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On Meromorphic Solutions of Algebraic Differential Equations[1])

by STEVEN BANK

1. Introduction

In this paper, we treat arbitrary first order algebraic differential equations (i.e. equations of the form $\Omega(z, y, dy/dz)=0$ where Ω is a polynomial in z, y and dy/dz). Such equations were treated by Valiron in [4], and it was shown that any entire function which satisfies a first order algebraic differential equation must be of finite order. (In fact, Valiron showed that for an entire transcendental solution with maximum modulus $M(r)$, there are positive constants k and λ, with λ rational, such that $\lim_{r\to+\infty} (\log M(r)/kr^{\lambda})=1$.) In this paper, we consider solutions of $\Omega(z, y, dy/dz)=0$, which are defined and meromorphic on the whole finite plane (i.e. meromorphic functions $h(z)$ such that $\Omega(z, h(z), h'(z))=0$ at each point z where h is analytic).

Our main result (§ 2 below) states that a meromorphic solution, which is not identically zero, cannot be written as the quotient of two entire functions where one is of finite order and the other is of infinite order. As a corollary (see §§ 4, 5), we show that if a meromorphic solution $h(z)$ has the property that for some rational function $R(z)$, the sequence of moduli of the zeros of $h(z)-R(z)$ has a finite exponent of convergence [1; p. 188], then h is a meromorphic function of finite order (where order is defined as in [2; p. 30] using the Nevanlinna characteristic). We also show that the same conclusion holds if the sequence of moduli of the poles of the solution $h(z)$ has a finite exponent of convergence. To the author's knowledge, no examples are known of meromorphic functions of infinite order which satisfy a first order algebraic differential equation, and the problem of proving or disproving the existence of such solutions for arbitrary first order algebraic differential equations is a very interesting one. Certain special cases have been treated as, for example, in the special case of the Riccati equation, it was shown by Wittich [6; p. 226] that all meromorphic solutions of the Riccati equation with polynomial coefficients are of finite order. However, the problem for arbitrary first order equations remains open, and we hope that our results here concerning this problem will aid in its eventual solution.

2.

We now state our main result:

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THEOREM: *A meromorphic function on the whole finite plane, which is not identically zero, and which is a solution of a first order algebraic differential equation, cannot be written as the quotient of two entire functions where one is of finite order and the other is of infinite order.*

3. Proof of the Theorem

Let $h(z)$ be meromorphic, and not identically zero on the plane, and satisfy the first order algebraic differential equation $\Omega(z, y, y')=0$ at each point of analyticity. We may write Ω in the form,

$$\Omega(z, y, y') = \sum Q_{kj}(z)\, y^k (y')^j \tag{1}$$

where the Q_{kj} are polynomials.

Let $h=f/g$ where f and g are entire functions. We will show that if either f or g is of finite order then both of them are of finite order. Thus suppose either f or g is of finite order. Without loss of generality, we may assume that

$$g \text{ is of finite order,} \tag{2}$$

since the function $1/h=g/f$ is a meromorphic function which satisfies the first order equation $\sum Q_{kj}(z)\,(-1)^j\, y^{N-(k+j)}(y')^j=0$ where $N=\max\{k+2j: Q_{kj}\neq 0\}$. Now if $f(z)$ is a polynomial, the result holds. Hence we may also assume that,

$$f \text{ is an entire transcendental function.} \tag{3}$$

Let $M(r)=\max_{|z|=r}|f(z)|$ and let $\sum_{n=0}^{\infty} c_n z^n$ be the power series expansion of $f(z)$. For each $r>0$, let $\nu(r)$ be the central index [1; p. 183] of f (i.e. $\nu(r)$ is the maximum j such that $|c_j|\, r^j=\max_{m\geq 0}|c_m|\, r^m$). Then in view of (3),

$$\nu(r) \text{ is an unbounded increasing function of } r, \tag{4}$$

and it is proved in [3; pp. 198, 210] (and also in [5; pp. 95, 103]) that there exists $\alpha\in(0, 1)$ such that if we exclude from the interval $(1, +\infty)$ an infinite sequence of exceptional finite open intervals (W_s, W_s') for which

$$\sum_{s=1}^{\infty} (\log W_s' - \log W_s) \quad \text{converges,} \tag{5}$$

and for which we may assume,

$$W_s' < W_{s+1} \quad \text{for all } s \quad \text{and} \quad \lim_{s\to\infty} W_s' = +\infty\,, \tag{6}$$

then in the remaining set $(1, +\infty)-E$, where $E=\bigcup_{s=1}^{\infty}(W_s, W_s')$, the following are true: There exists a number $R_1 \geqslant 1$ such that for $r > R_1$ and $r \notin E$, we have

$$\log M(r) > c(\nu(r))^{\alpha}, \quad \text{where} \quad c \text{ is a positive constant} \tag{7}$$

independent of r, and if z is any point on $|z| = r$ at which $|f(z)| = M(r)$, then

$$f'(z) = (\nu(r)/z)(1 + \varepsilon(z)) f(z), \quad \text{where} \quad |\varepsilon(z)| < (\nu(r))^{-\delta} \tag{8}$$

for some fixed $\delta > 0$. (The elements of $(1, +\infty) - E$ are called *ordinary* values of index α in [3, 4, 5].)

Since g is of finite order (by (2)), let λ be any number greater than the order of g. Let $a_1, a_2, \ldots$ be the zeros (if any) of g lying outside the closed unit disk and let D be the domain outside the closed unit disk exterior to all the disks $|\zeta - a_n| < |a_n|^{-\lambda}$. Then it is proved in [5; p. 74], by using the representation for g given by the Hadamard Factorization Theorem, that there exist $R_2 > 1$ and $\sigma > 0$ such that $|g'(z)/g(z)| < r^{\sigma}$ for $z \in D$ and $|z| = r > R_2$. Thus if we let F be the union of all the open intervals, $(|a_n| - |a_n|^{-\lambda}, |a_n| + |a_n|^{-\lambda})$ for $n = 1, 2, \ldots$, then clearly,

$$|g'(z)/g(z)| < r^{\sigma} \quad \text{on} \quad |z| = r \quad \text{if} \quad r > R_2 \quad \text{and} \quad r \notin F. \tag{9}$$

Now the series $\sum_{n=1}^{\infty} |a_n|^{-\lambda}$ converges since λ is greater than the order of g (see [5; pp. 51, 52]), and so it is clear that the set F can be written as the union of a sequence of finite open intervals (T_s, T_s') such that

$$T_s' < T_{s+1} \quad \text{for all } s, \quad \text{and} \quad \sum_{s \geqslant 1} (T_s' - T_s) \quad \text{converges.} \tag{10}$$

In view of (5), (6) and (10), clearly we may write $E \cup F$ as the union of a sequence of finite open intervals,

$$E \cup F = \bigcup_{s=1}^{\infty} (V_s, V_s'), \quad \text{where} \tag{11}$$

$$V_s' < V_{s+1} \quad \text{for all } s \quad \text{and} \quad \lim_{s \to \infty} V_s' = +\infty \tag{12}$$

and

$$\sum_{s=1}^{\infty} (\log V_s' - \log V_s) \text{ converges.} \tag{13}$$

Now since g is of finite order, there exist $d > 0$ and $R_3 > 1$ such that when $r > R_3$,

$$|g(z)| \leqslant \exp(r^d) \quad \text{on} \quad |z| = r. \tag{14}$$

Also, since the coefficients $Q_{kj}(z)$ are polynomials, there exist $b > 0$ and $R_4 > 1$ such that when $r > R_4$,

$$|Q_{kj}(z)| \leqslant r^b \quad \text{on} \quad |z| = r \quad \text{for all} \quad k, j. \tag{15}$$

Now define,

$$a = 1 + \max\{\sigma + 1, b + 1, d/\alpha\} \quad (\text{where } \alpha \text{ is as in (7)}), \tag{16}$$

and define

$$A = \{r \mid r > 1 \quad \text{and} \quad \nu(r) > r^a\}. \tag{17}$$

We now prove,

LEMMA A: *There exists a number* $r^* > 1$ *such that* $A \cap (r^*, +\infty) \subset E \cup F$.

Proof: Assume the contrary. Then there exists a sequence of distinct values of r in $(1, +\infty)$ tending to $+\infty$ such that

$$r \in A \quad \text{but} \quad r \notin E \cup F. \tag{18}$$

Let B be the set of values of r comprising this sequence. Now $h = f/g$ satisfies the relation,

$$\sum Q_{kj}(z)\,(h(z))^k\,(h'(z))^j = 0, \tag{19}$$

at all points z where h is analytic. Let

$$p = \max\{k + j : Q_{kj} \not\equiv 0\} \quad \text{and} \quad m = \max\{j : Q_{p-j,j} \not\equiv 0\}. \tag{20}$$

Let $r \in B$ and let z be a point on $|z| = r$ at which $|f(z)| = M(r)$. Clearly $f(z) \neq 0$ and since $r \notin F$, $g(z) \neq 0$. Thus $h(z) \neq 0$ and $h(z) \neq \infty$ and so by dividing equation (19) by $(h(z))^p$ (where p is as in (20)), and noting that $h'/h = (f'/f) - (g'/g)$, we can write equation (20) in the form,

$$\Lambda(z) = \Phi(z), \quad \text{where} \tag{21}$$

$$\Lambda(z) = \sum_{j=0}^{m} Q_{p-j,j}(z)\left(\frac{f'(z)}{f(z)} - \frac{g'(z)}{g(z)}\right)^j, \tag{22}$$

and

$$\Phi(z) = -\sum_{k+j<p} Q_{kj}(z)\left(\frac{f'(z)}{f(z)} - \frac{g'(z)}{g(z)}\right)^j \left(\frac{f(z)}{g(z)}\right)^{k+j-p}. \tag{23}$$

We now assert that there exist real numbers $r' > 1$ and $K > 0$ such that if $r \in B$ and $r > r'$, then,

$$|\Phi(z)| \leq K(M(r))^{-1/2} \exp(\mathrm{p}r^d), \tag{24}$$

at each point of $|z| = r$ at which $|f(z)| = M(r)$.

To prove (24), we recall first from (18) that if $r \in B$ then $r \notin E$ and $r \notin F$. Since $\nu(r) \to +\infty$ as $r \to +\infty$ (by (4)), we see that $\epsilon(z)$ (in (8)) tends to zero as $r \to +\infty$ in B.

Since $M(r)$ also tends to $+\infty$, there exists $R_5>1$ such that for $r\in B$ and $r>R_5$, we have

$$M(r)>1, \quad \nu(r)>1 \quad \text{and} \quad |\varepsilon(z)|<\tfrac{1}{2}, \tag{25}$$

at every point of $|z|=r$ at which $|f(z)|=M(r)$. Let $R_6=\max_{1\leqslant j\leqslant 5}R_j$ and let r be any element of B such that $r>R_6$. Let z be any point on $|z|=r$ at which $|f(z)|=M(r)$. We refer to the right side of (23). If $k+j<p$ then $p-(k+j)\geqslant 1$, so $|f(z)|^{k+j-p}=$ $=(M(r))^{k+j-p}\leqslant(M(r))^{-1}$ since $M(r)>1$ by (25). Also $|g(z)|\leqslant\exp(r^d)$ by (14) so clearly $|g(z)|^{p-(k+j)}\leqslant\exp(\mathrm{pr}^d)$. Thus $|f(z)/g(z)|^{k+j-p}\leqslant(M(r))^{-1}\exp(\mathrm{pr}^d)$. Also, by (9) and (15), we have $|g'(z)/g(z)|\leqslant r^\sigma$ and $|Q_{kj}(z)|\leqslant r^b$. Since $|\varepsilon(z)|<\frac{1}{2}$ by (25), we have by (8) that $|f'(z)/f(z)|<2(\nu(r)/r)<2\nu(r)$ since $r>1$. Thus noting that $j<p<p+1$ if $k+j<p$, we have from (23) and the above estimates that

$$|\Phi(z)|\leqslant Kr^b(2\nu(r)+r^\sigma)^{p+1}(M(r))^{-1}\exp(\mathrm{pr}^d), \tag{26}$$

for some constant $K>0$. Now let $u(r)=r^b(2\nu(r)+r^\sigma)^{p+1}(M(r))^{-1/2}$. We show that $u(r)\to 0$ as $r\to+\infty$ through elements of B. Now we may write,

$$u(r)=\left[\frac{r^{4b}}{M(r)}\right]^{1/4}\left\{\frac{2\nu(r)}{(M(r))^{1/(4p+4)}}+\left[\frac{r^{4(p+1)\sigma}}{M(r)}\right]^{1/(4p+4)}\right\}^{p+1}. \tag{27}$$

Since f is an entire transcendental function (by (3)), we have by Cauchy's estimate ([5; p. 6]) that as $r\to+\infty$,

$$r^{4b}/M(r) \quad \text{and} \quad r^{4(p+1)\sigma}/M(r) \quad \text{both tend to zero.} \tag{28}$$

Now if $r\in B$ and $r>R_6$, then by (7), $\nu(r)<(c^{-1}\log M(r))^{1/\alpha}$. Hence,

$$0<\frac{2\nu(r)}{(M(r))^{1/(4p+4)}}<\frac{2(c^{-1}\log M(r))^{1/\alpha}}{(M(r))^{1/(4p+4)}}. \tag{29}$$

Since $M(r)\to+\infty$ as $r\to+\infty$, clearly the term on the right in (29) tends to zero as $r\to+\infty$. Thus as $r\to+\infty$ through elements of B, we have by (29) that $2\nu(r)/(M(r))^{1/(4p+4)}\to 0$. Hence in view of (27) and (28), we have $u(r)\to 0$ as $r\to+\infty$ through elements of B. Thus there exists $r'>R_6$ such that for $r\in B$ and $r>r'$, we have $u(r)<1$. In view of (26) and the definition of $u(r)$, we obtain (24).

We now consider $\Lambda(z)$ given by (22). Since $Q_{p-m,m}$ is a polynomial which is not identically zero (by (20)), clearly there exist constants $R_7>r'$ and $L>0$ such that for $r>R_7$,

$$|Q_{p-m,m}(z)|\geqslant L \quad \text{on} \quad |z|=r. \tag{30}$$

Case I: $m=0$. Then $\Lambda(z)=Q_{p-m,m}(z)$. Since $\Lambda(z)=\Phi(z)$ by (21), we have by (24) and (30) that $L \leqslant K(M(r))^{-1/2} \exp(\mathrm{pr}^d)$ if $r \in B$ and $r > R_7$. Thus

$$M(r)\exp(-2\mathrm{pr}^d) \leqslant (K/L)^2 \quad \text{if} \quad r \in B \quad \text{and} \quad r > R_7. \tag{31}$$

But if $r \in B$ then $r \in A$ and so $\nu(r) > r^a$ by (17). Hence by (7), $\log M(r) > \mathrm{cr}^{a\alpha}$, and so by (31), we have

$$\exp(\mathrm{cr}^{a\alpha} - 2\mathrm{pr}^d) \leqslant (K/L)^2 \quad \text{if} \quad r \in B \quad \text{and} \quad r > R_7. \tag{32}$$

But by (16), $a > d/\alpha$, and since $d > 0$, $\alpha > 0$ and $c > 0$, clearly the left side of (32) tends to $+\infty$ as $r \to +\infty$. Thus (32) is impossible (since by our assumption (18), there exist r-values in B tending to $+\infty$). This contradiction proves Lemma A in the case $m = 0$.

Case II: $m > 0$. By (25), if $r \in B$ and $r > R_7$ then $|\varepsilon(z)| < \frac{1}{2}$ at each point of $|z| = r$ at which $|f(z)| = M(r)$, and so by (8), $|zf'(z)/(f(z)\nu(r))| \geqslant 1 - |\varepsilon(z)| > \frac{1}{2}$. Thus, $|f'(z)/f(z)| > \frac{1}{2}(\nu(r)/r)$. Now if $r \in B$ then $r \in A$ and so $\nu(r) > r^a$ by (17). Hence if $r \in B$ and $r > R_7$ then

$$|f'(z)/f(z)| > \tfrac{1}{2} r^{a-1} \tag{33}$$

at every point of $|z| = r$ at which $|f(z)| = M(r)$. Since $r \in B$ implies $r \notin F$ (by (18)), we also have by (9) that $|g'(z)/g(z)| \leqslant r^\sigma$. Thus with (33), we obtain

$$\left|\frac{f'(z)}{f(z)} - \frac{g'(z)}{g(z)}\right| \geqslant \tfrac{1}{2} r^{a-1} - r^\sigma = r^{a-1}\left(\tfrac{1}{2} - r^{\sigma-(a-1)}\right). \tag{34}$$

By definition of a, $\sigma - (a-1) < 0$, and so by (34), there exists $R_8 > R_7$ such that if $r \in B$ and $r > R_8$ then

$$\left|\frac{f'(z)}{f(z)} - \frac{g'(z)}{g(z)}\right| > \tfrac{1}{4} r^{a-1}, \tag{35}$$

at every point of $|z| = r$ at which $|f(z)| = M(r)$.

We now assert that there exist constants $r^{\#} > R_8$ and $K^{\#} > 0$ such that if $r \in B$ and $r > r^{\#}$, then

$$|A(z)| \geqslant K^{\#} \quad \text{at all points of} \quad |z| = r \quad \text{at which} \quad |f(z)| = M(r). \tag{36}$$

To prove (36), we consider the quotients,

$$\Psi_j(z) = (Q_{p-j,j}(z)/Q_{p-m,m}(z))\left(\frac{f'(z)}{f(z)} - \frac{g'(z)}{g(z)}\right)^{j-m} \tag{37}$$

for $j = 0, 1, \ldots, m-1$, at points on $|z| = r$ at which $|f(z)| = M(r)$, where $r \in B$ and $r > R_8$. By (15), (30) and (35), we have

$$|\Psi_j(z)| \leqslant (4^{m-j}/L)\, r^{b+(j-m)(a-1)}. \tag{38}$$

Now $b>0$, and by (16), $a-1>b$ and so if $j<m$, we have $(m-j)(a-1)\geqslant(a-1)>b$. Thus $b+(j-m)(a-1)<0$ and so the right side of (38) tends to zero as $r\to+\infty$. Hence there exists a real number $r^{\#}>R_8$ such that for $r>r^{\#}$ and $r\in B$ we have

$$|\Psi_j(z)| \leqslant (1/(m+1)) \quad \text{for} \quad j=0,1,\ldots,m-1. \tag{39}$$

Now in view of (22) and (37) we have,

$$A(z) = Q_{p-m,m}(z)\left(\frac{f'(z)}{f(z)} - \frac{g'(z)}{g(z)}\right)^m \left(1 + \sum_{j=0}^{m-1} \Psi_j(z)\right)$$

and so

$$|A(z)| \geqslant |Q_{p-m,m}(z)| \left|\frac{f'(z)}{f(z)} - \frac{g'(z)}{g(z)}\right|^m \left(1 - \sum_{j=0}^{m-1} |\Psi_j(z)|\right).$$

Thus in view of (30), (35) and (39), we have that if $r>r^{\#}$ and $r\in B$ then

$$|A(z)| \geqslant L4^{-m}(1/(m+1))\, r^{m(a-1)} \tag{40}$$

at every point of $|z|=r$ at which $|f(z)|=M(r)$. Since $a-1>b>0$ and $m>0$ we have that $m(a-1)>0$. Thus since $r^{\#}>1$ we see that if $r>r^{\#}$ then $r^{m(a-1)}>1$. Hence (36) follows from (40) by setting $K^{\#}=L4^{-m}(1/(m+1))$.

Since $A(z)=\Phi(z)$ by (21), we have by (24) and (36) that $K^{\#}\leqslant K(M(r))^{-1/2}\exp(\mathrm{pr}^d)$ if $r>r^{\#}$ and $r\in B$. Thus

$$M(r)\exp(-2\mathrm{pr}^d) \leqslant (K/K^{\#})^2 \quad \text{if} \quad r>r^{\#} \quad \text{and} \quad r\in B. \tag{41}$$

But if $r\in B$ then $r\in A$ and so $\nu(r)>r^a$ by (17). Hence by (7), $\log M(r)>\mathrm{cr}^{a\alpha}$ and so by (41),

$$\exp(\mathrm{cr}^{a\alpha} - 2\mathrm{pr}^d) \leqslant (K/K^{\#})^2 \quad \text{if} \quad r>r^{\#} \quad \text{and} \quad r\in B. \tag{42}$$

But by (16), $a>d/\alpha$ and since $d>0$, $\alpha>0$ and $c>0$, clearly the left side of (42) tends to $+\infty$ as $r\to+\infty$. Thus (42) is impossible (since by our assumption (18), there exist r-values in B tending to $+\infty$). This contradiction proves the lemma in Case II, and thus the proof of Lemma A is complete.

We now prove,

LEMMA B: *There exists a real number $r_0>1$ such that $\nu(r)<2r^a$ for all $r>r_0$.*

Proof: By Lemma A, there exists $r^*>1$ such that

$$A \cap (r^*, +\infty) \subset E \cup F. \tag{43}$$

Now by (11), (12) and (13), $E\cup F$ is the union of a sequence of open intervals (V_s, V_s') where $V_s'<V_{s+1}$, $\lim_{s\to\infty} V_s'=+\infty$, and $\sum_{s=1}^{\infty}\log(V_s'/V_s)$ converges. Since the series

converges, $\lim_{s\to\infty}(V_s'/V_s)=1$ and hence $\lim_{s\to\infty}(V_s'/V_s)^a=1$. Since also $\lim_{s\to\infty}V_s'=+\infty$, clearly there exists s_0 such that

$$V_s'>r^* \quad \text{and} \quad (V_s'/V_s)^a<2 \quad \text{for} \quad s\geqslant s_0. \tag{44}$$

We now show that if r_0 is taken to be V_{s_0}', then the conclusion of Lemma B holds. Let $r>r_0$.

If $r\notin A$, then by (17), $v(r)\leqslant r^a$ and so $v(r)<2r^a$.

Suppose now $r\in A$. Then by (43), $r\in E\cup F$. Hence for some s_1, $r\in(V_{s_1}, V_{s_1}')$. Since $r<V_{s_1}'$ while $r>V_{s_0}'$ clearly (by (12)), $s_1>s_0$ so (44) holds for s_1. Thus

$$V_{s_1}'>r^* \quad \text{and} \quad (V_{s_1}'/V_{s_1})^a<2. \tag{45}$$

Now the endpoint V_{s_1}' is clearly not in $E\cup F$ and so by (43), $V_{s_1}'\notin A$. Thus $v(V_{s_1}')\leqslant (V_{s_1}')^a$, and so by (45), $v(V_{s_1}')<2(V_{s_1})^a$. Since v is increasing, we thus have $v(r)\leqslant v(V_{s_1}')<2(V_{s_1})^a$. But $V_{s_1}<r$ and so we obtain $v(r)<2r^a$ which proves Lemma B completely.

In view of Lemma B, we have $\limsup_{r\to+\infty}(\log v(r)/\log r)\leqslant a$. But in [5; pp. 33–34], it is shown that $\log v(r)/\log r$ and $\log\log M(r)/\log r$ have the same limit superior as $r\to+\infty$. Hence $\limsup_{r\to+\infty}(\log\log M(r)/\log r)\leqslant a$, thus proving that the entire function f is of finite order at most a. This concludes the proof of the theorem stated in § 2.

4. Corollary

Let $h(z)$ be a meromorphic function on the whole finite plane which is a solution of a first order algebraic differential equation. Then $h(z)$ can be written as the quotient of two entire functions where both are of finite order, if either of the following two conditions is satisfied:

(A) *For some rational function $R(z)$, the sequence of moduli of the zeros of the function $h(z)-R(z)$ has a finite exponent of convergence* [1; p. 188].

(B) *The sequence of moduli of the poles of $h(z)$ has a finite exponent of convergence.*

Proof: Suppose (A) holds for a rational function $R(z)$. Letting $\varphi(z)$ be the canonical product [1; p. 195] corresponding to the sequence of zeros of $h(z)-R(z)$, we have that $\varphi(z)$ is an entire function of finite order by [1; p. 195], and that the function $\psi(z)=\varphi(z)/(h(z)-R(z))$ is entire. Now if $h(z)$ satisfies the equation $\sum Q_{kj}(z)\,y^k(y')^j=0$ (where the Q_{kj} are polynomials) then clearly $h(z)-R(z)$ satisfies the equation

$$\sum Q_{kj}(z)\,(y+R(z))^k\,(y'+R'(z))^j=0. \tag{46}$$

Since $R(z)$ is a rational function and the Q_{kj} are polynomials, it is clear that by multiplying equation (46) through by a suitable polynomial, this equation becomes a first order algebraic differential equation. Since $h(z)-R(z)$ is a solution, and since $h(z)-$

$-R(z)=\varphi(z)/\psi(z)$ where $\varphi(z)$ is of finite order, we have by the theorem in § 2 that $\psi(z)$ is also of finite order. Writing $R(z)=T_1(z)/T_2(z)$ where T_1 and T_2 are polynomials, we have $h(z)=(T_1\psi+T_2\varphi)/(T_2\psi)$ which is a representation of h as the quotient of two entire functions of finite order.

If (B) holds, we form the canonical product $\varphi(z)$ corresponding to the poles of $h(z)$. Then $\varphi(z)$ is of finite order and $\psi(z)=\varphi(z)h(z)$ is entire. By the theorem in § 2, $\psi(z)$ is of finite order also and so $h(z)=\psi(z)/\varphi(z)$ is the desired representation.

5. Remark

By the corollary, if either (A) or (B) is satisfied, then h is the quotient of two entire functions of finite order. It easily follows (using [2; p. 15]) that h is a meromorphic function of finite order (where order is defined as in [2; p. 30] using the Nevanlinna characteristic). In this case, (B) must hold by [2; p. 31], and since $h(z)-R(z)$ will also be a meromorphic function of finite order (for any rational function $R(z)$ by [2; pp. 15–16]), we see that (A) must hold for *all* rational functions $R(z)$ by [2; p. 31].

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