  $n$ -dimensional smooth Riemannian manifold  $M$  which is open (i.e. has no compact components) and on  $M$  a field  $\sigma$  of tangent  $k$ -planes. This note gives a sufficient condition for  $\sigma$  to be homotopic to an integrable field. The condition is stated in terms of the complementary  $q$ -plane field  $\sigma^\perp$  ( $q=n-k$ ), which we may consider as a  $q$ -dimensional subbundle of  $TM$ , the tangent bundle of  $M$ .

**THEOREM:** *If the structural group of  $\sigma^\perp$  can be reduced to a discrete group, then  $\sigma$  is homotopic to an integrable field.*

**REMARKS:** This theorem was suggested by the following result of Ehresmann ([1, p. 38], [2, p. 364]): Let  $N \subset M$  be an embedded submanifold; then  $N$  is a leaf of a foliation of a neighborhood of  $N$  in  $M$  if and only if the structural group of the normal bundle of  $N$  in  $M$  can be reduced to a discrete group. It gives a partial answer to a question posed by Reeb [7], Haefliger [2] and Thomas [8]. Since the one-dimensional orthogonal group is discrete, this gives a simpler proof of [6], Theorem 1.2: every  $(n-1)$ -plane field on  $M$  is homotopic to an integrable field. The restriction to open manifolds allows the use of submersion theory.

The proof of this theorem is given in the next two sections; the last section contains an example.

Early drafts of this note contain a much more restricted theorem. I am very grateful to André Haefliger for pointing out this generalisation.

## 2. Proof of Theorem

Let  $\tilde{M}$  be the universal cover of  $M$ , with  $p: \tilde{M} \rightarrow M$  the projection, and denote by  $\alpha: \tilde{M} \rightarrow \tilde{M}$  the covering transformation corresponding to the element  $\alpha$  of the fundamental group  $\pi$  of  $M$ . We will use the notation  $\alpha_*: T\tilde{M} \rightarrow T\tilde{M}$ , etc., for the differential of  $\alpha: \tilde{M} \rightarrow \tilde{M}$ , etc. A field  $\tau$  of tangent planes on  $\tilde{M}$  satisfying

$$\alpha_*\tau = \tau \quad \text{for } \alpha \in \pi \tag{*}$$

projects on  $M$  to a field  $p_*\tau$  which is integrable if and only if  $\tau$  is. The fields  $\sigma, \sigma^\perp$  are lifted up by  $p$  to fields  $p^*\sigma, p^*\sigma^\perp$  on  $\tilde{M}$  satisfying (\*). Let us give  $\tilde{M}$  the Riemannian metric pulled back from  $M$  by  $p$ ; then  $p^*\sigma^\perp = (p^*\sigma)^\perp$ .

The universal covering  $p: \tilde{M} \rightarrow M$  is a principal bundle over  $M$  with group  $\pi$ . The bundle  $\sigma^\perp$ , having discrete structural group, is isomorphic to  $\tilde{M} \times_r R^q$ , the  $q$ -plane bundle associated to  $p: \tilde{M} \rightarrow M$  by a representation  $r: \pi \rightarrow O(q)$ . (Compare [3], Lemma 1.) To simplify notation, let  $r_\alpha = r(\alpha)$ . As usual, we construct  $\tilde{M} \times_r R^q$  by dividing  $\tilde{M} \times R^q$  by the relation  $(x, y) \sim (\alpha(x), r_\alpha(y))$ , for  $\alpha \in \pi$ . Let  $\varphi: \sigma^\perp \rightarrow \tilde{M} \times_r R^q$  be the isomorphism, and  $\tilde{p}: \tilde{M} \times R^q \rightarrow \tilde{M} \times_r R^q$  the canonical projection.

There exists a unique trivialisation  $\Phi: p^*\sigma^\perp \rightarrow \tilde{M} \times R^q$  making the following square of bundle maps commute, as can easily be verified.

$$\begin{array}{ccc} p^*\sigma^\perp & \xrightarrow{\Phi} & \tilde{M} \times R^q \\ p_* \downarrow & & \downarrow \tilde{p} \\ \sigma^\perp & \xrightarrow{\varphi} & \tilde{M} \times_r R^q \end{array}$$

The reason for defining  $\Phi$  is to obtain the tangent bundle map  $H: T\tilde{M} \rightarrow TR^q$  by the composition

$$T\tilde{M} \longrightarrow p^*\sigma^\perp \xrightarrow{\Phi} \tilde{M} \times R^q \xrightarrow{\text{proj.}} R^q = TR_0^q \subset TR^q$$

(the first map is orthogonal projection, and  $TR_0^q$  denotes the tangent space at the origin.) This map has kernel  $\ker H = p^*\sigma$ ; furthermore it is easy to check that  $H$  is a  $\pi$ -equivariant epimorphism in the sense that  $H \circ \alpha_* = (r_\alpha)_* \circ H$ , for  $\alpha \in \pi$ , and that  $\text{rank}(H) = q$ .

**LEMMA:** *The map  $H$  is homotopic through  $\pi$ -equivariant epimorphisms to the differential  $f_*$  of a submersion  $f: \tilde{M} \rightarrow R^q$ .*

This lemma is proved in the next section. It yields the proof of the theorem, as follows. If  $H_t$  is the homotopy,  $0 \leq t \leq 1$ , then  $\ker H_t$  defines a homotopy between  $p^*\sigma$  and the  $k$ -plane field  $\ker(f_*)$ , which is tangent to the foliation given by the manifolds  $\{f = \text{constant}\}$ . By equivariance each  $\ker H_t$  satisfies (\*), so  $p_*\ker H_t$ ,  $0 \leq t \leq 1$ , gives a homotopy between  $\sigma$  and the integrable field  $p_*\ker(f_*)$ .

### 3. Proof of Lemma

First, following [5], Corollary 1.2, realize  $M$  as  $M = \bigcup_{i=1}^\infty U_i$ , an expanding union of compact manifolds with boundary, such that  $U_0$  is an  $n$ -disc and either a)  $U_{i+1}$  retracts into  $U_i$  through embeddings which leave  $U_{i-2}$  fixed, or b)  $U_{i+1}$  is  $U_i$  with a handle of index  $\leq n-1$  attached. We may assume that the  $p^{-1}U_i$  give a similar decomposition for  $\tilde{M}$ ; of course now in case b)  $p^{-1}U_{i+1}$  will be  $p^{-1}U_i$  with one handle attached for each element of  $\pi$ . This can be guaranteed, for instance, by taking neighborhoods of the simplexes of an  $(n-1)$ -dimensional spine of  $M$  (which exists by [9], Theorem 3.2) in a sufficiently fine triangulation.

For convenience in indexing, we will parametrize the homotopy by  $[0, \infty]$ . Let  $U_0^*$  be one component of  $p^{-1}U_0$ ; then  $U_0^*$  is an  $n$ -disc and, by Lemma 2.1 of [5],  $H|U_0^*$  is homotopic through bundle epimorphisms to the differential  $f'_*$  of a submersion  $f':U_0^*\rightarrow R^q$ . Let  $V_0$  be an open set containing  $U_0^*$  and such that  $V_0\cap\alpha(V_0)=\emptyset$  for  $\alpha\in\pi$ , and let  $H'_t, t\in[0, 1]$ , be a homotopy, fixed outside  $V_0$ , between  $H$  and a bundle epimorphism  $H'_1$  equal to  $f'_*$  on  $U_0^*$ . Such a homotopy, through bundle epimorphisms, exists by [5], Lemma 5.1. A  $\pi$ -equivariant homotopy  $H_t:TM\rightarrow TR^q, 0\leqq t\leqq 1$ , is now defined by  $\tilde{H}_t=H$  outside  $p^{-1}p(V_0)$ , and  $H_t|_{\alpha(x)}=(r_\alpha)_*\circ H'_t|_x\circ\alpha_*^{-1}$  for  $x\in V_0$ , and  $\alpha\in\pi$ . Also define  $f_0:p^{-1}U_0\rightarrow R^q$  by  $f_0\circ\alpha(x)=r_\alpha\circ f'(x)$ , for  $x\in U_0^*, \alpha\in\pi$ .

Induction hypothesis: between  $t=0$  and  $t=k$  we have deformed  $H$  through  $\pi$ -equivariant epimorphisms to a map  $H_k$  which over  $p^{-1}U_k$  is the differential of a submersion  $f_k$ . Observe that the deformation between  $k$  and  $k+1$  will leave this map fixed on  $p^{-1}U_{k-2}$ . The induction step will thus imply the existence of a well-defined homotopy, since for any  $x\in p^{-1}U_k, H_t|_x=H_{k+2}|_x$  for  $t\geqq k+2$ .

Proof of induction step: cases a) and b) must be distinguished. In case a) there is a homotopy  $h_t, t\in[0, 1]$ , of embeddings of  $U_{k+1}$  in itself, joining the identity map of  $U_{k+1}$  to  $h_1:U_{k+1}\rightarrow U_k$ , and such that each  $h_t$  is the identity on  $U_{k-2}$ . Covering this homotopy defines a unique similar homotopy  $\tilde{h}_t$  on  $p^{-1}U_{k+1}$ . Let  $f_{k+1}=f_k\circ\tilde{h}_1$ . Extend the homotopy  $(h_t)_*:TU_{k+1}\rightarrow TU_{k+1}$  to a homotopy of bundle epimorphisms  $L_t:TM\rightarrow TM$  (see [5], Lemma 5.1); cover this homotopy to define  $\tilde{L}_t:TM\rightarrow TM$ , and finally let  $H_{k+t}=H_k\circ\tilde{L}_t$ , for  $t\in[0, 1]$ .

In case b), let  $U_{k+1}^*$  be one of the components of  $p^{-1}U_{k+1}-p^{-1}U_k$ . It follows from the proof of [5], Lemma 6.2 that  $H_k|U_{k+1}^*$  is homotopic through epimorphisms fixed near  $p^{-1}U_k$  to  $f'_*$ , where  $f':U_{k+1}^*\rightarrow R^q$  is a submersion extending  $f_k$ . Proceed as in the case  $k=0$  to define  $f_{k+1}$  and a homotopy from  $H_k$  to  $H_{k+1}$  satisfying the induction hypothesis.

#### 4. An Example

Consider the punctured projective space  $P^6-x$ . This manifold is doubly covered by  $S^5\times R$ . On  $S^5\times R$  the 2-plane field spanned by  $v$  (the pullback of a non-zero vectorfield on  $P^5$  via the projections  $S^5\times R\rightarrow S^5\rightarrow P^5$ ) and  $w=\partial/\partial t$  is invariant under the antipodal map and so defines a 2-plane field on  $P^6-x$ . This bundle clearly has discrete structural group; it now follows that  $P^6-x$  has a 4-dimensional foliation.

The existence of such a foliation cannot be directly deduced from submersion theory, for the following argument shows there can be no submersion from  $P^6-x$  to a 2-dimensional manifold  $M$ . Such a submersion  $f$  would split  $T(P^6-x)$  as  $\eta^2\oplus\xi^4$ , where  $\eta=f^*TM$ . Note first that for  $i\leqq 5$  the  $i^{\text{th}}$  Stiefel-Whitney class [4] of  $P^6-x$  is  $w_i(P^6-x)=a^i$ , where  $a$  generates  $H^1(P^6-x; Z_2)=Z_2$ . The equation  $w_5(P^6-x)=w_1\eta w_4\xi+w_2\eta w_3\xi$  shows that  $w_1\eta$  and  $w_2\eta$  cannot both be 0. Suppose  $w_1\eta\neq 0$ , that

is  $w_1\eta = a$ . This would imply  $a^3 = w_1(f^*TM)^3 = f^*(w_1TM)^3 = 0$ . Similarly  $w_2\eta \neq 0$  is also impossible.

Finally, note that the 2-plane field spanned by  $v$  and  $w$  on  $S^5 \times R$  is integrable, and thus projects to give an integrable 2-plane field on  $P^6 - x$ . This manifold therefore carries foliations of dimensions 1 (integrate a non-zero vectorfield), 2, 4 and 5 (see the Remarks in § 1). On the other hand, a straightforward argument with Stiefel-Whitney classes shows that there can exist no 3-plane field on  $P^6 - x$ .

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