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The Changes of Sign of Certain Arithmetical Error-Terms

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1. Introduction

If $\pi(x)$ denotes the number of primes not exceeding x, and $\operatorname{li} x = \int_2^x (\log t)^{-1}\, dt\ (x \geqslant 2)$, then the prime number theorem states that $\pi(x) \sim \operatorname{li} x$, as $x \to \infty$. The error-term in this asymptotic relation is $\pi(x) - \operatorname{li} x$, and it was for long conjectured that $\pi(x) - \operatorname{li} x < 0$ for all large x. J. E. LITTLEWOOD proved this conjecture false by showing [8] that $\pi(x) - \operatorname{li} x = \Omega_\pm(x^{1/2} \log\log\log x / \log x)$. The prime number theorem is equivalent to the assertion that $\psi(x) \sim x$, where ψ is the well-known Chebyshev function. The error-term here is $\psi(x) - x$, and it changes sign an infinity of times, as shown by PHRAGMÉN [9]. PHRAGMÉN's result is a corollary of a general theorem of LANDAU's [7] on Dirichlet integrals. PÓLYA [11] refined LANDAU's theorem, and considered, as a particular case, the problem of estimating the number of changes of sign of $\psi(x) - x$ in the interval $1 < x \leqslant t$. If $N(t)$ denotes that number, then PÓLYA's result implies that

$$\overline{\lim_{t \to \infty}} \frac{N(t)}{\log t} > 0. \tag{1.1}$$

The original proof of PÓLYA's theorem contains a gap, first recognized by PÓLYA himself, which is filled in § 2. PÓLYA's theorem can be applied, as we shall show in § 2.3, to the error-term associated with the arithmetical function $r_k(n)$, which is the number of representations of an integer n as a sum of k squares ($k \geqslant 2$). This application is made possible by the fact that the Dirichlet series $\sum_{n=1}^{\infty} r_k(n) \cdot n^{-s}$ represents the Epstein zeta-function $\zeta_k(s)$, which satisfies HECKE's functional equation, namely

$$\pi^{-s} \Gamma(s)\, \zeta_k(s) = \pi^{s-k/2}\, \Gamma\left(\frac{k}{2} - s\right) \zeta_k\left(\frac{k}{2} - s\right), \tag{1.2}$$

and this equation implies a fundamental identity given by K. CHANDRASEKHARAN and RAGHAVAN NARASIMHAN [2].

More generally, we consider in § 3 the functional equation of CHANDRASEKHARAN and NARASIMHAN, which includes (1.2), and study the problem of change of sign of the "error-term" associated with the coefficients of Dirichlet series which satisfy such an equation. Thus, given an equation such as

$$\Delta(s)\, \varphi(s) = \Delta(\delta - s)\, \varphi(\delta - s), \tag{1.3}$$

where δ is a real number, $\Delta(s)$ is a product of a finite number of gamma functions, say $\Delta(s)=\prod_{\nu=1}^{N}\Gamma(\alpha_\nu s+\beta_\nu)$, and $\varphi(s)=\sum_{n=1}^{\infty}a_n\lambda_n^{-s}$, we define for $\varrho\geqslant 0$,

$$A_\lambda^\varrho(x)=\frac{1}{\Gamma(\varrho+1)}\sum_{\lambda_n\leqslant x}{}' a_n(x-\lambda_n)^\varrho,$$

which is the fractional integral of order ϱ of the summatory function $A_\lambda^0(x)\equiv A(x)=\sum_{\lambda_n\leqslant x}{}' a_n$. CHANDRASEKHARAN and NARASIMHAN have shown [3] that corresponding to the equation (1.3), there exists a "residual function" $S_\varrho(x)$, such that

$$\operatorname{Re}\{A_\lambda^\varrho(x)-S_\varrho(x)\}=\Omega_\pm(x^\Theta), \tag{1.4}$$

where $\Theta=\{A\delta+(2A-1)\varrho-\frac{1}{2}\}/2A$, with $A=\sum_{\nu=1}^{N}\alpha_\nu$. (A similar result holds for the imaginary part of $A_\lambda^\varrho(x)-S_\varrho(x)$.) The proof of this general Ω-theorem rests on the fact that equation (1.3) implies, for sufficiently large ϱ, the formula

$$x^c\{A_\lambda^\varrho(x)-S_\varrho(x)\}=\sum_{n=1}^{\infty}c_n\cdot\cos(\gamma_n x^{1/2A}+D)+g(x), \tag{1.5}$$

where c is a real constant, $\sum_{n=1}^{\infty}|c_n|<\infty$, $0<\gamma_1<\gamma_2<\cdots<\gamma_n\to\infty$, A is as in (1.4), D is a real constant, and $g(x)=o(1)$, as $x\to\infty$.

Clearly, (1.4) implies that the real part of the "error-term" $A_\lambda^\varrho(x)-S_\varrho(x)$ has an infinity of changes of sign in the interval $0<x<\infty$. In this paper, we obtain a lower bound for the number of changes of sign of $\operatorname{Re}\{A_\lambda^\varrho(x)-S_\varrho(x)\}$, and of $\operatorname{Im}\{A_\lambda^\varrho(x)-S_\varrho(x)\}$, in a given interval (Theorem 4.1). This is achieved by combining asymptotic formula (1.5) with an argument introduced by LIOUVILLE, and later applied by PÓLYA [12] to the study of the changes of sign of certain trigonometrical series. In the case $\varrho=0$, Theorem 4.1 gives a lower bound for the number of changes of sign, in any interval, of the error-term associated with such arithmetical functions as $d(n)$, the number of positive divisors of the positive integer n, or $r_k(n)$, or RAMANUJAN'S function $\tau(n)$.

It may be remarked that the results obtained by appealing to PÓLYA'S theorem are weaker than those obtained in § 4 from asymptotic formula (1.5), since PÓLYA'S theorem gives only a "lim sup result", as in (1.1), for an interval $0<x\leqslant t$, whereas Theorem 4.1 gives a lower bound for the *number* of changes of sign of the error-term under consideration in *any* given interval.

The problems discussed in this paper were suggested to me by Professor K. CHANDRASEKHARAN; I take pleasure in recording here my gratitude for his advice and constant encouragement.

2. Pólya's Theorem

2.1. If ω is a function of the real variable u, bounded and integrable over every finite interval $1 \leqslant u \leqslant U$, and s is a complex number, written $s=\sigma+it$, where σ and t are real, and $i^2=-1$, then the integral $\int_1^\infty \omega(u)u^{-s}du$ is called a Dirichlet integral.

A theorem of LANDAU's ([7]; [6], p. 88) states that if $\omega(u)$ is real-valued, and is of constant sign for all sufficiently large u, and if the integral $f(s)=\int_1^\infty \omega(u)u^{-s}du$ has a finite abscissa of convergence $\sigma=\sigma_0$, then the real point $s=\sigma_0$ of its line of convergence is a singularity of the function $f(s)$ which it represents.

In order to state this theorem in a more convenient form, we introduce a function $W(x)$ associated with the sign of $\omega(u)$. We assume that $\omega(u)$ is either of constant sign for $u>1$, or that there exists a sequence (u_n), $1=u_0<u_1<u_2<\cdots$, with no finite point of accumulation, such that

$$(-1)^n\,\omega(u) \geqslant 0 \quad \text{for} \quad u_{n-1}<u<u_n, \tag{2.1}$$

and such that $\omega(u)$ is not identically zero in any of the intervals $u_{n-1}<u<u_n$. If $\omega(u)$ is of constant sign for $u>1$, then $W(x)=0$; otherwise we define $W(x)=n$ for $u_{n-1}\leqslant x<u_n$. Thus $W(x)$ is simply the number of changes of sign of $\omega(u)$ in the interval $1<u\leqslant x$. LANDAU's theorem then takes the following form.

THEOREM A (LANDAU). *Let* $\int_1^\infty \omega(u)u^{-s}du$ *have a finite abscissa of convergence* σ_0. *Let* $\Phi(s)=\int_1^\infty \omega(u)u^{-s}du$ *be regular in the half-plane* $\sigma>\theta$, *but in no larger half-plane* $\sigma>\theta-\varepsilon$ $(\varepsilon>0)$. If $\Phi(s)$ *is regular at* $s=\theta$, *then* $\lim_{x\to\infty} W(x)=+\infty$.

PÓLYA's extension [11] of LANDAU's theorem is as follows.

THEOREM B (PÓLYA). *Let the integral* $\int_1^\infty \omega(u)u^{-s}du$ *have a finite abscissa of convergence* σ_0. *Let*

$$\Phi(s)=\int_1^\infty \omega(u)u^{-s}du \tag{2.2}$$

be regular in the half-plane $\sigma>\theta$, *but in no larger half-plane* $\sigma>\theta-\varepsilon$ $(\varepsilon>0)$. *Further, let* $\Phi(s)$ *be meromorphic in* $\sigma\geqslant\theta-b$, *for some* $b>0$. *Then,*

$$\overline{\lim_{x\to\infty}}\frac{W(x)}{\log x}\geqslant\frac{\gamma}{\pi}, \tag{2.3}$$

where γ *is defined as follows: If* $\Phi(s)$ *has poles on the line* $\sigma=\theta$, *then* $s=\theta+i\gamma$ *is the pole with the smallest non-negative imaginary part; otherwise,* $\gamma=+\infty$.

2.2. We shall now indicate how the gap in PÓLYA's original proof of Theorem B can be filled. PÓLYA's proof applies previous results of his on entire functions of exponential type [10]. Use is also made of certain properties of plane convex sets. The argument runs as follows.

If $\omega(u)$ is of constant sign for $u>u_0$, Theorem A implies that $\gamma=0$, and then (2.3) is trivial. We therefore suppose that $\omega(u)$ has an infinity of changes of sign. Let (u_n) be the points of change of sign of $\omega(u)$, and let $W(x)$ be the number of its changes of sign in the interval $1<u\leqslant x$, as defined earlier. Set

$$d=\overline{\lim_{x\to\infty}}\,\frac{W(x)}{\log x},$$

and suppose at first that $d<\infty$; then PÓLYA shows that the infinite product

$$\prod_{n=1}^{\infty}\left\{1-\frac{z^2}{(\log u_n)^2}\right\}=F(z)$$

is absolutely convergent, and that F is an entire function of exponential type ([11], p. 22). He sets

$$F(z)=a_0+\frac{a_2 z^2}{2!}+\frac{a_4 z^4}{4!}+\cdots,$$

defines

$$f(z)=\frac{a_0}{z}+\frac{a_2}{z^3}+\frac{a_4}{z^5}+\cdots, \tag{2.4}$$

and shows that series (2.4) converges (at least) for $|z|>\pi d$ ([10], p. 578). Let J be the convex hull of the singularities of f. Since f is an odd function, and the coefficients (a_n) are real, J is symmetric with respect to both real and imaginary axes. Let $s=\kappa$ be the point at which the positive real axis intersects the boundary of J.

Theorem B is established by applying LANDAU's Theorem A to the function

$$\Phi^*(s)=\int_1^{\infty}\omega(u)\,F(\log u)\,u^{-s}\,du. \tag{2.5}$$

Because of the definition of F, combined with inequality (2.1), the integrand in (2.5) satisfies the inequality

$$\omega(u)\,F(\log u)\leqslant 0\quad(u>1). \tag{2.6}$$

A theorem proved by PÓLYA in [10] (*Satz* V, p. 598) implies that $\Phi^*(s)$ is regular in the half-plane $\sigma>\theta+\kappa$ ([11], p. 25). The behaviour of $\Phi^*(s)$ for $\sigma\leqslant\theta+\kappa$ depends on the behaviour of $\Phi(s)$, defined by (2.2), on the line $\sigma=\theta$. Accordingly, PÓLYA distinguishes two cases:

(a) Φ has poles on $\sigma=\theta$;

(b) Φ is regular on $\sigma=\theta$.

In Case (a), he shows that if $\pi d<\gamma$, then Φ^* is regular at $s=\theta+\kappa$, but has a singularity at another point of the line $\sigma=\theta+\kappa$. But this is impossible, because of (2.6) and Theorem A. Therefore $\pi d\geqslant\gamma$.

The gap in PÓLYA's proof occurs in Case (b). The argument here is that if $d<\infty$, then Φ^* is regular at $s=\theta+\kappa$, but has singularities arbitrarily near the line $\sigma=\theta+\kappa$, in the half-plane $\sigma<\theta+\kappa$. Again, this is impossible because of (2.6) and Theorem A. Hence $d=+\infty$.

In order to establish this part of Theorem B, we require the following result on plane convex sets:

LEMMA 2.1. *Let $\mathfrak{A}$ be a closed, bounded, plane convex set, whose boundary consists only of extreme points*[1]*. Let l be a supporting line*[2] *of $\mathfrak{A}$ through $\xi\in\mathfrak{A}$. If $\mathfrak{B}$ is a translate of $\mathfrak{A}$, such that $\xi\in\mathfrak{B}$ and $\mathfrak{B}$ lies on the same side of l as $\mathfrak{A}$, then $\mathfrak{A}$ and $\mathfrak{B}$ coincide.*

Proof. Let τ be the translation $\tau:\mathfrak{A}\rightarrow\mathfrak{B}$. Since $\xi\in\mathfrak{B}$, $\xi\in l$, and $\mathfrak{B}$ is entirely on one side of l, l is a supporting line of $\mathfrak{B}$ through ξ. Since the boundary of $\mathfrak{A}$ consists only of extreme points, there are exactly two points on the boundary such that the supporting lines through these points are parallel to a given direction[3]. Now $\xi'=\tau(\xi)$ is a point on the boundary of $\mathfrak{B}$ such that there is a supporting line l' of $\mathfrak{B}$ through ξ' which is parallel to l. Since $\mathfrak{B}$ is on the same side of l as $\mathfrak{A}$, l' must be that one of the two supporting lines of $\mathfrak{B}$ parallel to l which is closest to l. Therefore $l=l'$ and $\xi=\xi'$, so that $\mathfrak{A}$ and $\mathfrak{B}$ coincide.

The problem which must be solved in order to establish Case (b) of Theorem B may be stated geometrically as follows[4].

Let (c_ν) be a sequence of points in the complex s-plane, with the following properties:

$$\left.\begin{array}{l}\textit{The points } c_\nu \textit{ lie pairwise symmetric to the real axis. They have no}\\ \textit{point of accumulation in the finite part of the plane.}\end{array}\right\} \tag{2.7}$$

$$\operatorname{Re}(c_\nu)<\theta \quad \textit{for all } \nu, \textit{ and } \quad \overline{\lim_{\nu\rightarrow\infty}}\operatorname{Re}(c_\nu)=\theta. \tag{2.8}$$

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1) An extreme point of a closed plane convex set K is a boundary point which is not an interior point of any line segment belonging to K (for example, $\mathfrak{A}$ can be a circle, or an ellipse, but not a rectangle).

2) A supporting line of a closed plane convex set K is a line which contains at least one point of K, and such that K lies entirely on one side of this line. A supporting line contains at most two extreme points. There are exactly two supporting lines parallel to a given direction.

3) This follows from the remarks in Footnote (2): there are two supporting lines of $\mathfrak{A}$ in each direction, and since all boundary points of $\mathfrak{A}$ are extreme points, each supporting line contains exactly one boundary point.

4) The (c_ν) are the poles of $\Phi(s)$ in the strip $\theta-b\leqslant\sigma\leqslant\theta$; J is the convex hull of the singularities of f [c.f. (2.4)]. With the notation introduced in [11], and according to the *Hilfssatz* on p. 24 of [11], ζ is a singularity of $X^*(s)$. Because of *Satz* V of [10], ζ is a regular point of $\Psi^*(s)$. Hence, ζ is a singularity of $\Phi^*=\Psi^*+X^*$ ([11], pp. 24–25).

Further, let J be a bounded, closed, plane convex set, which is symmetric with respect to both the real and the imaginary axis, and is contained in the disc $|s| \leqslant \pi d$. Let $\sigma = \kappa$ be the supporting line of J which is perpendicular to the positive real axis. Consider the sets $c_\nu + J$[5] $(\nu = 1, 2, \ldots)$. We have to prove that for each $\varepsilon > 0$, there exists a point $\zeta = \zeta(\varepsilon)$, which satisfies the following conditions:

$$\theta + \kappa - \varepsilon \leqslant \mathrm{Re}(\zeta) < \theta + \kappa, \tag{2.9}$$

$$\zeta \textit{ is an extreme point of some set } c_n + J, \tag{2.10}$$

and

$$\zeta \notin c_\nu + J \quad \textit{for} \quad \nu \neq n. \tag{2.11}$$

For that purpose, we consider the convex hull $\mathfrak{H}$ of all the sets $c_\nu + J$ with $\mathrm{Im}(c_\nu) > 0$. Let h be (one of) the extreme point(s) of $\mathfrak{H}$ on its supporting line parallel to the real axis; and let H be that part of the boundary of $\mathfrak{H}$ which is in the half-plane $\sigma > \mathrm{Re}(h)$. Then, $\mathfrak{H}$ and H have the following properties:

Each extreme point of $\mathfrak{H}$ belongs to the boundary of one of the sets $c_\nu + J$. (2.13)

Indeed, suppose, if possible, that p is an extreme point of $\mathfrak{H}$ such that $p \notin c_\nu + J$, for all ν. Then, since the sets $c_\nu + J$ are closed, we can find a circle $\mathfrak{C}$ with p as centre, such that $\mathfrak{C} \cap (c_\nu + J) = \emptyset$, for all ν. If $\mathfrak{C}$ is small enough, the set obtained by removing $\mathfrak{C} \cap \mathfrak{H}$ from $\mathfrak{H}$ is contained in a proper convex subset $\mathfrak{H}^*$ of $\mathfrak{H}$, since p is an extreme point of $\mathfrak{H}$[6]). But since $(\mathfrak{C} \cap \mathfrak{H}) \cap (c_\nu + J)$ is empty for all ν, we would have $\mathfrak{H} \subset \mathfrak{H}^*$, which is absurd.

H contains no half-line. (2.13)

Indeed, because of (2.8), H could contain a half-line only if this line were on the vertical $\sigma = \theta + \kappa$. Now there cannot be a point $q = (\theta + \kappa) + i\tau$ on the line $\sigma = \theta + \kappa$ such that all points of H with imaginary part greater than τ lie on this line, while those with imaginary part smaller than τ lie to the left of it (Fig. 1). For if this were the case, q would be an extreme point of $\mathfrak{H}$ and would therefore, by (2.12), belong to one of the $c_\nu + J$. But this is impossible, since $\mathrm{Re}(c_\nu) < \theta$.

There are extreme points of $\mathfrak{H}$ with arbitrarily large imaginary part. (2.14)

Indeed, let $t_0 > 0$ be given. Because of (2.7) and (2.8), we can find a point $p_1 \in H$ such that $\mathrm{Im}(p_1) \geqslant t_0$. Because of (2.13), p_1 does not lie on any half-line belonging to H. Therefore, if p_1 is not itself an extreme point of $\mathfrak{H}$, the supporting line of $\mathfrak{H}$ through p_1 contains two extreme points of $\mathfrak{H}$. One of these, say p_2, is such that $\mathrm{Im}(p_2) > \mathrm{Im}(p_1) \geqslant t_0$.

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[5]) By $c_\nu + J$, we understand the translate of J through the vector c_ν.

[6]) For a proof of this property of extreme points, see for instance [10], pp. 577–578.

Now it follows from the conditions on the (c_ν) and from the convexity of $\mathfrak{H}$ that if $h_1 \in H$ and $\theta+\kappa-\varepsilon \leqslant \mathrm{Re}(h_1) < \theta+\kappa$, all points $h_2 \in H$ for which $\mathrm{Im}(h_2) > \mathrm{Im}(h_1)$ also lie in the strip $\theta+\kappa-\varepsilon \leqslant \sigma < \theta+\kappa$. From this remark, and from (2.8) and (2.14), it easily follows that there is a point ζ^*, such that

$$\zeta^* \in H, \tag{2.15}$$

$$\theta + \kappa - \varepsilon \leqslant \mathrm{Re}(\zeta^*) < \theta + \kappa, \tag{2.16}$$

$$\zeta^* \textit{ lies on the boundary of a set } c_n + J \textit{ with } \mathrm{Im}(c_n) > 2\pi d. \tag{2.17}$$

In order to locate a point ζ with properties (2.9), (2.10) and (2.11), we shall consider two cases, according as the boundary of J consists only of extreme points, or not.

A. If the boundary of J consists only of extreme points, we may choose $\zeta = \zeta^*$. Indeed, ζ^* is an extreme point of $c_n + J$. Since J is contained in the disc $|s| \leqslant \pi d$, it follows from (2.17) that $\zeta^* \notin c_\nu + J$ if $\mathrm{Im}(c_\nu) \leqslant 0$. By applying Lemma 2.1 with $\mathfrak{A} = c_n + J$ and $\xi = \zeta^*$, we see that $\zeta^* \notin c_\nu + J$ if $\mathrm{Im}(c_\nu) > 0$ and $\nu \neq n$.

B. If the boundary of J does not consist entirely of extreme points, ζ^* need not be an extreme point of $c_n + J$. Also, ζ^* may belong to the boundary of some other translate of J, say of $c_{n_1} + J$ (Fig. 2). But if ζ^* is not an extreme point of $c_n + J$, the

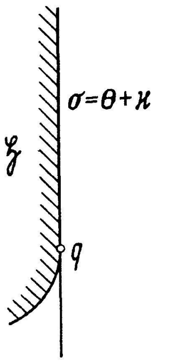

Fig. 1.

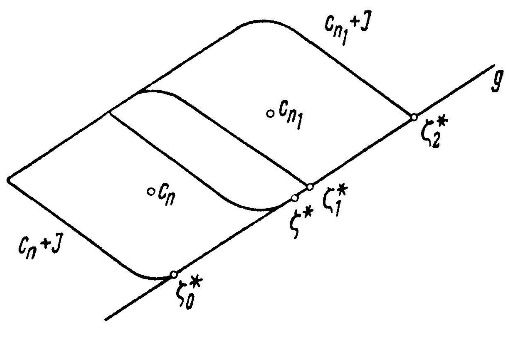

Fig. 2.

supporting line g of $c_n + J$ through ζ^* contains two extreme points of $c_n + J$, say ζ_0^* and ζ_1^*. One of these has a real part greater than that of ζ^*; suppose that $\mathrm{Re}(\zeta_1^*) > \mathrm{Re}(\zeta^*)$. Then, $\theta+\kappa-\varepsilon < \mathrm{Re}(\zeta_1^*) \leqslant \mathrm{Re}(c_n)+\kappa < \theta+\kappa$, so that ζ_1^* lies in the strip $\theta+\kappa-\varepsilon \leqslant \sigma < \theta+\kappa$. If ζ_1^* lies on the boundary of $c_{n_1} + J$ (as in Fig. 2), consider its translate ζ_2^*; ζ_2^* lies on g, is an extreme point of $c_{n_1} + J$, and is in the strip $\theta+\kappa-\varepsilon \leqslant \sigma < \theta+\kappa$.

Should ζ_2^* lie on the boundary of some other translate $c_{n_2} + J$ of J, we can find in the same manner an extreme point ζ_3^* of $c_{n_2} + J$ on g and in the strip $\theta+\kappa-\varepsilon \leqslant \sigma <$

$\theta+\kappa$. Proceeding in this manner, we must finally obtain a set $c_{n_r}+J$ and an extreme point ζ_{r+1}^* of this set which is exterior to all the other sets $c_\nu+J$, for otherwise the points $c_n, c_{n_1}, c_{n_2}, \ldots$, which lie on a parallel to g, would have a point of accumulation in the finite part of the plane, in contradiction with condition (2.7). Then, we may take $\zeta=\zeta_{r+1}^*$.

This concludes the proof of Case (b), and hence of PÓLYA's theorem.

2.3. *A new application of Pólya's theorem*

Let $r_k(n)$ denote the number of representations of the positive integer n as a sum of k integral squares ($k\geqslant 2$), representations which differ only in sign, or order, being counted as distinct.

The generating function of $r_k(n)$ is $\zeta_k(s)$, EPSTEIN's zeta-function of order k [5], which has the representation

$$\zeta_k(s)=\sum_{n=1}^{\infty}\frac{r_k(n)}{n^s}$$

in the half-plane $\mathrm{Re}(s)>k/2$, and satisfies the functional equation

$$\pi^{-s}\Gamma(s)\,\zeta_k(s)=\pi^{s-k/2}\,\Gamma\left(\frac{k}{2}-s\right)\zeta_k\left(\frac{k}{2}-s\right). \tag{2.18}$$

Let

$$P_k^\varrho(x)=\frac{1}{\Gamma(\varrho+1)}\sum_{n\leqslant x}{}' r_k(n)\,(x-n)^\varrho-\frac{\pi^{k/2}\,x^{\varrho+k/2}}{\Gamma(\varrho+k/2+1)}+\frac{x^\varrho}{\Gamma(\varrho+1)}\qquad(\varrho\geqslant 0),$$

the dash meaning that if $\varrho=0$ and x is an integer, the last term in the sum must be multiplied by $\frac{1}{2}$. If $\varrho=0$, $P_k^0(x)$ is the error-term in the lattice-point problem for the sphere in k-dimensional space. Indeed, if we define $r_k(0)=1$, we have

$$P_k^0(x)=\sum_{0\leqslant n\leqslant x}{}' r_k(n)-\frac{(\pi x)^{k/2}}{\Gamma(k/2+1)}; \tag{2.19}$$

if x is not an integer, $\sum'_{0\leqslant n\leqslant x} r_k(n)$ is equal to the number of lattice-points in a sphere of radius $\sqrt{x}$, whose centre is a lattice-point, and $(\pi x)^{k/2}/\Gamma(k/2+1)$ is the volume of this sphere.

K. CHANDRASEKHARAN and RAGHAVAN NARASIMHAN have shown [2] that functional equation (2.18) implies the identity

$$\sqrt{\pi}\,2^{2r+1}\int_0^\infty F_k^\varrho(x^2)\,x^{2r}e^{-sx}\,dx=(2\pi)^{k/2}\sum_{n=1}^{\infty} r_k(n)\,g_n(s), \tag{2.20}$$

where $F_k^\varrho(x)=P_k^\varrho(2x)$, $\mathrm{Re}(s)>0$, r is a sufficiently large integer, and

$$g_n(s)=\sum_{\nu=0}^{r}(-1)^{\nu+r}e_\nu\frac{s^{2\nu}\Gamma(\gamma+\nu+r)}{2^{2\nu}(s^2+8\pi^2n^2)^{\gamma+\nu+r}},$$

where $\gamma=\varrho+\frac{1}{2}k+\frac{1}{2}$, and the e_ν are constants.

If k is *odd*, this identity allows the application of Pólya's theorem to $P_k^0(x)$.

Indeed, if we make the change of variable $x\to\log x$ in the integral on the left-hand side of (2.20), we obtain an identity which shows that the function of s defined by the integral

$$\int_1^\infty \log^{2r}x\cdot F_k^\varrho(\log^2 x)\,x^{-1-s}\,dx$$

is regular in the half-plane $\sigma>0$, and has singularities on the imaginary axis, at the points $s=\pm\pi n\sqrt{-8}$ $(n=1,2,\ldots)$.

If k is odd, these singularities are poles. Pólya's theorem can then be applied to obtain the following result, announced in [13]:

If k is odd, and if $W_k(t)$ denotes the number of changes of sign of $P_k^0(x)$ in the interval $0<x\leqslant t$, then

$$\overline{\lim_{t\to\infty}}\frac{W_k(t)}{\sqrt{t}}\geqslant 2. \tag{2.21}$$

If k is even, identity (2.20) cannot be used to estimate the number of changes of sign of $P_k^0(x)$; in this case, Pólya's theorem gives information only on the changes of sign of those $P_k^\varrho(x)$ for which $\varrho\equiv\frac{1}{2}$ (mod 1). This is curious, since results on representations of an integer as a sum of an odd number of squares are usually more difficult to obtain than results on representations as a sum of an even number of squares.

Identities analogous to (2.20), which involve the error-terms arising from other solutions of the functional equation $\Gamma(s)\,\varphi(s)=\Gamma(\delta-s)\,\psi(\delta-s)$, can be deduced from a general identity given in [2] (Lemma 3, p. 491), of which (2.20) is a particular case. However, these identities permit the application of Pólya's theorem only when $\delta+\varrho+\frac{1}{2}$ is an integer, and this condition often precludes the possibility of obtaining a result in the case $\varrho=0$.

Identities of this type in the case of functional equations with more than one gamma factor are not known. In the following sections, we shall apply a different method, and obtain a lower bound for the number of changes of sign, in a given interval, of the error-terms arising from any given instance of functional equation (1.3).

3. The Functional Equation

3.1. We begin by defining, after CHANDRASEKHARAN and NARASIMHAN [3], the general functional equation with which we shall be concerned.

DEFINITION 3.1. Let (λ_n) and (μ_n) be two sequences of real numbers such that

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty,$$
$$0 < \mu_1 < \mu_2 < \cdots < \mu_n \to \infty,$$

and let (a_n), (b_n) be two sequences of complex numbers, not all zero. Let δ be a real number, and s a complex variable with real part σ and imaginary part t. Let

$$\Delta(s) = \prod_{\nu=1}^{N} \Gamma(\alpha_\nu s + \beta_\nu), \tag{3.1}$$

where $N \geqslant 1$, $\alpha_\nu > 0$ and β_ν is complex, and let $A = \sum_{\nu=1}^{N} \alpha_\nu$. We say that the functional equation

$$\Delta(s)\,\varphi(s) = \Delta(\delta - s)\,\psi(\delta - s) \tag{3.2}$$

holds, if the functions φ and ψ are representable by the Dirichlet series

$$\varphi(s) = \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s}, \quad \psi(s) = \sum_{n=1}^{\infty} \frac{b_n}{\mu_n^s}, \tag{3.3}$$

each of which is absolutely convergent in some half-plane, and if there is a domain D in the s-plane, which is the exterior of a bounded, closed set S, and in which there exists a holomorphic function χ with the properties

$$\lim_{|t| \to \infty} \chi(\sigma + it) = 0,$$

uniformly in every interval $-\infty < \sigma_1 \leqslant \sigma \leqslant \sigma_2 < +\infty$, and

$$\chi(s) = \Delta(s)\,\varphi(s), \quad \text{for} \quad \sigma > c_1,$$
$$\chi(s) = \Delta(\delta - s)\,\psi(\delta - s), \quad \text{for} \quad \sigma < c_2,$$

where c_1 and c_2 are some constants.

3.2. For $\varrho \geqslant 0$, we define

$$A_\lambda^\varrho(x) = \frac{1}{\Gamma(\varrho + 1)} \sum_{\lambda_n \leqslant x}{}' a_n (x - \lambda_n)^\varrho, \tag{3.4}$$

the accent indicating that the last term of the sum is to be multiplied by $\frac{1}{2}$ if $\varrho = 0$ and $x = \lambda_n$. We shall restrict our considerations to the case where ϱ is an integer.

Let

$$S_\varrho(x)=\frac{1}{2\pi i}\int_{\mathscr{C}}\frac{\Gamma(s)\,\varphi(s)}{\Gamma(s+\varrho+1)}\,x^{s+\varrho}\,ds\,,\quad \varrho\geqslant 0\,, \tag{3.5}$$

where $\varphi(s)=\sum_{n=1}^{\infty} a_n\,\lambda_n^{-s}$, as in (3.3), and where $\mathscr{C}$ is a curve enclosing all the singularities of the integrand.

Further, let

$$c_\varrho=\frac{A\,\delta+\varrho}{2A}-\varepsilon\,,\quad 0<\varepsilon<\frac{1}{4A}\,,$$

and let

$$I_\varrho(x)=\frac{1}{2\pi i}\int_{\mathscr{C}'}\frac{\Gamma(\delta-s)\,\Delta(s)}{\Gamma(\delta+\varrho+1-s)\,\Delta(\delta-s)}\,x^{\delta+\varrho-s}\,ds\,, \tag{3.6}$$

where $\mathscr{C}'$ is a curve formed by the lines $\sigma=c_\varrho+it$, with $|t|>R$, together with three sides of the rectangle whose vertices are $c_\varrho-iR$, $c_\varrho+r-iR$, $c_\varrho+r+iR$ and $c_\varrho+iR$. We assume that

$$c_\varrho>\max\left(-\operatorname{Re}\frac{\beta_\nu}{\alpha_\nu}\right),\quad \nu=1,2,\ldots,N\,,$$

and choose r and R in such a manner that all the poles of the integrand in (3.6) are to the left of $\mathscr{C}'$.

It is shown in [3, §4] that the identity

$$A_\lambda^\varrho(x)-S_\varrho(x)=\sum_{n=1}^{\infty}\frac{b_n}{\mu_n^{\delta+\varrho}}\,I_\varrho(\mu_n x) \tag{3.7}$$

holds for $\varrho\geqslant 2A\beta-A\delta-\frac{1}{2}$, where β is such that $\sum_{n=1}^{\infty}|b_n|\,\mu_n^{-\beta}<\infty$.

For $I_\varrho(x)$ we have, as in [3, §4], the asymptotic formula

$$I_\varrho(x)=c\cdot x^{(\omega-1/2)/2A}\cos(h\,x^{1/2A}+D)+o(x^{(\omega-1/2)/2A})\,, \tag{3.8}$$

as $x\to\infty$, where c and D are real constants, $\omega=A\delta+(2A-1)\varrho$ and $h=2e^{-\theta/2A}$ with $\theta=2\{\sum_{\nu=1}^{N}\alpha_\nu\log\alpha_\nu-A\log A\}$, and δ, α_ν and A are as in Definition 3.1. A more precise asymptotic formula for $I_\varrho(x)$ is given in [4, Lemma 1], but (3.8) is sufficient for our purposes.

By combining (3.7) and (3.8) we obtain, for integral $\varrho\geqslant 2A\beta-A\delta-\frac{1}{2}$, the asymptotic formula

$$A_\lambda^\varrho(x)-S_\varrho(x)=c\sum_{n=1}^{\infty}\frac{b_n}{\mu_n^{\delta+\varrho}}(\mu_n x)^{(\omega-1/2)/2A}\cos\left(h(\mu_n x)^{1/2A}+D\right)+o(x^{(\omega-1/2)/2A})\,, \tag{3.9}$$

as $x\to\infty$.

In order to obtain a lower bound for the number of changes of sign of the real and imaginary parts of $A_\lambda^\varrho(x)-S_\varrho(x)$ in a given interval, for a given non-negative integer ϱ, we shall require, beside (3.9), the following trivial extension of ROLLE's theorem, which we state without proof.

LEMMA 3.1. *Let f be continuous in the closed interval $[a, b]$ and differentiable in the open interval (a, b), except perhaps at $c\in(a, b)$ at which, however, the left and right hand derivatives $f'^{-}(c)$ and $f'^{+}(c)$ exist. Let $f(a)=f(b)$. Then, f'^{-}, f'^{+} and $\frac{1}{2}(f'^{-}+f'^{+})$ all change sign at least once in (a, b).*

We also state as a lemma some properties of $A_\lambda^\varrho(x)$ and $S_\varrho(x)$ which are easily verified with the definitions (3.4) and (3.5) of these functions.

LEMMA 3.2. *For $\varrho\geqslant 0$,*

$$S_\varrho(x)=\frac{d}{dx}\,S_{\varrho+1}(x).$$

For $\varrho>0$,

$$A_\lambda^\varrho(x)=\frac{d}{dx}\,A_\lambda^{\varrho+1}(x).$$

For $\varrho=0$ and $x\neq\lambda_n$,

$$A_\lambda^0(x)=\frac{d}{dx}\,A_\lambda^1(x),$$

whereas if $x=\lambda_n$,

$$A_\lambda^0(x)=\frac{1}{2}\left(\frac{d^+}{dx}+\frac{d^-}{dx}\right)A_\lambda^1(x),$$

since

$$\frac{d^+}{dx}\,A_\lambda^1(x)=\sum_{\lambda_n\leqslant x}a_n \quad \text{and} \quad \frac{d^-}{dx}\,A_\lambda^1(x)=\sum_{\lambda_n<x}a_n .$$

4. A Lower Bound for the Change of Sign Function

4.1. We are now in a position to prove the following result on the changes of sign of the real and imaginary parts of $A_\lambda^\varrho(x)-S_\varrho(x)$.

THEOREM 4.1. *Suppose that the functional equation*

$$\Delta(s)\,\varphi(s)=\Delta(\delta-s)\,\psi(\delta-s)$$

is satisfied as in Definition 3.1. Let ϱ be a non-negative integer. Let $W_\lambda^\varrho(t)$ denote the number of changes of sign of the function $\operatorname{Re}\{A_\lambda^\varrho(x)-S_\varrho(x)\}$ in the interval $0<x\leqslant t$. If $\operatorname{Re}(b_n)\neq 0$ for at least one value of n, then[7]

$$W_\lambda^\varrho(t)\geqslant\left[\frac{h(\mu_1 t)^{1/2A}}{\pi}\right]-C, \tag{4.1}$$

[7]) Here, $[\xi]$ denotes the largest integer $\leqslant\xi$.

where C is a number independent of t, and $h=2e^{-\theta/2A}$, with

$$\theta = 2\left\{\sum_{\nu=1}^{N} \alpha_\nu \log\alpha_\nu - A\log A\right\}.$$

Let $V_\lambda^\varrho(t)$ denote the number of changes of sign of $\operatorname{Im}\{A_\lambda^\varrho(x)-S_\varrho(x)\}$ *in the interval* $0<x\leqslant t$. *If* $\operatorname{Im}(b_n)\neq 0$ *for at least one value of n, then*

$$V_\lambda^\varrho(t) \geqslant \left[\frac{h(\mu_1 t)^{1/2A}}{\pi}\right] - C',$$

where C' is independent of t.

Proof. We shall assume that $\operatorname{Re}(b_1)\neq 0$, as we may. The idea of the proof is to obtain first a lower bound for the number of changes of sign of $\operatorname{Re}\{A_\lambda^{\varrho+m}(x)-S_{\varrho+m}(x)\}$ in $0<x\leqslant t$, where m is a sufficiently large non-negative integer. This is achieved by applying relation (3.9). Then, by differentiating m times, and applying Lemmas 3.1 and 3.2, we get (4.1).

Given $\varrho\geqslant 0$, we choose an integer $m\geqslant 0$ which is so large, that

$$\varrho + m \geqslant 2A\beta - A\delta - \tfrac{1}{2}, \tag{4.2}$$

and that also

$$\frac{|\operatorname{Re}(b_1)|}{\mu_1^\vartheta} > \sum_{n=2}^{\infty} \frac{|\operatorname{Re}(b_n)|}{\mu_n^\vartheta}, \tag{4.3}$$

where

$$\vartheta = (A\delta + \varrho + m + \tfrac{1}{2})/2A. \tag{4.4}$$

Then, because of (4.2), (3.9) holds, with $\varrho+m$ in place of ϱ, and we have the relation

$$c'\cdot x^{\vartheta-\delta-\varrho-m}\operatorname{Re}\{A_\lambda^{\varrho+m}(x) - S_{\varrho+m}(x)\} = \sum_{n=1}^{\infty} \frac{\operatorname{Re}(b_n)}{\mu_n^\vartheta}\cos(h(\mu_n x)^{1/2A} + D) + g(x), \tag{4.5}$$

where c' is a real constant, ϑ is defined by (4.4), and

$$g(x) = o(1), \tag{4.6}$$

as $x\to\infty$.

Because of (4.3) and (4.6), we can find an X such that

$$|g(x)| < \frac{|\operatorname{Re}(b_1)|}{\mu_1^\vartheta} - \sum_{n=2}^{\infty}\frac{|\operatorname{Re}(b_n)|}{\mu_n^\vartheta}, \quad \text{for} \quad x\geqslant X. \tag{4.7}$$

Now let the sequence $x_0<x_1<x_2<\cdots$ be such that $x_0\geqslant X$, and

$$\cos(h(\mu_1 x_\nu)^{1/2A} + D) = (-1)^\nu, \quad \nu = 0, 1, 2, \ldots. \tag{4.8}$$

Then, we have

$$\operatorname{sgn} \operatorname{Re}\{A_\lambda^{\varrho+m}(x_\nu) - S_{\varrho+m}(x_\nu)\} = (-1)^\nu \operatorname{sgn} \operatorname{Re}(b_1) \tag{4.9}$$

for $\nu=0, 1, 2,\ldots$.

Indeed, on setting $x=x_\nu$, the right-hand side of (4.5) becomes

$$(-1)^\nu \frac{\operatorname{Re}(b_1)}{\mu_1^\vartheta} + \sum_{n=2}^{\infty} \frac{\operatorname{Re}(b_n)}{\mu_n^\vartheta} c_n(x_\nu) + g(x_\nu) = G(x_\nu),$$

say, where $c_n(x)=\cos(h(\mu_n x)^{1/2A}+D)$. We have

$$-|\operatorname{Re}(b_n)| \leqslant \operatorname{Re}(b_n)\cdot c_n(x_\nu) \leqslant |\operatorname{Re}(b_n)|, \tag{4.10}$$

and it is easily seen that inequalities (4.7) and (4.10) imply that

$$(-1)^\nu \frac{\operatorname{Re}(b_1)}{\mu_1^\vartheta} - \frac{|\operatorname{Re}(b_1)|}{\mu_1^\vartheta} < G(x_\nu) < (-1)^\nu \frac{\operatorname{Re}(b_1)}{\mu_1^\vartheta} + \frac{|\operatorname{Re}(b_1)|}{\mu_1^\vartheta}, \tag{4.11}$$

whence (4.9) follows immediately.

Therefore, $\operatorname{Re}\{A_\lambda^{\varrho+m}(x)-S_{\varrho+m}(x)\}$ changes sign at least once in each of the open intervals $(x_\nu, x_{\nu+1})$, $\nu=0, 1, 2,\ldots$. Consequently, its number of changes of sign in the interval $0<x\leqslant t$ is not less than

$$\left[\frac{h(\mu_1 t)^{1/2A}}{\pi}\right] - k,$$

where k is a number independent of t. By applying Lemma 3.2 and Rolle's theorem (or Lemma 3.1, if $\varrho=0$), we conclude that $\operatorname{Re}\{A_\lambda^\varrho(x)-S_\varrho(x)\}$ has at least

$$\left[\frac{h(\mu_1 t)^{1/2A}}{\pi}\right] - k - m$$

changes of sign, as x varies from 0 to t; this proves our theorem.

4.2. We have actually proved slightly more in the case $\varrho=0$. Because of Lemma 3.1, it follows from our proof that besides $\operatorname{Re}\{A_\lambda^0(x)-S_0(x)\}$, the functions

$$\operatorname{Re}\left\{\sum_{\lambda_n<x} a_n - S_0(x)\right\} \quad \text{and} \quad \operatorname{Re}\left\{\sum_{\lambda_n\leqslant x} a_n - S_0(x)\right\}$$

also change sign at least $[h(\mu_1 t)^{1/2A}/\pi]-C$ times in $(0, t]$.

4.3. It is clear that the method used to prove Theorem 4.1 can be applied to prove

THEOREM 4.1′. *Under the same assumptions as in Theorem 4.1, there exists a positive constant X such that if $a\geqslant X$, then* $\operatorname{Re}\{A_\lambda^\varrho(x)-S_\varrho(x)\}$ *changes sign at least*

$[h\mu_1^{1/2A}(b^{1/2A}-a^{1/2A})/\pi]-C''$ *times in the interval* $a\leq x\leq b$, *where* C'' *depends neither on* a *nor on* b.

[In fact, X is the real number which appears in inequality (4.7), and $C''=m$, the integer which satisfies inequalities (4.2) and (4.3).]

It may be of interest to remark that a result of this sort, for an arbitrary interval $[a, b]$, does not follow from PÓLYA's theorem.

4.4. In the same manner as above, results analogous to Theorems 4.1 and 4.1′ can be proved for the changes of sign of the real and imaginary parts of $B_\mu^\varrho(x)-S_\varrho(x)$.

4.5. We shall now apply Theorem 4.1 to the error-terms connected with the arithmetical functions $r_k(n)$ and $d(n)$.

The lattice-point function $r_k(n)$. As we have already seen in § 2, the generating function of $r_k(n)$ is the Epstein zeta-function $\zeta_k(s)$. It is regular in the finite part of the plane, except for a simple pole with residue $\pi^{k/2}/\Gamma(k/2)$ at $s=k/2$. It vanishes at $s=-1, -2, \ldots$, and has the value -1 at $s=0$. Functional equation (3.2) is satisfied by $\varphi(s)=\psi(s)=\pi^{-s}\zeta_k(s)$, with $a_n=b_n=r_k(n)$, $\lambda_n=\mu_n=\pi n$ and $\delta=k/2$. We have $A=1$ and $h=1$. It follows that

$$A_\lambda^\varrho(x)-S_\varrho(x)=\sum_{\pi n\leq x}{}' r_k(n)\,(x-\pi n)^\varrho-\frac{x^{k/2+\varrho}}{\Gamma(\varrho+k/2+1)}+\frac{x^\varrho}{\Gamma(\varrho+1)}.$$

In the case $\varrho=0$, if we make the substitution $x\to\pi x$, and set $r_k(0)=1$, Theorem 4.1 implies that as x varies from 0 to t, the error-term

$$P_k^0(x)=\sum_{0\leq n\leq x}{}' r_k(n)-\frac{(\pi x)^{k/2}}{\Gamma(k/2+1)}$$

changes sign at least $2\sqrt{t}-A_1$ times, where A_1 is independent of t. This result is obviously stronger than inequality (2.21), which we deduced from PÓLYA's theorem, and which holds only for *odd* k.

The divisor function $d(n)$. Let $d(n)$ denote the number of positive divisors of n. Its generating function is $\zeta^2(s)$, the square of RIEMANN's zeta-function, and functional equation (3.2) is satisfied by $\varphi(s)=\psi(s)=\pi^{-s}\zeta^2(s)$, $a_n=b_n=d(n)$, $\lambda_n=\mu_n=\pi n$, and $\delta=1$. We have $N=2$, $\alpha_1=\alpha_2=\frac{1}{2}$, whence $A=1$ and $h=2$. Further,

$$S_\varrho(x)=\frac{\pi^\varrho\zeta^2(0)}{\Gamma(\varrho+1)}\left(\frac{x}{\pi}\right)^\varrho-\pi^\varrho\sum_{1\leq 2\nu+1\leq\varrho}\left(\frac{x}{\pi}\right)^{\varrho-2\nu-1}\frac{\zeta^2(-2\nu-1)}{\Gamma(2\nu-1)\,\Gamma(2\nu+2)}$$
$$+\frac{\pi^\varrho}{\Gamma(\varrho+2)}\left(\frac{x}{\pi}\right)^{\varrho+1}\left(\gamma-\frac{\Gamma'}{\Gamma}(2+\varrho)+\log\frac{x}{\pi}\right),$$

where γ is Euler's constant (see [3], p. 130). If we consider the case $\varrho = 0$, and make the substitution $x \to \pi x$, it follows from Theorem 4.1 that the error-term

$$\sum_{n \leqslant x}' d(n) - \{x \log x + (2\gamma - 1)x + \tfrac{1}{4}\}$$

has at least $4\sqrt{t} - A_2$ changes of sign in the interval $0 < x \leqslant t$, where A_2 is independent of t.

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