Differential Structures on a Product of Spheres.

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1. Introduction

In this paper we give a classification under the relation of orientation preserving diffeomorphism, of all differential structures on a simply connected product of spheres $S^k \times S^p$ of dimension greater than five. In particular, we prove the following theorem.

THEOREM 1. *Let M^n be a differential n-manifold of dimension greater than five that is homeomorphic to a product of spheres $S^k \times S^p$, where $2 \leqslant k \leqslant p$. Then there are homotopy spheres A^p and V^n such that M^n is diffeomorphic to $(S^k \times A^p) + V^n$. Furthermore, we have the following conclusions.*

(a) *If B^p and U^n are homotopy spheres such that M^n is also diffeomorphic to $(S^k \times B^p) + U^n$, then $S^k \times A^p$ and $S^k \times B^p$ are diffeomorphic. If, in addition, either* $k \equiv 2, 4, 5, 6 \pmod 8$ *or $k \geqslant p-3$, then V^n and U^n are diffeomorphic.*

(b) *If either $k \geqslant p-3$ or* $p=4, 5, 6, 7$, *then there is one and only one homotopy n-sphere V^n such that M^n is diffeomorphic to $(S^k \times S^p) + V^n$.*

Here S^n denotes the unit n-sphere with its usual differential structure in euclidean $(n+1)$-space R^{n+1} and $+$ denotes the connected sum operation. Now let Θ_n denote the group of homotopy n-spheres and let Φ_p^{k+1} denote the subgroup of Θ_p consisting of these homotopy p-spheres that embed in R^{p+k+1} with a trivial normal bundle. Let $H_{p,k}$ denote the subset of Θ_p consisting of those homotopy p-spheres A^p such that $S^k \times A^p$ is diffeomorphic to $S^k \times S^p$. It is known that if $k \geqslant 2$, then $H_{p,k} = \Phi_p^{k+1}$. In the next section we prove the following.

THEOREM 2. *Let A^p and B^p be homotopy p-spheres such that $p > 4$ and let k be an integer greater than one. Then $S^k \times A^p$ and $S^k \times B^p$ are diffeomorphic if and only if* $A^p \equiv \pm B^p \operatorname{mod} H_{p,k}$.

This theorem follows from Lemmas 4, 5, and 6 of the next section. Theorems 1 and 2 combine to give a classification of differential structures on $S^k \times S^p$ in the case where either $k \equiv 2, 4, 5, 6 \pmod 8$ or $k \geqslant p-3$. In fact, let $C_{p,k}$ be the set obtained from the quotient group $\Theta_p / H_{p,k}$ by identifying $x \in \Theta_p / H_{p,k}$ with its inverse $-x$. Then Theorem 1 has the following corollary.

COROLLARY 1.1. *The diffeomorphism classes of n-manifolds $(n \geqslant 6)$ that are homeomorphic to $S^k \times S^p$ $(k \leqslant p)$ are in a one-to-one correspondence with the product $C_{p,k} \times \Theta_n$, provided that either* $k \equiv 2, 4, 5, 6 \pmod 8$ *or $k \geqslant p-3$.*

This one-to-one correspondence is given by Theorem 1. In particular, conclusion (a) of Theorem 1 asserts that the group Θ_{p+k} acts nontrivially *via* connected sum on the differential structure of M^n, provided that either $k \equiv 2, 4, 5, 6 \pmod 8$ or $k \geqslant p-3$. This is not always true in the remaining case where $k < p-3$ and $k \equiv 0, 1, 3, 7 \pmod 8$, although we do show here that Θ_{p+k} acts nontrivially on $S^k \times S^p$ in all cases (Lemma 2 below). In a subsequent paper we shall prove the following result.

THEOREM. *Let*

$$\tau_{p,k}: \theta_p \otimes \pi_k(SO(p-1)) \to \theta_{p+k}$$

denote the pairing of Milnor-Munkres-Novikov and let A^p and V^{p+k} be homotopy spheres. Then $(S^k \times A^p) + V^{p+k}$ is diffeomorphic to $S^k \times A^p$ if and only if there exists $\alpha \in \pi_k(SO(p-1))$ such that $V^{p+k} = \tau_{p,k}(A^n \otimes \alpha)$. In particular, if $k \geqslant p-3$, then $\tau_{p,k} = 0$.

Thus the differential structures on $S^k \times S^p$ can be classified in terms of $\Theta_p / H_{p,k}$ and the pairing $\tau_{p,k}$. We remark that the pairing $\tau_{p,k}$ corresponds to composition in the stable homotopy groups of spheres and is sometimes nontrivial. Therefore the complete classification in the case where $k < p-3$ and $k \equiv 0, 1, 3, 7 \pmod 8$ is more complicated than that given by Corollary 1.1.

We can make some remarks that relate to the structure of the groups $H_{p,k}$. Let bP_{p+1} denote the subgroup of Θ_p consisting of those homotopy p-spheres that bound parallelizable manifolds. It is known that $bP_{p+1} \subset \Phi_p^2$ and hence, since $\Phi_p^{k+1} = H_{p,k}$ for $k \geqslant 2$, it follows that $bP_{p+1} \subset H_{p,k} \subset H_{p,k+1}$, provided that $k \geqslant 2$. Theorem 1 asserts that $H_{p,k} = \Theta_p$ if $k \geqslant p-3$, although this is not true for $k < p-3$. In fact, if Σ^{16} represents the nonzero element in $\Theta_{16} \approx Z_2$, then according to [5, Corollary 1.5] $S^{12} \times \Sigma^{16}$ is not diffeomorphic to $S^{12} \times S^{16}$; that is, $H_{16,12} = 0$. Finally, it can be shown that $H_{p,k}/bP_{p+1}$ is isomorphic to the cokernel of the Hopf-Whitehead homomorphism $J: \pi_p(SO(k+1)) \to \pi_{p+k+1}(S^{k+1})$, provided that $2k > p-1$ and $p \neq 2^a - 2$ (c.f. [5, Th. 1.7]). This isomorphism is induced by the Pontrjagin-Thom construction and is of interest here since the groups bP_{p+1} have been determined in many cases.

We include here a result on the action of Θ_{k+p} on the total space E of a differential k-sphere bundle over a homotopy p-sphere. Precisely, $\pi: E \to A^p$ is a k-sphere bundle over A^p with the special orthogonal group $SO(k+1)$ as structural group such that the homeomorphisms which specify the local product structure are diffeomorphisms, and where the fibre is S^k.

PROPOSITION 1. *Each element of Θ_{k+p} acts nontrivially on the differential structure of the total space E of a differential k-sphere bundle over a homotopy p-sphere A^p, provided that $k < p-1$ and $k \equiv 2, 4, 5, 6 \pmod 8$.*

The proof of Theorem 1 is given in Section 3; following the proof there are some remarks on the case where $k=1$ or $n<6$. The results on the action of Θ_n are proved in the next section. All manifolds are assumed to be smooth of class C^∞, and oriented;

diffeomorphisms are assumed to be orientation preserving and of class C^∞. Finally, D^n denotes the unit n-disc with the standard differential structure in euclidean n-space R^n.

2. Products of Homotopy Spheres

We begin with the lemmas that are needed in proving the theorems.

LEMMA 1. *If A^p is a homotopy p-sphere, then $A^p \times D^k$ is diffeomorphic to $S^p \times D^k$, provided that $p \neq 3$ and $k \geqslant p-2$.*

REMARK. If $k \geqslant p+2 \geqslant 7$, then this is a result of Mazur.

Proof. If $p=1, 2$, then this is a classical result; if $p=4$, then this is a result of HIRSCH [4, Theorem 6]. Since A^p and S^p are diffeomorphic for $p=5, 6$ we can assume that $p \geqslant 7$. It follows from the theorems of HAEFLIGER [3] that A^p may be differentiably embedded in R^{p+k}, provided that $k \geqslant p-2$. Furthermore, it follows from [5, Th. 1.10] that the normal tube of this embedding is diffeomorphic to $A^p \times D^k$. But it is known that the normal tube of an embedded homotopy p-sphere in R^{p+k} is diffeomorphic to $S^p \times D^k$, provided that $k \geqslant 3$, and the lemma is proved.

LEMMA 2. *Let A^p and B^p be homotopy p-spheres and let V^{p+k} be a homotopy $(p+k)$-sphere, where $p+k \geqslant 6$. If $k < p-3$ and $k \equiv 0, 1, 3, 7$ (mod 8), then assume that $S^k \times B^p$ is diffeomorphic to $S^k \times S^p$. Then, if $(S^k \times A^p) + V^{p+k}$ is diffeomorphic to $S^k \times B^p$, then V^{p+k} is diffeomorphic to the standard $(p+k)$-sphere S^{p+k} (and hence $S^k \times A^p$ and $S^k \times B^p$ are diffeomorphic).*

This lemma is also true for $p+k=5$ since $\Theta_5 = 0$.

Proof. In the first place if $p=k \geqslant 3$, then the lemma follows from [1, Th. B]. If $k \geqslant p-3$, then Lemma 1 above implies that $S^k \times A^p$ and $S^k \times B^p$ are both diffeomorphic to $S^k \times S^p$, provided that $p \neq 3$. Thus if $k=p-1$, then Lemma 2 follows from [2, Lem. 1]. Therefore we can assume that $k < p-1$, which implies that $p \geqslant 4$. If $p=4$, then $k=2$ and there is nothing to prove since $\Theta_6 = 0$. Therefore assume that $p > 4$.

Let

$$h:(S^k \times A^p) + V^{p+k} \to S^k \times B^p$$

be a diffeomorphism. It is known that there is a diffeomorphism $f: S^{p-1} \to S^{p-1}$ such that A^p is diffeomorphic to $D_1^p \cup_f D_2^p$, the disjoint union of two copies D_1^p, D_2^p of the p-disc D^p indentified along the boundaries *via* the diffeomorphism f (that is, $x \in \partial D_2^p$ is identified with $f(x) \in \partial D_1^p$ and $D_1^p \cup_f D_2^p$ is given the orientation of D_2^p). Similarly, B^p is diffeomorphic to $D_1^p \cup_g D_2^p$, where $g: S^{p-1} \to S^{p-1}$ is a diffeomorphism. Thus we can write $S^k \times A^p$ as a disjoint union of two copies of $S^k \times D^p$, in the form

$$S^k \times A^p = (S^k \times D_1^p) \cup_{i \times f} (S^k \times D_2^p), \tag{1}$$

with points identified along $S^k \times S^{p-1}$ *via* the diffeomorphism $i \times f$, where $i: S^k \to S^k$ is the identity map. Similarly,

$$S^k \times B^p = (S^k \times D_1^p) \cup_{i \times g} (S^k \times D_2^p). \tag{2}$$

Now let $0 \in D_1^p$ denote the center of the p-disc D_1^p. The k-sphere $S^k \times 0$ is embedded in $S^k \times A^p$ and in $S^k \times B^p$. We can assume that the connected sum $(S^k \times A^p) + V^{p+k}$ is made far away from the sphere $S^k \times 0$ and hence $S^k \times 0$ is also embedded in $(S^k \times A^p) + V^{p+k}$. The next step is to show that we can assume that the diffeomorphism h is the identity on the k-sphere $S^k \times 0$. In fact, since $k < p-1$ h maps the homotopy class of $S^k \times 0$ in $(S^k \times A^p) + V^{p+k}$ onto either the homotopy class of $S^k \times 0$ in $S^k \times B^p$ or the negative of the homotopy class of $S^k \times 0$ in $S^k \times B^p$. In the latter case we can compose h with the (orientation preserving) diffeomorphism $\varrho \times i: S^k \times B^p \to S^k \times (-B^p)$, where $\varrho: S^k \to S^k$ is a diffeomorphism of degree -1 and $i: B^p \to -B^p$ is the identity map ($-B^p$ is the manifold B^p with the orientation reversed), to obtain a diffeomorphism $(\varrho \times i) \circ h: (S^k \times A^p) + V^{p+k} \to S^k \times (-B^p)$ that maps the homotopy class of $S^k \times 0$ in $(S^k \times A^p) + V^{p+k}$ into the homotopy class of $S^k \times 0$ in $S^k \times (-B^p)$. Thus we can assume that the restriction $h \mid S^k \times 0$ is homotopic to the inclusion $S^k \times 0 \subset S^k \times B^p$. It follows from the theorems of HAEFLIGER [3] that $h \mid S^k \times 0$ is diffeotopic to the inclusion $S^k \times 0 \subset S^k \times B^p$, and hence by application of the diffeotopy extension theorem we can assume that $h(u, 0) = (u, 0)$ for each $(u, 0) \in S^k \times 0$. By the tubular neighborhood theorem we may further suppose that $h(S^k \times D_1^p) = S^k \times D_1^p$ such that for each $(u, v) \in S^k \times D_1^p$, $h(u, v) = (u, v \cdot \alpha(u))$, where $\alpha: S^k \to SO(p)$ is a differentiable map and $v \cdot \alpha(u)$ denotes the action of $\alpha(u) \in SO(p)$ on $v \in D_1^p$. Now perform the spherical modification on $(S^k \times A^p) + V^{p+k}$ that removes the k-sphere $S^k \times 0$ with product structure $S^k \times D_1^p$. The following proposition implies that the result of this modification is V^{p+k}.

PROPOSITION 2. *Let $A^p = D_1^p \cup_f D_2^p$ be a homotopy p-sphere and let $\varphi: S^k \times D_1^p \subset S^k \times A^p$ be the inclusion. Then, the result of the spherical modification on $S^k \times A^p$ based on φ is always S^{k+p}.*

Proof. The result of the modification is

$$(D^{k+1} \times S^{p-1}) \cup_{i \times f} (S^k \times D_2^p), \tag{3}$$

which is clearly diffeomorphic to

$$S^{p+k} = (D^{k+1} \times S^{p-1}) \cup_i (S^k \times D^p) \tag{4}$$

(where $i: S^k \times S^{p-1} \to S^k \times S^{p-1}$ is the identity) by virtue of the map that sends $(u, v) \in S^k \times D^p$ into $(u, v) \in S^k \times D_2^p$ and $(u, v) \in D^{k+1} \times S^{p-1}$ into $(u, f(v)) \in D^{k+1} \times S^{p-1}$ (this diffeomorphism goes from (4) to (3)). Q.E.D.

Returning to the lemma we perform the corresponding modification (under h)

on $S^k \times B^p$ to remove the k-sphere $S^k \times 0$ with product structure $h(S^k \times D_1^p)$ in $S^k \times B^p$. From the latter modification we obtain the manifold

$$(D^{k+1} \times S^{p-1}) \cup_\psi (S^k \times D_2^p), \tag{5}$$

where $\psi = (h^{-1} \mid S^k \times S^{p-1}) \circ (i \times g)$ (see (2)), which is clearly diffeomorphic to V^{p+k} because of the way that this modification was defined (using h). We complete the proof of Lemma 2 by showing that (5) is diffeomorphic to S^{p+k}. This is done by constructing a diffeomorphism from (5) to (4) as follows, recalling that $k<p-1$. If $k<p-3$ and $k \equiv 2, 4, 5, 6 \pmod 8$, then $\pi_k(SO(p))=0$ and hence we can apply Proposition 2 to conclude that (5) is diffeomorphic to S^{p+k}. If $k=p-3$ or $p-2$ then by Lemma 1 we can assume that g is the identity; if $k<p-3$ and $k \equiv 0, 1, 3, 7 \pmod 8$, then by hypothesis $S^k \times B^p$ is diffeomorphic to $S^k \times S^p$ and we can again assume that g is the identity. Thus in these cases $\psi = h^{-1} \mid S^k \times S^{p-1}$ and we have the diffeomorphism that sends

$$(u, v) \in D^{k+1} \times S^{p-1} \quad \text{into} \quad (u, v) \in D^{k+1} \times S^{p-1}$$

and $(u, v) \in S^k \times D_2^p$ into

$$h(u, v) = (u, v \cdot \alpha(u)) \in S^k \times D^p.$$

The proof of Proposition 1 is similar to the preceding and is left to the reader.

The following lemma is a weakened form of Lemma 2 but removes the special assumption which was made there in the case where $k<p-3$ and $k \equiv 0, 1, 3, 7 \pmod 8$.

Lemma 3. *Let A^p and B^p be homotopy p-spheres such that for some integer k, $A^p \times S^k$ and $B^p \times S^k$ are diffeomorphic up to a point. Then $A^p \times S^m$ and $B^p \times S^m$ are diffeomorphic for all $m \geq \max(k, 2)$.*

Proof. If $k \geq p-3$, then the lemma is a trivial consequence of Lemma 1. Thus we can assume that $k<p-3$. If $h: A^p \times S^k \to B^p \times S^k$ is a diffeomorphism up to a point, then we can compose h with the inclusion $B^p \times S^k \subset B^p \times D^{m+1}$ and we obtain a differentiable embedding $A^p \to B^p \times D^{m+1}$ with a trivial normal bundle. We show that the embedding $A^p \to B^p \times D^{m+1}$ is also a homotopy equivalence, by an elementary argument. In fact, let $y \in A^p$ and $z \in B^p$ such that the k-spheres $y \times S^k$ and $z \times S^k$ do not contain the singularity of h. Now $k<p-3$ and hence by standard arguments (theorems of Haefliger and diffeotopy extension) we can assume that h maps $y \times S^k$ diffeomorphically onto $z \times S^k$. It follows that the induced homomorphism

$$h_*: \pi_p(A^p \times S^k) = \pi_p(A^p) + \pi_p(S^k) \to \pi_p(B^p \times S^k) = \pi_p(B^p) + \pi_p(S^k)$$

maps $\pi_p(S^k)$ isomorphically onto $\pi_p(S^k)$ and hence h_* maps the generator of $\pi_p(A^p)$ into a generator of $\pi_p(B^p)$ plus an element of $\pi_p(S^k)$. Consequently the composition $A^p \times S^k \to B^p \times S^k \subset B^p \times D^{m+1}$ maps the generator of $\pi_p(A^p)$ into a generator of $\pi_p(B^p \times D^{m+1})$ and it follows that the embedding $A^p \to B^p \times D^{m+1}$ is a homotopy equivalence with a trivial normal bundle. We can apply [7, Th. 4.1] to conclude that $A^p \times D^{m+1}$ is diffeomorphic to $B^p \times D^{m+1}$, provided that $m \geqslant \max(k, 2)$.

The remainder of the present section is devoted to the study of the diffeomorphism classes of manifolds of the form $S^k \times A^p$, where A^p is an arbitrary homotopy p-sphere. Let $p \geqslant 4$ and $k \geqslant 2$ be a given pair of integers. Then two homotopy p-spheres A^p and B^p are called *k-equivalent* if and only if $S^k \times A^p$ and $S^k \times B^p$ are diffeomorphic. Thus the group Θ_p is divided into k-equivalence classes. It is clear that the k-equivalence class of an element $A^p \in \Theta_p$ contains its inverse $-A^p$ in the group Θ_p. Lemma 4 below asserts that the k-equivalence class of S^p is a subgroup of Θ_p. This subgroup is denoted by $H_{p,k}$.

LEMMA 4. *The set $H_{p,k}$ of those homotopy p-spheres A^p such that $S^k \times A^p$ is diffeomorphic to $S^k \times S^p$ forms a subgroup of Θ_p, provided that $p \neq 3$ and $k \geqslant 2$.*

This lemma follows from the next lemma. Lemma 5 implies that any k-equivalenc class is the union of cosets of the subgroup $H_{p,k}$ of the group Θ_p.

LEMMA 5. *Let A^p and B^p be homotopy p-spheres such that $A^p \in H_{p,k}$. Then $S^k \times (A^p + B^p)$ is diffeomorphic to $S^k \times B^p$.*

Proof. Since $A^p \in H_{p,k}$ it follows that A^p may be embedded in the interior of a $(p+k+1)$-disc in $D^{k+1} \times B^p$ with a trivial normal bundle. But B^p is embedded in $D^{k+1} \times B^p$ in the obvious way with a trivial normal bundle, and hence we can form the connected sum $A^p + B^p$ in $D^{k+1} \times B^p$ so that $A^p + B^p$ has a trivial normal bundle. Furthermore, the resulting embedding $A^p + B^p \to D^{k+1} \times B^p$ is a homotopy equivalence and hence by [7, Th. 4.1] $D^{k+1} \times (A^p + B^p)$ is diffeomorphic to $D^{k+1} \times B^p$, provided that $k \geqslant 2$. Q.E.D.

In general it does not seem likely that each k-equivalence class contains exactly one coset of $H_{p,k}$. If this is the case, then the k-equivalence classes are in a one-to-one correspondence with the quotient group $\Theta_p / H_{p,k}$; in particular, this would imply that $\Theta_p / H_{p,k}$ consists entirely of elements of order two. The next lemma is the best that we can do in this direction.

LEMMA 6. *If A^p and B^p are homotopy p-spheres such that $S^k \times A^p$ and $S^k \times B^p$ are diffeomorphic, then either $S^k \times (A^p + B^p)$ or $S^k \times (A^p + (-B^p))$ is diffeomorphic to $S^k \times S^p$, provided that $k \geqslant 2$.*

Proof. If $k \geqslant p-3$, then the lemma is a consequence of Lemma 1. Thus we can assume that $k < p-3$. The hypothesis implies that A^p may be embedded in $D^{k+1} \times B^p$ with a trivial normal bundle. Furthermore, it follows from an argument given in the proof of Lemma 3 that the embedding $A^p \to D^{k+1} \times B^p$ is a homotopy equivalence. Let us assume that the embedding maps the orientation class of A^p onto the orientation class of B^p in $D^{k+1} \times B^p$ (otherwise we replace B^p by $-B^p$). Now $-B^p$ is embedded in $D^{k+1} \times B^p$ in the obvious way with a trivial normal bundle and we can assume that A^p and $-B^p$ are disjoint in $D^{k+1}\times B^p$. Thus we can form the connected sum $A^p+(-B^p)$ in $D^{k+1}\times B^p$ such that the resulting embedding $A^p+(-B^p)\to D^{k+1}\times B^p$ has a trivial normal bundle and is homotopically trivial. Now the engulfing result of [9, Chapter 7] applies to conclude that $A^p+(-B^p)$ is embedded in the interior of a $(p+k+1)$-disc in $D^{k+1}\times B^p$. But the normal tube of a homotopy p-sphere embedded in the interior of a $(p+k+1)$-disc is diffeomorphic to $D^{k+1}\times S^p$, provided that $k\geqslant 2$ and $p\neq 3, 4$, and hence it follows that $D^{k+1}\times(A^p+(-B^p))$ is diffeomorphic to $D^{k+1}\times S^p$ (if $p=3$ or 4, then apply [4, Th. 6]).

Lemma 6 implies that the k-equivalence class of an element $A^p\in\Theta_p$ is exactly the union of the cosets $A^p+H_{p,k}$ and $-A^p+H_{p,k}$. That is, each k-equivalence class consists of at most two cosets. Furthermore, a k-equivalence class consists of exactly one coset if and only if it contains an element A^p such that $A^p+H_{p,k}$ is of order two in the group $\Theta_p/H_{p,k}$. This completes the proof of Theorem 2.

3. Classification

Proof of Theorem 1. Since M^n is homeomorphic to $S^k\times S^p$, where $n\geqslant 6$ and $p\geqslant k\geqslant 2$, it follows that M^n is simply connected and $H_3(M^n; Z)$ has no 2-torsion. Therefore the "Hauptvermutung" of D. SULLIVAN [8] implies that there is a combinatorial equivalence

$$h: M^n\to S^k\times S^p, \tag{5}$$

where the combinatorial structures are compatible with the differential structures and $S^k\times S^p$ has the usual combinatorial structure. We now apply the obstruction theory of MUNKRES [6]. The application is particularly simple since we are dealing with a product of spheres. We note that the combinatorial equivalence h is a diffeomorphism mod the $n-1$ skeleton L_{n-1} of M^n; suppose that h is a diffeomorphism mod the $n-q$ skeleton L_{n-q} of M^n, where $1\leqslant q<k-1$. The obstruction to an approximation $g: M^n\to S^p\times S^k$ of h such that g is a diffeomorphism mod the $n-q-1$ skeleton L_{n-q-1}, is a simplicial $(n-q)$-cycle $\lambda_{n-q}h$ of L_{n-q} with coefficients in the group Γ^q (see [6, § 3]; g is called a smoothing of h), where Γ^m is the group of diffeomorphisms of S^{m-1} modulo those diffeomorphisms that are extendable to diffeomorphisms of D^m. If $\lambda_{n-q}h=0$, then the smoothing g exists according to [6, § 4]. Since $\Gamma^1=0$ it follows that the smoothing g does exist if $q=1$. Furthermore, if $\lambda_{n-q-1}g$ is homologous to zero in $H_{n-q-1}(L_{n-q}; \Gamma^{q+1})$, then it follows from [6, § 5] that there is a smoothing f of h such that $\lambda_{n-q-1}f=0$. But $H_{n-q-1}(L_{n-q}; \Gamma^{q+1})\approx H_{n-q-1}(M^n; \Gamma^{q+1})=0$ for $1\leqslant q<k-1$ and hence by induction there exists a map $g: M^n\to S^k\times S^p$ that is a diffeomorphism mod the $n-k$ skeleton of M^n. Thus the first obstruction to deforming h into a diffeomorphism is a well defined homology class γh (called the obstruction class) in $H_p(M^n; \Gamma^k)$. We first consider the case where $k<p$; then $H_p(M^n; \Gamma^k)\approx\Gamma^k$ and hence we can consider the obstruction class γh to be an element of Γ^k. Let

$\varphi: S^{k-1} \to S^{k-1}$ be a diffeomorphism that represents γh and let $N(\gamma h)$ denote the homotopy k-sphere $D_1^k \cup_\varphi D_2^k$. There is the combinatorial equivalence $j: S^k \to N(\gamma h)$ of degree $+1$, defined by writing $S^k = D_1^k \cup_i D_2^k$ and letting j be the identity on D_1^k and the radial extension of φ^{-1} on D_2^k, and hence we have a combinatorial equivalence $j \times i: S^k \times S^p \to N(\gamma h) \times S^p$, where i is the identity. It follows from [6, Def. 3.4] that the first obstruction $\gamma(j \times i)$ to deforming $j \times i$ to a diffeomorphism is $-\gamma h$. Furthermore, by [6, 3.8] the first obstruction to deforming the composition $(j \times i) \circ h: M^n \to N(\gamma h) \times S^p$ into a diffeomorphism is

$$\begin{aligned} \gamma((j \times i) \circ h) &= \gamma(j \times i) + \gamma h \\ &= -\gamma h + \gamma h \\ &= 0 \end{aligned}$$

and hence we can assume that there is a map

$$h': M^n \to N(\gamma h) \times S^p,$$

that is a diffeomorphism mod the k-skeleton of M^n. By Lemma 1 $N(\gamma h) \times S^p$ is diffeomorphic to $S^k \times S^p$ since $p > k$ (if $k=3$, then $N(\gamma h)$ is diffeomorphic to S^3 since $\Gamma^3 = 0$, as is well known) and hence we have a map (also denoted by h') $h': M^n \to S^k \times S^p$ that is a diffeomorphism mod the k-skeleton of M^n. The first obstruction to deforming h' into a diffeomorphism is a class $\gamma h' \in H_k(M^n; \Gamma^p) \approx \Gamma^p$. Let $\psi: S^{p-1} \to S^{p-1}$ be a diffeomorphism that represents $\gamma h'$, let $N(\gamma h') = D_1^p \cup_\psi D_2^p$, and let $j': S^p \to N(\gamma h')$ be the combinatorial equivalence of degree $+1$ as defined above for $N(\gamma h')$. Then we have the combinatorial equivalence $i \times j': S^k \times S^p \to S^k \times N(\gamma h')$ and the first obstruction to smoothing $(i \times j') \circ h': M^n \to S^k \times N(\gamma h')$ is $\gamma(i \times j') + \gamma h'$, which is zero since $\gamma(i \times j') = -\gamma h'$. It follows that there is a map $h'': M^n \to S^k \times N(\gamma, h')$ that is a diffeomorphism mod a point of M^n. Under these circumstances it is known that there is a homotopy n-sphere V^n such that M^n is diffeomorphic to $(S^k \times N(\gamma, h')) + V^n$.

Now suppose that $k = p$. The first obstruction to deforming the combinatorial equivalence (5) into a diffeomorphism is a class $\gamma h \in H_k(M^n; \Gamma^k) \approx \Gamma^k \oplus \Gamma^k$; write $\gamma h = \gamma^1 \oplus \gamma^2$, where $\gamma^1, \gamma^2 \in \Gamma^k$, and let $\varphi_1, \varphi_2: S^{k-1} \to S^{k-1}$ be diffeomorphisms that represent γ^1, γ^2. As before we have the homotopy spheres $N(\gamma^1) = D_1^k \cup_{\varphi_1} D_2^k$, $N(\gamma^2) = D_1^k \cup_{\varphi_2} D_2^k$ and the combinatorial equivalences $j_1: S^k \to N(\gamma^1)$, $j_2: S^k \to N(\gamma^2)$. The first obstruction to smoothing $j_1 \times j_2: S^k \times S^k \to N(\gamma^1) \times N(\gamma^2)$ is the class $\gamma(j_1 \times j_2) = (-\gamma^1) \oplus (-\gamma^2)$ and hence the first obstruction to deforming the composition

$$(j_1 \times j_2) \circ h: M^n \to N(\gamma^1) \times N(\gamma^2)$$

is

$$\gamma(j_1 \times j_2) \circ h) = \gamma(j_1 \times j_2) + \gamma h = 0.$$

It follows that there is a map $h': M^n \to N(\gamma^1) \times N(\gamma^2)$ that is a diffeomorphism mod a point. Now a result of WALL applies to conclude that $N(\gamma^1) \times N(\gamma^2)$ and $S^k \times S^k$ are diffeomorphic up to a point (in fact, see [1, Th. B]) and hence there is a homotopy $2k$-sphere V^n such that M^n is diffeomorphic to $(S^k \times S^k) + V^n$.

The proof of Theorem 1 is now completed by applying Lemmas 1 and 2.

In this theorem it is assumed that $2 \leqslant k \leqslant p$ and $n = p + k \geqslant 6$. We conclude with some remarks on the excluded cases.

CASE 1. $k = 1, n = p + k \geqslant 6$. The Hauptvermutung of [8] does not apply in this case and so consider a differential n-manifold M^n that is combinatorially equivalent to $S^1 \times S^p$. We can apply the obstruction theory of [6] as was done in the proof of Theorem 1 to conclude that M^n is diffeomorphic to $(S^1 \times A^p) + V^n$, where A^p and V^n are homotopy spheres that are combinatorially equivalent to the standard spheres. Then by application of Lemma 2 we have the following theorem: *If M^n is a differential n-manifold that is combinatorially equivalent to $S^1 \times S^p$, where $n = p + 1 \geqslant 6$, then there are homotopy spheres A^p and V^n such that M^n is diffeomorphic to $(S^1 \times A^p) + V^n$. If $S^1 \times A^p$ is diffeomorphic to $S^1 \times S^p$, then V^n is uniquely determined by M^n.* On the other hand, if we assume only that there is a homeomorphism h between M^n and $S^1 \times S^p$, then according to [8] there is an integer q such that the homeomorphism $h \times$identity between $M^n \times R^q$ and $S^1 \times S^p \times R^q$ is homotopic to a combinatorial equivalence (R^q is euclidean q-space). One can try to smooth the combinatorial equivalence between $M^n \times R^q$ and $S^1 \times S^p \times R^q$ by applying [6].

CASE 2. $p + k \leqslant 6$. Since $\Gamma^q = 0$ for $q \leqslant 6$, it follows from MUNKRES' obstruction theory that combinatorial equivalences can be smoothed to diffeomorphisms for manifolds of dimension $\leqslant 6$. Thus if M^n is combinatorially equivalent to $S^k \times S^p$, where $p + k \leqslant 6$, then M^n is diffeomorphic to $S^k \times S^p$.

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