# On Parametric Representation for Quasisymmetric Frunctions.

Autor(en): **Agard, Stephen / Kelingos, John**

Objekttyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **44 (1969)**

PDF erstellt am: **23.07.2024**

Persistenter Link: https://doi.org/10.5169/seals-33787

# On Parametric Representation for Quasisymmetric Functions

by STEPHEN AGARD and JOHN KELINGOS[1])

In 1956, A. Beurling and L. Ahlfors [4] gave a necessary and sufficient condition for a homeomorphism of the real line onto itself to be the boundary correspondence of a quasiconformal mapping of the upper half plane. If the boundary function is denoted by $u=f(x)$, the condition is simply that for some $k \geqq 1$,

$$1/k \leqq \frac{f(x_3) - f(x_2)}{f(x_2) - f(x_1)} \leqq k, \tag{1}$$

for all $x_1 < x_2 < x_3$, with $(x_3 - x_2)/(x_2 - x_1) = 1$. A function satisfying property (1) is called $k$-quasisymmetric. The infimum of the numbers $k$ for which (1) is satisfied is denoted by $k[f]$. The importance of quasisymmetric functions to the theory of quasiconformal mappings cannot be overstated.

Qualitatively, the properties of quasisymmetric functions parallel rather closely those of quasiconformal mappings [8]. Quantitatively, however, they have some shortcomings not shared by quasiconformal mappings. For example, the inverse of a $k$-quasisymmetric function, although quasisymmetric, need not be $k$-quasisymmetric; the composition of a $k_1$-quasisymmetric function with a $k_2$-quasisymmetric function need not be $k_1 k_2$-quasisymmetric.

It is our intention to introduce a new dilatation $K[f]$ for quasisymmetric functions. Superficial advantages will include the relations $K[f] = K[f^{-1}]$ and $K[f_1 \circ f_2] \leqq \leqq K[f_1]\, K[f_2]$. Further, if $f$ is the boundary correspondence of a $K$-quasiconformal mapping, then $K[f] \leqq K$. However, our primary purpose is to build sufficient quantitative precision into the definition to enable us to characterize the infinitesimal generators for the class of quasisymmetric functions.

## 1. A Sufficient Condition for the Generation of Quasisymmetric Functions

It is instructive at this time to state and prove a simple theorem on parametric representation for quasisymmetric functions. The theorem will simultaneously serve to illustrate the type of problem we are interested in, to expose the difficulties connected with the definition of $k[f]$, and to indicate the direction one might take in redefining the maximal dilatation.

Suppose $\zeta(u, t)$ is continuously differentiable for all real $u$ and $0 \leqq t \leqq T$. For each

[1]) Work done with partial support of National Science Foundation Grant GP 7041 X.

fixed $x$, $-\infty < x < +\infty$, let $u = f_t(x)$ denote the solution to the initial value problem

$$du/dt = \zeta(u, t); u(0) = x. \tag{2}$$

THEOREM. *Let* $\zeta(u, t)$ *satisfy the global condition*

$$-M \leqq \frac{\zeta(u_3, t) - \zeta(u_2, t)}{u_3 - u_2} - \frac{\zeta(u_2, t) - \zeta(u_1, t)}{u_2 - u_1} \leqq M \tag{3}$$

*for some* $M > 0$, *all* $u_1 < u_2 < u_3$, *and all* $t$. *Then for each* $t$, $u = f_t(x)$ *is* $e^{Mt}$-*quasisymmetric.*

*Proof.* Fix $x_1 < x_2 < x_3$, and let $u_i(t) = f_t(x_i)$, $i = 1, 2, 3$, be the solution to (2) with $x = x_i$. The existence of the solutions for all $t$ is proved in the more general setting of Theorem 1 in Section 2. From (3),

$$-M \leqq \frac{u_3'(t) - u_2'(t)}{u_3(t) - u_2(t)} - \frac{u_2'(t) - u_1'(t)}{u_2(t) - u_1(t)} \leqq M.$$

Integrating from 0 to $t$ and then exponentiating, we find

$$e^{-Mt}\left(\frac{x_3 - x_2}{x_2 - x_1}\right) \leqq \frac{u_3(t) - u_2(t)}{u_2(t) - u_1(t)} \leqq e^{Mt}\left(\frac{x_3 - x_2}{x_2 - x_1}\right). \tag{4}$$

The result follows if we take $x_3 - x_2 = x_2 - x_1$.

From the symmetry of (4) we observe that for fixed $t$, the inverse of the function $f_t(x)$ is also $e^{Mt}$-quasisymmetric. Hence it is impossible to prove a converse to the above theorem. Specifically, there exists an $e^{Mt}$-quasisymmetric function $\phi(x)$ for which it is impossible to find a family $f_t(x)$ as above, such that $\phi(x) = f_T(x)$.

Furthermore our hypothesis that (3) hold for all real $u$ is too liberal in that we only use those values of $u$ along the solution curves $u_i(t)$. Nevertheless it seems very difficult to prove the theorem under weaker conditions on $\zeta$.

Finally, the form of (4) indicates that the difficulty with definition (1) is the requirement that $(x_3 - x_2)/(x_2 - x_1) = 1$. The observation that ratios of this type are certain cross ratios leads us to a new definition of quasisymmetry, motivated by results of Teichmüller [11].

## 2. A New Definition of Quasisymmetry

Denote by $C(0, 1, \infty)$ the extended $z$-plane minus the three points $0, 1, \infty$. Let $z = z(w)$ denote the familiar elliptic modular function [9], which maps the upper half plane $\mathrm{Im}(w) > 0$ onto $C(0, 1, \infty)$. The hyperbolic density $\varrho(z)$ in $C(0, 1, \infty)$ is invariantly defined by the relation

$$\varrho(z)\,|dz| = |dw|/\mathrm{Im}(w). \tag{5}$$

The hyperbolic distance in $C(0, 1, \infty)$ is then defined as

$$\sigma(z_1, z_2) = \inf_{\gamma} \int_{\gamma} \varrho(z)\,|dz| \tag{6}$$

where the infimum is taken over all arcs $\gamma$ joining $z_1$ and $z_2$ in $C(0, 1, \infty)$ for which the integral has meaning.

Suppose now that $u=f(x)$ is a sense preserving self homeomorphism of the real line. Let $x_1<x_2<x_3$ and let $u_i=f(x_i)$, $i=1, 2, 3$. Set

$$\left.\begin{aligned} a &= (\infty, x_1, x_2, x_3) = -(x_3 - x_2)/(x_2 - x_1) \\ a' &= (\infty, u_1, u_2, u_3) = -(u_3 - u_2)/(u_2 - u_1). \end{aligned}\right\} \tag{8}$$

Given $K\geqq 1$, we say that $K$ is admissible for $f$ if

$$-\log K \leqq \int_a^{a'} \varrho(x)\,dx \leqq \log K, \tag{9}$$

for all $x_1<x_2<x_3$. The maximal dilatation of $f(x)$, denoted now by $K[f]$, is the infimum of all $K$ which are admissible for $f$.

Since $a$ and $a'$ are negative, and since the negative real axis is a geodesic for the hyperbolic density in $C(0, 1, \infty)$, condition (9) is equivalent to

$$\sigma(a, a') \leqq \log K. \tag{10}$$

It is easy to show [2] that

$$\sigma(a, a') = \left| \log \frac{\mu\left(\dfrac{1}{\sqrt{1-a'}}\right)}{\mu\left(\dfrac{1}{\sqrt{1-a}}\right)} \right|, \tag{11}$$

where $\mu(r)$, $0<r<1$, denotes as usual the conformal modulus of the unit disk slit along the real axis from 0 to $r$. $\mu$ is a continuous, strictly decreasing function with limits $\infty$ and 0 at 0 and 1 respectively. Consequently, condition (9) is simply the familiar quadrilateral condition $1/K\leqq(\operatorname{mod} Q')/(\operatorname{mod} Q)\leqq K$, where $Q$ is the quadrilateral whose interior is the upper half plane, and whose vertices are $x_1$, $x_2$, $x_3$, and $\infty$. An immediate result of this form of the definition is that for the boundary correspondence of a $K$-quasiconformal self mapping of the upper half plane, $K$ is admissible. Hence $K[f]\leqq K$, and the bound is best possible in the class of $K$-quasiconformal mappings.

It is a trivial consequence of the definition that $K[f]=K[f^{-1}]$ and that $K[f_1 \circ f_2] \leqq K[f_1]\, K[f_2]$. Furthermore, the crucial fact that the total collection of quasisymmetric functions remains the same under either definition (1) or (9) follows from the observation that $K[f]<\infty$ if and only if $k[f]<\infty$.

Indeed, suppose first that $K[f]=K<\infty$. Let $x_1<x_2<x_3$, and using (10) and (11) to solve for $-a'$, we get

$$\frac{1}{\left(\mu^{-1}\left(\frac{1}{K}\mu\left(\frac{1}{\sqrt{1-a}}\right)\right)\right)^2}-1 \leqq \frac{u_3-u_2}{u_2-u_1} \leqq \frac{1}{\left(\mu^{-1}\left(K\mu\left(\frac{1}{\sqrt{1-a}}\right)\right)\right)^2}-1.$$

From the functional relation [7], $\mu(r)\,\mu(\sqrt{1-r^2})=\pi^2/4$, it follows that the upper and lower bounds are reciprocals when $a=-1$, and consequently,

$$k[f] \leqq (\mu^{-1}(K\mu(1/\sqrt{2})))^{-2}-1=\lambda(K). \tag{12}$$

Using a well known result of Teichmüller [11], it can be shown that this bound is best possible.

Conversely, suppose $k[f]=k<\infty$. Then the Beurling-Ahlfors extension [4] of $f$ is a $k^2$-quasiconformal mapping, and it follows that $K[f] \leqq k^2$.

We make two additional observations concerning this new definition. The first concerns the role played by $\infty$ in (8) and therefore (9). If instead of using only cross ratios of the form $a=(\infty, x_1, x_2, x_3)$, we use cross ratios $a=(x_1, x_2, x_3, x_4)$ for all quadruples $x_1<x_2<x_3<x_4$, then $a$ and the corresponding image cross ratios are negative and hence condition (9) still makes sense. For a given $K \geqq 1$, there are of course fewer functions satisfying the more stringent requirement, although as above the total collection of such functions as $K$ ranges over all values is the same as the collection of all functions which satisfy (1) for some $k \geqq 1$. The advantage of generalizing in this way is that now condition (9) remains invariant with respect to linear fractional transformations of the extended real line. Thus, for example, the function $f(x)=1/x$ would have $K[f]=1$. But because of the simpler form our main result takes in section 3 when the point at $\infty$ is singled out, we prefer not to generalize (9) to the fullest extent, although a corresponding theorem can be proved in this more general setting.

Finally, although the definition is admittedly an adaptation of properties inherited from quasiconformal mappings, it is reasonable to consider its formal application to self homeomorphisms of the finite plane. This leads to a coefficient of quasiconformality $\hat{K}$, the infimum of numbers $K \geqq 1$ for which (10) holds. Again, $\hat{K}[\phi]<\infty$ if and only if the mapping $\phi$ is quasiconformal. It is an open question how large the ordinary dilatation $D$ of a mapping $\phi$ subject to the inequality $\hat{K}[\phi] \leqq K$ can be. In [1] the bound $D \leqq \lambda(\sqrt{K})$ (cf. equation (12)) is established.

## 3. Parametric Representation

We consider families $f_t(x)$ with the property that for each $t$, $f_t$ is a sense-preserving homeomorphism of the real line onto itself. Denoting the inverse function by $x=f_t^{-1}(u)$, we require further that $f_t(x)$, $f_t^{-1}(u)$, and $\partial f_t(x)/\partial t$ be continuous in both variables. Finally, if $M(t)$ is a non-negative integrable function, we will say that $f_t$ is of *exponential type-$M(t)$* if for $t_0<t_1$,

$$K[f_{t_1}\circ f_{t_0}^{-1}] \leqq \exp\left\{\int_{t_0}^{t_1} M(\tau)\,d\tau\right\}. \tag{14}$$

Suppose next that $\zeta(u, t)$ is a continuous function defined for all real $u$ and $0\leqq t\leqq T$. Suppose in addition there exists a non-negative continuous function $M(t)$ such that for all $0\leqq t\leqq T$, and every triple $u_1<u_2<u_3$,

$$\frac{-M(t)}{|a|\,\varrho(a)} \leqq \frac{\zeta(u_3, t)-\zeta(u_2, t)}{u_3-u_2} - \frac{\zeta(u_2, t)-\zeta(u_1, t)}{u_2-u_1} \leqq \frac{M(t)}{|a|\,\varrho(a)}, \tag*{\{*\}}$$

where as usual $a=(\infty, u_1, u_2, u_3)$ and $\varrho(a)$ is the hyperbolic density in $C(0, 1, \infty)$ defined by (5).

THEOREM 1. *The solution $u=f_t(x)$ to the initial value problem*

$$du/dt=\zeta(u, t); \qquad u(0)=x \tag{2}$$

*is of exponential type-$M(t)$ if and only if $\zeta(u, t)$ satisfies $\{*\}$.*

*Proof.* We remark first that the continuity of $\zeta(u, t)$ together with condition $\{*\}$ imply that the solutions $f_t(x)$ are unique and exist for all $0\leqq t\leqq T$. To see this, we need the asymptotic behavior of $\varrho(a)$ [10, page 246].

$$\log\varrho(a)=\begin{cases}\log 1/|a| - \log\log 1/|a| + \psi_0(a)\\ -\log|a| - \log\log|a| + \psi_\infty(a)\end{cases} \tag{15}$$

where $\psi_0$ and $\psi_\infty$ are continuous at $a=0$ and $\infty$ respectively. Now condition $\{*\}$ implies

$$\frac{|\zeta(u_3, t)-\zeta(u_2, t)|}{u_3-u_2} \leqq \frac{M(t)}{|a|\,\varrho(a)} + \frac{|\zeta(u_2, t)-\zeta(u_1, t)|}{u_2-u_1}.$$

Consequently, if we fix a point $(t_0, u_0)$ in the right half $(t, u)$-plane, and a value $u_1<u_0$, then in a sufficiently small rectangle about $(t_0, u_0)$ it follows from (15) and the continuity of $\zeta(u, t)$ and $M(t)$ that $|\zeta(u_3, t)-\zeta(u_2, t)|\leqq A\,|u_3-u_2|\log 1/|u_3-u_2|$ for some

constant $A$. Hence the solution through the point $(t_0, u_0)$ is unique by Theorem 2.1, page 48 of [5].

Similarly, if values for $u_1$ and $u_2$ are fixed and $t$ varies in a finite interval $0 \leqq t \leqq T$, then for sufficiently large values of $u$, $|\zeta(u, t)| \leqq B|u| \log|u|$. Hence the solutions are of order $\exp\{e^{Bt}\}$, and therefore must exist for all $0 \leqq t \leqq T$.

Now suppose $\zeta(u, t)$ satisfies $\{*\}$. We first show that the solutions $f_t(x)$ are quasisymmetric. For this purpose fix $x_1 < x_2 < x_3$ and let the corresponding solutions to (2) be denoted by $u_i(t) = f_t(x_i)$, $i = 1, 2, 3$. Denote by $a'_t$ the cross ratio $a'_t = (\infty, u_1(t), u_2(t), u_3(t))$. Note that $a'_0 = (\infty, x_1, x_2, x_3) = a$. From $\{*\}$ we have

$$\frac{-M(t)}{|a'_t|\,\varrho(a'_t)} \leqq \frac{\dfrac{du_3(t)}{dt} - \dfrac{du_2(t)}{dt}}{u_3(t) - u_2(t)} - \frac{\dfrac{du_2(t)}{dt} - \dfrac{du_1(t)}{dt}}{u_2(t) - u_1(t)} \leqq \frac{M(t)}{|a'_t|\,\varrho(a'_t)}.$$

Equivalently

$$\frac{-M(t)}{|a'_t|\,\varrho(a'_t)} \leqq \frac{d\log|a'_t|}{dt} \leqq \frac{M(t)}{|a'_t|\,\varrho(a'_t)},$$

or

$$-M(t) \leqq \varrho(a'_t)\, d|a'_t|/dt \leqq M(t).$$

Integrating from 0 to $t$, making the substitution $s = a'_t$, and noting that $|a'_t| = -a'_t$, we arrive at

$$\left|\int_a^{a'_t} \varrho(s)\,ds\right| \leqq \int_0^t M(\tau)\,d\tau.$$

It follows from (9) that $K[f_t] \leqq \exp\int_0^t M(\tau)\,d\tau$.

To show that the family $f_t$ is of exponential type-$M(t)$, fix $t_0 \geqq 0$ and set $g_t = f_{t_0+t} \circ f_{t_0}^{-1}$. Then

$$dg_t/dt = \zeta(g_t, t_0 + t) = \xi(g_t, t),$$

and $g_0(x) = f_{t_0} \circ f_{t_0}^{-1}(x) \equiv x$, so $g_t$ is the solution to the initial value problem $du/dt = \xi(u, t)$; $u(0) = x$, and $\xi(u, t)$ has property $\{*\}$ with $M(t)$ replaced by $M(t_0 + t)$. Therefore

$$K[f_{t_1} \circ f_{t_0}^{-1}] = K[g_{t_1 - t_0}] \leqq \exp \int_0^{t_1 - t_0} M(t_0 + \tau)\,d\tau = \exp \int_{t_0}^{t_1} M(t)\,dt,$$

which completes the proof of the first half of the theorem.

For the converse we assume $f_t(x)$ is a family of exponential type-$M(t)$. If we define the velocity profile

$$\zeta(u, t) = \left.\frac{\partial f_t(x)}{\partial t}\right|_{x=f^{-1}_t(u)}$$

then certainly $f_t(x)$ is the solution to the initial value problem (2). That $\zeta(u, t)$ satisfies $\{*\}$ is shown simply by reversing the steps above. The details are omitted.

## 4. A Class of Functions with Property $\{*\}$

The following question quite naturally arises as a consequence of Theorem 1. For a given quasisymmetric function $f(x)$ how does one find a family $f_t(x)$ of exponential type which includes $f(x)$? (See [6], where this question is considered and solved for quasiconformal mappings of the disk.) In this section we solve this problem for quasisymmetric functions which are already the boundary correspondences of a certain class of quasiconformal mappings of the upper half plane.

We begin by finding an explicit class of functions $\zeta(u, t)$ with property $\{*\}$.

THEOREM 2. *Let $b_t(\xi)$, $t \geqq 0$, be bounded and measurable in the complex $\xi$-plane. If $b_t$ is symmetric $(b_t(\bar{\xi}) = \overline{b_t(\xi)})$, then*

$$\zeta(u, t) = \frac{-u(u-1)}{\pi} \iint_{\xi\text{-plane}} \frac{b_t(\xi)\, d\sigma_\xi}{\xi(\xi-1)(\xi-u)} \tag{16}$$

*is real if $u$ is real, and satisfies $\{*\}$ with*

$$M(t) = 2 \sup_{\xi} |b_t(\xi)|\,. \tag{17}$$

*Proof.* Fix $u_1 < u_2 < u_3$ and $t \geqq 0$. Set

$$Q = \frac{\zeta(u_3, t) - \zeta(u_2, t)}{u_3 - u_2} - \frac{\zeta(u_2, t) - \zeta(u_1, t)}{u_2 - u_1}. \tag{18}$$

One shows routinely from (16) that

$$Q = \frac{-1}{\pi} \iint_{\xi\text{-plane}} \frac{b_t(\xi)(u_3 - u_1)\, d\sigma_\xi}{(\xi - u_1)(\xi - u_2)(\xi - u_3)}. \tag{19}$$

Let us make the change in variable $\eta = (\xi - u_1)(u_2 - u_3)/(\xi - u_3)(u_2 - u_1)$, and compute the expressions $a = (u_2 - u_3)/(u_2 - u_1)$, $a - 1 = (u_1 - u_3)/(u_2 - u_1)$, $\eta - 1 = (\xi - u_2)(u_1 - u_3)/(\xi - u_3)(u_2 - u_1)$, $\eta - a = (u_3 - u_2)(u_1 - u_3)/(\xi - u_3)(u_2 - u_1)$,

$$d\sigma_\eta = \left|\frac{d\eta}{d\xi}\right|^2 d\sigma_\xi = \frac{|u_2 - u_3|^2\, |u_1 - u_3|^2}{|\xi - u_3|^4\, |u_2 - u_1|^2}\, d\sigma_\xi,$$

so that

$$\frac{|a-1|\,d\sigma_\eta}{|\eta|\,|\eta-1|\,|\eta-a|}=\frac{|u_1-u_3|\,d\sigma_\xi}{|\xi-u_1|\,|\xi-u_2|\,|\xi-u_3|}.$$

Consequently, from (19),

$$|Q|\leqq\frac{1}{\pi}\sup_\xi|b_t(\xi)|\iint\limits_{\xi\text{-plane}}\frac{|u_3-u_1|\,d\sigma_\xi}{|\xi-u_1|\,|\xi-u_2|\,|\xi-u_3|}$$
$$=M(t)\frac{1}{|a|}\left\{\frac{|a|\,|a-1|}{2\pi}\iint\limits_{\eta\text{-plane}}\frac{d\sigma_\eta}{|\eta|\,|\eta-1|\,|\eta-a|}\right\}.$$

But in [1], page 7, it is shown that the quantity in braces is exactly $1/\varrho(a)$, which completes the proof of the theorem.

For the moment let us relax the requirements of regularity on the family $f_t(x)$ which we imposed in Section 3. Suppose $w=g(z)$ is a normalized $(g(0)=0,\ g(1)=1)$ $K$-quasiconformal mapping from $\mathrm{Im}(z)\geqq 0$ onto $\mathrm{Im}(w)\geqq 0$ with boundary correspondence $u=f(x)$, and complex dilatation $\mu(z)$, ess sup $|\mu(z)|=\hat{k}<1$, $K=(1+\hat{k})/(1-\hat{k})$. By reflection in the real axes we can assume $g$ is quasiconformal in the entire plane and that $g(\bar{z})=\overline{g(z)}$, $\mu(\bar{z})=\overline{\mu(z)}$. Define $\mu_t(z)=t\mu(z)$, $0\leqq t\leqq 1$, and let $g_t(z)$ be the normalized quasiconformal mapping of the plane with dilatation $\mu_t(z)$ [3]. By the symmetry of $\mu_t$, $g_t$ maps the real axis onto itself and induces a boundary correspondence $u=f_t(x)$ with $K[f_t]\leqq(1+t\hat{k})/(1-t\hat{k})$, $f_0(x)\equiv x$, and $f_1(x)\equiv f(x)$.

It is well known that $g_{t_1}\circ g_{t_0}^{-1}$ is a $K(t_1,t_0)$-quasiconformal mapping, where

$$K(t_1,t_0)=\frac{(1+t_1\hat{k})(1-t_0\hat{k})}{(1-t_1\hat{k})(1+t_0\hat{k})}=\exp\left\{\log\frac{1+t_1\hat{k}}{1-t_1\hat{k}}-\log\frac{1+t_0\hat{k}}{1-t_0\hat{k}}\right\}$$
$$=\exp\left\{\int_{t_0}^{t_1}\frac{2\hat{k}\,dt}{1-t^2\hat{k}^2}\right\}$$

and therefore the family $f_t(x)$ is of exponential type-$2\hat{k}/(1-t^2\hat{k}^2)$, $0\leqq t\leqq 1$. Because a quasisymmetric function with $k[f]=k$ admits a $k^2$-quasiconformal extension, we have thus proved that *a k-quasisymmetric function according to definition* (1) *is reached at time* $t=1$ *in a family of exponential type* $-M(t)$, *with*

$$M(t)=\frac{2(k^4-1)}{(1-t^2)(k^4+1)+4t^2k^2}.$$

Returning to the family $g_t$, it is of interest to note that the associated function $\zeta(u,t)$ is of the type constructed in Theorem 2 provided $\mu$ is sufficiently smooth.

Indeed, Lemmas 19 and 21 of [3] would then ensure that $g_t(z)$ is differentiable in $t$ and in fact

$$\partial g_t(z)/\partial t = (Pb_t)(w) - w(Pb_t)(1); \quad w = g_t(z), \tag{20}$$

where

$$b_t(w) = \frac{\mu(z)}{1 - t^2|\mu(z)|^2} \frac{\dfrac{\partial g_t(z)}{\partial z}}{\overline{\dfrac{\partial g_t(z)}{\partial z}}}; \quad w = g_t(z) \tag{21}$$

and where $P$ is the transform

$$(Pb)(w) = \frac{-w}{\pi} \iint\limits_{\xi\text{-plane}} \frac{b(\xi)\, d\sigma_\xi}{\xi(\xi - w)}. \tag{22}$$

We remark that (20) is true in general, but the derivative must be taken as a limit in $L^p$($z$-plane) for some $p>2$. Our added hypotheses on $\mu$ are merely included to ensure that the derivative is also a pointwise limit on the $x$-axis. Since $g$ and $\mu$ are symmetric, so is $b_t$, and hence if $z=x$ is real, we get from (20) and (22),

$$\zeta(u, t) = \left.\frac{\partial f_t(x)}{\partial t}\right|_{u=f_t(x)} = \frac{-u(u-1)}{\pi} \iint\limits_{\xi\text{-plane}} \frac{b_t(\xi)\, d\sigma_\xi}{\xi(\xi-1)(\xi-u)}. \tag{23}$$

The class of quasiconformal mappings with, say, $\mu$ of class $C^2$ with compact support, is dense with respect to uniform convergence on compact sets in the class of all quasiconformal mappings. This smoothness of $\mu$ imposes sufficient regularity on $g_t$ and hence on $b_t$ (cf. equation (21)) to guarantee that $\zeta(u, t)$ (cf. equation (16)) is continuous. Summarizing, we have proved

THEOREM 3. *The class of functions generated by continuous functions $\zeta(u, t)$ constructed according to Theorem 2 from functions $b_t(\xi)$ having continuous derivatives and compact support is dense with respect to uniform convergence on compact sets in the class of normalized quasisymmetric functions.*

## 5. Quasiconformal Extensions and Examples

There remains a dearth of quasiconformal extensions for a given boundary correspondence. The Beurling-Ahlfors extension has found numerous applications because of its explicit form. In this section, we point out a way of obtaining quasiconformal extensions for the quasisymmetric functions generated by the functions $\zeta(u, t)$ of Theorem 2.

We suppose that $F(w, t)$, $0 \leqq t \leqq T$, is a complex valued function of the complex variable $w$, with $F_{\bar{w}}$ bounded, and sufficient regularity to justify the following computations.

Let $w = f_t(z)$ be the solution to the initial value problem

$$dw/dt = F(w, t); \quad w(0) = z. \tag{23}$$

By the uniqueness, for each fixed $t, f_t(z)$ is a homeomorphism. We determine its complex dilatation $\mu_t(z)$ defined by

$$\frac{\partial f_t(z)}{\partial \bar{z}} = \mu_t(z) \frac{\partial f_t(z)}{\partial z}. \tag{24}$$

Differentiating with respect to $z$, $\bar{z}$ in (23) and with respect to $t$ in (24), we find

$$\frac{\partial^2 f}{\partial z\, \partial t} = F_w \frac{\partial f}{\partial z} + F_{\bar{w}} \frac{\partial \bar{f}}{\partial z}, \quad \frac{\partial^2 f}{\partial \bar{z}\, \partial t} = F_w \frac{\partial f}{\partial \bar{z}} + F_{\bar{w}} \frac{\partial \bar{f}}{\partial \bar{z}}, \quad \frac{\partial^2 f}{\partial t\, \partial \bar{z}} = \frac{\partial \mu}{\partial t} \frac{\partial f}{\partial z} + \mu \frac{\partial^2 f}{\partial t\, \partial z},$$

from which it follows upon equating the mixed partials that

$$\frac{\left|\dfrac{\partial \mu_t(z)}{\partial t}\right|}{1 - |\mu_t(z)|^2} = |F_{\bar{w}}(f_t(z), t)|. \tag{25}$$

Then setting $M(t) = 2 \sup_w |F_{\bar{w}}(w, t)|$, and noting $\mu_0(z) = 0$, we find from (25)

$$\log\left(\frac{1 + |\mu_t|}{1 - |\mu_t|}\right) = \int_0^t \frac{2\, d|\mu_\tau|}{1 - |\mu_\tau|^2} \leqq \int_0^t \frac{2\left|\dfrac{\partial \mu_\tau}{\partial \tau}\right| d\tau}{1 - |\mu_\tau|^2} \leqq \int_0^t M(\tau)\, d\tau. \tag{26}$$

As an application, for sufficiently regular symmetric $b_t$, define $F(w, t)$ by the right side of (16) where $u$ is replaced by the complex variable $w$. Since the expression coincides with $(Pb_t)(w) - w(Pb_t)(1)$, it follows that $F_{\bar{w}} = b_t$. The complex solution of (23) is an extension of the real solution $f_t(x)$ generated by $\zeta(u, t)$ as defined in (16), and from (26) we see that the dilatation does not exceed $\exp\{\int_0^t M(\tau)\, d\tau\}$, the same bound noted for $K[f_t]$.

In order to take advantage of these observations, it becomes essential to determine all functions $b_t(w)$ with the property that $(Pb_t)(u) - u(Pb_t)(1)$ coincides with a prescribed function $\zeta(u, t)$ satisfying {*}. The existence of at least one follows from the Beurling-Ahlfors extension. For variational purposes it becomes desirable to describe all symmetric functions $b_t(w)$ such that $(Pb_t)(u) - u(Pb_t)(1)$ vanishes for real $u$.

We conclude with two simple examples to illustrate these ideas, the second having the variational property mentioned above.

EXAMPLE 1: $F(w, t) = w \log|w|$. This leads to $f_t(z) = z|z|^{e^t - 1}$ and results from a choice of $b_t(w) = w/2\bar{w}$.

EXAMPLE 2: $F(w, t) = (\bar{w} - w)/(1 - t^2)$, $0 \leqq t < 1$. This leads to $f_t(z) = (z + t\bar{z})/(1 + t)$, and results from a choice of $b_t(w) \equiv 1/(1 - t^2)$.

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*University of Minnesota*
*Minneapolis, Minn., U.S.A.*

Received November 22, 1968