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Locally Flat Imbeddings in the Metastable Range

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Introduction

With any imbedding f of a topological manifold M^n in the euclidean space E^q, there is associated the continuous map $\tilde{f}$ of $\tilde{M}=M\times M-\Delta_M$ in the unit sphere S^{q-1}, given by $\tilde{f}(x,y)=\dfrac{fx-fy}{\|fx-fy\|}$. The map $\tilde{f}$ is equivariant, that is, $\tilde{f}(x,y)=-\tilde{f}(y,x)$.

The following classification theorem has been proved by HAEFLIGER for differentiable manifolds and by WEBER for piecewise linear manifolds (in fact, for complexes). See [2], [6].

(1) Let $2q\geqslant 3(n+1)$ and suppose $F:\tilde{M}\to S^{q-1}$ is an equivariant map.

Then there is an imbedding $f:M\to E^q$ such that $\tilde{f}$ is homotopic through equivariant maps to F.

(2) Let $2q\geqslant 3(n+1)+1$ and suppose f and g are two imbeddings of M in E^q such that $\tilde{f}$ and $\tilde{g}$ are homotopic through equivariant maps.

Then f and g are isotopic, that is, there is a proper imbedding $H:M\times I\to E^q\times I$ commuting with projection on I such that $H(x,0)=(fx,0)$ and $H(x,1)=(gx,1)$.

In the differentiable and piecewise linear cases, the imbeddings and isotopies above are differentiable and piecewise linear, respectively.

The object of this paper is to prove a somewhat weaker form of the above theorem for topological manifolds and locally flat imbeddings. See Theorems 1 and 2. Here, $2q>3(n+1)$ is required in (1) and $2q>3(n+1)+1$ in (2); note, however, the remarks preceding Theorem 1. Further, the imbeddings f and g in the conclusion of (2) are concordant rather than isotopic, that is, the map H of (2) need not commute with projection on I.

The proofs of Theorems 1 and 2 depend on piecewise linear methods, and polyhedral imbedding theorems of CERNAVSKII and WEBER. Since topological manifolds are not assumed triangulable these methods are not applied directly, but are applied in coordinate neighborhoods which do have triangulations. The proofs then proceed by an induction on the number of coordinate neighborhoods.

1. Preliminary Results

Let M^n be a topological manifold. Unless otherwise specified M will be without boundary. A coordinate neighborhood in M is a pair (U,ψ) where U is an open set

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in M and $\psi: E^n \to U$ is a homeomorphism. The symbol $|\psi|_\alpha$ denotes the set $\psi\ C^n(\alpha)$ ($C^n(\alpha)$ is the cube $\{|x_i| \leqslant \alpha,\ i=1,\dots n\}$ in E^n) but also implies that ψ is the homeomorphism. If A is a subspace of a topological space, $\bar{A}$ will denote the closure of A, $\mathring{A}$ the interior of A and $\dot{A}$ or ∂A the boundary of A.

Let M^n, Q^q be topological manifolds, possibly with boundary. Let $f: M \to Q$ be a proper map, that is, $f\mathring{M} \subset \mathring{Q}$, $f\dot{M} \subset \dot{Q}$. f is said to be locally flat if for each $x \in M$ there are neighborhoods U of fx and V of x in Q and M respectively with f 1−1 on V, and a homeomorphism $h:(U, fV) \to (E^q, E^n)$ if $x \in \mathring{M}$, $h:(U, fV) \to (H^q, H^n)$ if $x \in \dot{M}$. Here H^n denotes the half-space $\{x_1 \geqslant 0\} \subset E^n$.

Lemma 1. *Let T be a triangulation of $C^n(1)$ and K a subcomplex of T. Let O be an open subset of the topological manifold M^n and suppose ψ is a coordinate homeomorphism such that $\psi K \subset O$. Let f be a locally flat imbedding of O in E^q. Then the map $f(\psi \mid K)$ is locally flat on every open simplex of K.*

Lemma 2. *Let M be a topological manifold and suppose ψ is a coordinate homeomorphism. Let f be a map of $|\psi|_1$ in E^q such that $f\psi$ is a piecewise linear imbedding of $C^n(1)$ in E^q, $q-n \geqslant 3$. Then $f \mid |\mathring{\psi}|_1$ is a locally flat imbedding of $|\mathring{\psi}|_1$ in E^q.*
Proof of Lemmas 1 and 2.

Proofs of these Lemmas are given in [3].

Theorem of Cernavskii [1]. Let K be a compact polyhedron of dimension n with $2q > 3(n+1)$ and let $f: K \to E^q$ be a topological imbedding of K in E^q. Let f be locally flat on every open simplex of some triangulation T of the polyhedron K. Then for any $\varepsilon > 0$ there exists an ε-homeomorphism $h: E^q \to E^q$ which is the identity outside an ε neighborhood of fK, where the homeomorphism $hf: K \to E^q$ is piecewise linear on the triangulation T.

If A is a topological space and Δ the diagonal in $A \times A$, let $\tilde{A}$ denote $A \times A - \Delta$. A map $F: \tilde{A} \to S^{q-1}$ will be called equivariant if it commutes with the natural involutions on $\tilde{A}$ and S^q, that is, $F(x, y) = -F(y, x)$ for $(x, y) \in \tilde{A}$. A homotopy through equivariant maps will be denoted by $\underset{\text{eq.}}{\simeq}$. If $f: A \to E^q$ is an imbedding, define $\tilde{f}: A \to S^{q-1}$ by $f(x, y) = \dfrac{fx - fy}{\|fx - fy\|}$.

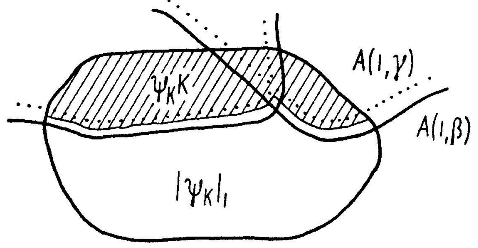
Theorem of Weber [6]. Let T^n be a finite complex, $2q \geqslant 3(n+1)$. Let $F: \tilde{T} \to S^{q-1}$ be an equivariant map. Let $K \subset T$ be a subcomplex and let $f: K \to E^q$ be an imbedding such that $\tilde{f} \underset{\text{eq.}}{\simeq} F \mid \tilde{K}$. Then there is an imbedding $f': T \to E^q$ such that $\tilde{f}' \underset{\text{eq.}}{\simeq} F$ and $f' \mid K = f$.

2. Locally Flat Imbeddings of Topological Manifolds

Let $|\psi_1|, \dots |\psi_k|$ be coordinate neighborhoods on a topological manifold M. If $\varrho > 0$, $1 \leqslant s \leqslant k$, let $A(s, \varrho) = |\mathring{\psi}_s|_\varrho \cup \dots \cup |\mathring{\psi}_{k-1}|_\varrho$ and $B(s, \varrho) = |\mathring{\psi}_s|_\varrho \cup \dots \cup |\mathring{\psi}_k|_\varrho$.

PROPOSITION 1. *Let M^n be a closed topological manifold, $2q>3(n+1)$, $n\geqslant 2$. Suppose $F:\tilde{M}\to S^{q-1}$ is an equivariant map. Let $\alpha<\beta\leqslant 1$, let $|\psi_1|,\ldots|\psi_k|$ be coordinate neighborhoods on M and suppose $f:A(1,\beta)\to E^q$ is a locally flat imbedding with $\tilde{f}\underset{\text{eq.}}{\simeq}F\mid\tilde{A}(1,\beta)$.*

Then there is a locally flat map $g:B(1,\gamma)\to E^q$ with $\alpha<\gamma<\beta$ and $g=f$ on $A(1,\gamma)$. Further, g is an imbedding of $|\psi_k|_\gamma$ in E^q with $\tilde{g}\mid|\tilde{\psi}_k|_\gamma\underset{\text{eq.}}{\simeq}F\mid|\tilde{\psi}_k|_\gamma$.



Proposition 1.

Proof of Proposition 1. Choose γ, $\alpha<\gamma<\beta$ and let T be a triangulation of $C(1)$ so fine that for any closed simplex σ of T, no $\psi_k\sigma$ meets both $|\psi_i|_\gamma$ and $|\psi_i|_\beta$ $i=1,\ldots(k-1)$.

Let $K=$ all $\sigma\in T$ such that $\psi_k\sigma$ does not meet $|\psi_k|_1-A(1,\beta)$. Note that $f(\psi_k\mid K)$ is an imbedding of K in E^q, locally flat on every open simplex of K by Lemma 1. By CERNAVSKII's Theorem, there is a homeomorphism h: $E^q\to E^q$ isotopic to the identity with $hf(\psi_k\mid K)$ a piecewise linear imbedding of K in E^q.

Now $\tilde{C}(1)\xrightarrow{\psi_k\times\psi_k}\tilde{M}\xrightarrow{F}S^{q-1}$ is an equivariant map.

Hence by WEBER's Theorem $hf(\psi_k\mid K)$ extends to a piecewise linear imbedding θ of $C(1)$ in E^q with $\tilde{\theta}\underset{\text{eq.}}{\simeq}F(\psi_k\times\psi_k)$.

Let g be defined by

$$f \text{ on } A(1,\gamma) \quad \text{and} \quad h^{-1}\theta\psi_k^{-1} \text{ on } |\psi_k|_\gamma .$$

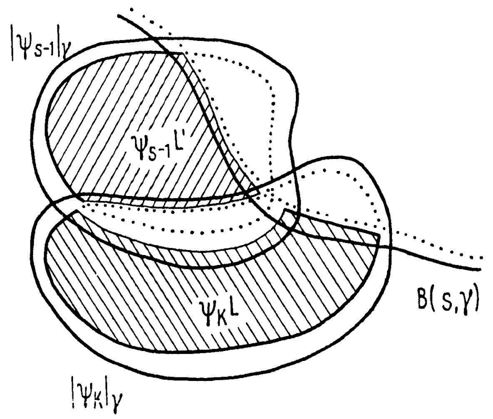
Then g is well defined. By Lemma 2, g is locally flat. Recall h is isotopic to the identity, thus

$$\tilde{g}\mid|\tilde{\psi}_k|_\gamma\underset{\text{eq.}}{\simeq}\widetilde{\theta\psi_k^{-1}}\mid|\tilde{\psi}_k|_\gamma\underset{\text{eq.}}{\simeq}F(\psi_k\times\psi_k)(\psi_k^{-1}\times\psi_k^{-1})\mid|\tilde{\psi}_k|_\gamma=F\mid|\tilde{\psi}_k|_\gamma .$$

This completes the proof of Proposition 1.

PROPOSITION 2. *Let M^n be a topological manifold, $2q>3(n+1)$, $n\geqslant 2$. Suppose $F:\tilde{M}\to S^{q-1}$ is an equivariant map. Let $\alpha<\gamma<1$, $1<s\leq k$ and suppose $g:B(1,\gamma)\to E^q$ is a locally flat map which is an imbedding on $A(1,\gamma)$ and $B(s,\gamma)$, with $F\mid\tilde{A}(1,\gamma)=\tilde{g}\mid\tilde{A}(1,\gamma)$ and $F\mid\tilde{B}(s,\gamma)\underset{\text{eq.}}{\simeq}\tilde{g}\mid\tilde{B}(s,\gamma)$. Then there are δ, g' with $\alpha<\delta<\gamma$ and g' a locally flat map of $B(1,\delta)$ in E^q with g' ambient isotopic to g on $A(1,\delta)$ and g' an imbedding of $B(s-1,\delta)$ in E^q such that $\tilde{g}'\mid\tilde{B}(s-1,\delta)\underset{\text{eq.}}{\simeq}F\mid\tilde{B}(s-1,\delta)$.*

We will prove Proposition 2 in section 4. Once this has been done, the main theorems of the paper follow; see Theorems 1 and 2 below. The strict inequality $2q>3(n+1)$ is required in order to apply CERNAVSKII's Theorem. However, R. H. BING has remarked that CERNAVSKII's result still holds when $2q\geqslant 3(n+1)$; see his mimeographed notes on "Radial Engulfing". If this is in fact the case, Propositions 1 and 2 and hence the main theorems below can be similarly improved.



Proposition 2.

THEOREM 1. *Let M^n be a closed topological manifold, $2q>3(n+1)$, $n\geqslant 2$. Suppose $F:\tilde{M}\to S^{q-1}$ is an equivariant map. Then there is a locally flat imbedding $f: M\to E^q$ with $\tilde{f}\underset{\text{eq.}}{\simeq} F$.*

Proof of Theorem 1. Let $|\psi_1|,\dots|\psi_r|$ be coordinate neighborhoods on M such that $M\subset|\psi_1|_\alpha\cup\cdots\cup|\psi_r|_\alpha$ with $0<\alpha<1$. Now $F\circ(\psi_1\times\psi_1)$ is an equivariant map of $\tilde{C}(1)$ in S^{q-1}. By WEBER's Theorem, there is a piecewise linear imbedding θ of $C(1)$ in E^q with $\tilde{\theta}\underset{\text{eq.}}{\simeq} F\circ(\psi_1\times\psi_1)$. Then by Lemma 2 $f_1=\theta\psi_1^{-1}$ is a locally flat imbedding of $|\psi_1|_1$ in E^q with $\tilde{f}_1\underset{\text{eq.}}{\simeq} F\,|\,|\tilde{\psi}_1|_1$.

Now apply an inductive argument based on Propositions 1 and 2. Suppose that β, f_{k-1} have been defined with $\alpha<\beta\leqslant 1$ and f_{k-1} a locally flat imbedding of $A(1,\beta)$ in E^q with $\tilde{f}_{k-1}\underset{\text{eq.}}{\simeq} F\,|\,\tilde{A}(1,\beta)$. By Proposition 1, f_{k-1} can be extended to a locally flat map $g:B(1,\gamma)\to E^q$ with $\alpha<\gamma<\beta$ such that $\tilde{g}\,|\,|\tilde{\psi}_k|_\gamma\underset{\text{eq.}}{\simeq} F\,|\,|\tilde{\psi}_k|_\gamma$. By repeated application of Proposition 2 with $s=k,\dots 2$, δ and f_k are obtained with $\alpha<\delta<\gamma$ and f_k a locally flat imbedding of $B(1,\delta)$ in E^q with $\tilde{f}_k\underset{\text{eq.}}{\simeq} F\,|\,\tilde{B}(1,\delta)$. This completes the inductive step. Thus there is a locally flat imbedding $f_r: M\to E^q$ with $\tilde{f}_r\underset{\text{eq.}}{\simeq} F$. The proof of Theorem 1 is completed by setting $f=f_r$.

LEMMA 3. *Let M be a topological manifold and let $h:\tilde{M}\times I\to S^{q-1}$ be an equivariant* *homotopy. Then there is an equivariant map H which makes the following diagram commutative for* $i=0, 1$.

$$\begin{array}{ccc} \widetilde{M \times \{i\}} & \xrightarrow{\widetilde{h\,|M \times \{i\}}} & S^{q-1} \\ \cap & & \cap \\ \widetilde{M \times I} & \xrightarrow{H} & S^q \end{array}$$

Proof of Lemma 3. Let

$$H(x, t, x', t') = \left(h\left(x, x', \frac{t+t'}{2}\right) \cos\frac{\pi}{2}(t-t'), \sin\frac{\pi}{2}(t-t')\right).$$

Since

$$\|H(x, t, x', t')\|^2 = \left\|h\left(x, x', \frac{t+t'}{2}\right)\right\|^2 \cos^2\frac{\pi}{2}(t-t') + \sin^2\frac{\pi}{2}(t-t')$$

$$= \cos^2\frac{\pi}{2}(t-t') + \sin^2\frac{\pi}{2}(t-t') = 1,$$

$H: \widetilde{M \times I} \to S^q \subset E^{q+1}$.

Since

$$H(x', t', x, t) = \left(h\left(x', x, \frac{t'+t}{2}\right) \cos\frac{\pi}{2}(t'-t), \sin\frac{\pi}{2}(t'-t)\right)$$

$$= \left(-h\left(x, x', \frac{t+t'}{2}\right) \cos\frac{\pi}{2}(t-t'), -\sin\frac{\pi}{2}(t-t')\right),$$

H is equivariant.

Note that $H(x, x', 0, 0) = (h(x, x', 0), 0)$ and $H(x, x', 1, 1) = (h(x, x', 1), 1)$. This completes the proof.

THEOREM 2. *Let M^n be a closed topological manifold, $2q > 3(n+1)+1$, $n \geq 2$. Suppose f and g are two locally flat imbeddings of M in E^q and that $\tilde{f} \underset{\text{eq.}}{\simeq} \tilde{g}: \tilde{M} \to S^{q-1}$. Then f and g are concordant, that is, there is a proper locally flat imbedding $h: M \times I \to E^q \times I$ with $h(x, 0) = (fx, 0)$ and $h(x, 1) = (gx, 1)$.*

Proof of Theorem 2. Let $\varepsilon > 0$, $0 < \alpha < 1$. Let $\ldots |\psi_{-m}|, \ldots |\psi_0|, |\psi_1|, \ldots |\psi_r|$ be a locally finite collection of coordinate neighborhoods on $M \times \mathring{I}$, such that $M \times \mathring{I} \subset \cdots \cup |\psi_{-m}|_\alpha \cup \cdots \cup |\psi_0|_\alpha \cup \cdots \cup |\psi_r|_\alpha$.

Further suppose that $M \times (0, \varepsilon] \cup [1-\varepsilon, 1) \subset \cdots \cup |\psi_{-m}|_\alpha \cup \cdots \cup |\psi_0|_\alpha$ and that $\cdots \cup |\psi_{-m}|_1 \cup \cdots \cup |\psi_0|_1 \subset M \times (0, 2\varepsilon] \cup [1-2\varepsilon, 1)$. By Lemma 3, since $\tilde{f} \underset{\text{eq.}}{\simeq} \tilde{g}: \tilde{M} \to S^{q-1} \subset S^q$, there is an equivariant map $F: \widetilde{M \times I} \to S^q$ with $F \mid \widetilde{M \times 0} = \tilde{f}$, and $F \mid \widetilde{M \times 1} = \tilde{g}$. Let $\Phi: \widetilde{M \times [0, 2\varepsilon]} \cup \widetilde{M \times [1-2\varepsilon, 1]} \to S^q$ be defined by $\tilde{f}$ on $\widetilde{M \times [0, 2\varepsilon]}$ and $\tilde{g}$ on $\widetilde{M \times [1-2\varepsilon, 1]}$. Now $F \mid \widetilde{M \times [0, 2\varepsilon]} \cup \widetilde{M \times [1-2\varepsilon, 1]} \underset{\text{eq.}}{\simeq} \Phi$ rel $\widetilde{M \times 0} \cup \widetilde{M \times 1}$ and by the homotopy extension property for $(\widetilde{M \times I}, \widetilde{M \times ([0, 2\varepsilon] \cup [1-2\varepsilon, 1])})$,

$F \underset{\text{eq.}}{\simeq} F'$ rel $\overbrace{M\times 0}\cup\overbrace{M\times 1}$ where $F' \mid \overbrace{M\times[0,2\varepsilon]}\cup\overbrace{M\times[1-2\varepsilon,1]}=\Phi$. Thus assume without loss of generality that $F\mid\overbrace{M\times[0,2\varepsilon]}=\tilde{f}$ and $F\mid\overbrace{M\times[1-2\varepsilon,1]}=\tilde{g}$. Let $h_0:\cdots\cup|\mathring{\psi}_{-m}|_1\cup\cdots\cup|\mathring{\psi}_0|_1\to E^q\times\mathring{I}$ be defined by the restriction of $f\times\text{id.}$ on $M\times(0,2\varepsilon]$, $g\times\text{id.}$ on $M\times[1-2\varepsilon,1)$. Then h_0, F' satisfy the hypotheses of Proposition 4. By repeated application of arguments used to prove Theorem 1, a locally flat imbedding h_r of $M\times\mathring{I}$ in $E^q\times\mathring{I}\approx E^{q+1}$ is obtained, with $h_r\mid M\times(0,\varepsilon]=f\times\text{id.}$ and $h_r\mid M\times[1-\varepsilon,1)=g\times\text{id.}$ Let $h:M\times I\to E^q\times I$ be defined by h_r on $M\times\mathring{I}$, $h(x,0)=(fx,0)$ and $h(x,1)=(gx,1)$. This completes the proof of Theorem 2.

3. Some Technical Results

We will first state some results from piecewise linear (p.l.) topology.

ZEEMAN's GENERAL POSITION THEOREM [8]. Let $X_0\subset X^n$, Y^k be polyhedra and Q^q a p.l. manifold with Y a subcomplex of Q.

Let $f:X\to Q$ be a p.l. imbedding such that $f(X-X_0)\subset\mathring{Q}$. Then there is an arbitrarily small ambient p.l. isotopy sending f into g keeping $\dot{Q}\cup fX_0$ fixed, such that $\dim g(X-X_0)\cap Y\leqslant n+k-q$.

If $\dot{Q}=\phi$ and $f(\overline{X-X_0})$ does not meet the closed set C, the isotopy can also be chosen fixed on C. $g\mid X-X_0$ is said to be in general position with respect to Y.

Suppose X is a subcomplex of a p.l. manifold Q^q, with respect to a given triangulation. Let $N(X,Q'')$ be the closed simplicial neighborhood of X in the barycentric 2nd derived of Q.

Theorem of WHITEHEAD [7]. If $X\searrow 0$ (X collapses to a point), then $N(X,Q'')$ is a q-ball.

ZEEMAN's Engulfing Theorem [8]. Let B be a p.l. q-ball and X^k a subcomplex of $\mathring{B}$. Then there exists a $k+1$ dimensional collapsible subspace $C\subset\mathring{B}$ with $X\subset C$.

Theorem of STALLINGS [5]. Let Q^q be a p.l. manifold without boundary, $q\geqslant 5$. Suppose Q is contractible and 1-connected at infinity. Then Q is p.l. homeomorphic to E^q.

LEMMA 4. *Let X be a polyhedron in E^q, $q\geqslant 5$. Suppose $X\subset B$ where B is an open subset of E^q, homeomorphic to E^q.*

Then $X\subset\mathring{B}'$ where $B'\subset B$ is a closed p.l. q-ball in E^q.

Proof of Lemma 4. As an open subset of E^q, B has a p.l. structure such that the inclusion $i:B\to E^q$ is a p.l. imbedding. Further, since B is homeomorphic to E^q, B is contractible and 1-connected at infinity. Hence by STALLING's Theorem, there is a p.l. homeomorphism $h:E^q\to B$.

Now $h^{-1}X$ is compact in E^q, hence is contained in the interior of a large standard closed q-ball Δ^q. Thus $X\subset h\mathring{\Delta}^q$, $h\Delta^q\subset B$. However, the inclusion of B in E^q is a p.l.

imbedding, hence $B' = ih\Delta^q$ is a closed p.l. q-ball p.l. imbedded in E^q satisfying the conclusion of the Lemma.

LEMMA 5. *Let X^k, Y^l be polyhedra with $l+k+1<q$, $2q>3(l+1)$, $q\geqslant 5$. Let B^q be a* p.l. *ball in E^q, with $X\subset \mathring{B}$.*

Suppose $f: Y\to E^q$ is a topological imbedding of Y in E^q, locally flat on every open simplex of Y, with $fY\cap X=\phi$.

Then there is a p.l. *ball $B_1^q\subset B\subset E^q$ with $X\subset \mathring{B}_1$ and $B_1\cap fY=\phi$.*

Proof of Lemma 5. *By* ZEEMAN's Engulfing Theorem, there is a collapsible C with $X\subset C\subset \mathring{B}$ and $\dim C\leqslant k+1$.

By CERNAVSKII's Theorem, there is a homeomorphism $h: E^q\to E^q$ fixed on X with $hf: Y\to E^q$ a p.l. imbedding, h so close to the identity that $h^{-1}C\subset \mathring{B}$.

By ZEEMAN's General Position Theorem, there is a homeomorphism $h': E^q\to E^q$ fixed on $hX=X$ with $h'hfY\cap C=\phi$, since $l+(k+1)<q$. h' can be chosen so close to the identity that $h^{-1}C\subset h\mathring{B}$. Write $g=h^{-1}h'^{-1}$ so that g is fixed on $X, fY\cap gC=\phi$, and $gC\subset \mathring{B}$.

Now let J be a triangulation of E^q with C as a subcomplex, so fine that if $N=N(C, J'')$, then $gN\subset \mathring{B}$ and $gN\cap fY=\phi$. By WHITEHEAD's Theorem, N is a p.l. q-ball. Note that $g^{-1}X=X\subset C\subset \mathring{N}$ so that $g\mathring{N}$ is an open topological q-ball in E^q containing X. By Lemma 4, there is a closed p.l. q-ball $B_1\subset E^q$ with $X\subset B_1$, $B_1\subset g\mathring{N}\subset B$. Note that $B_1\cap fY\subset gN\cap fY=\phi$. This concludes the proof of Lemma 5.

Let B^k, B^l, B^q be p.l. balls and let $\phi: B^k\cup B^l\to B^q$ be a proper p.l. map such that $\phi\mid B^k$ and $\phi\mid B^l$ are p.l. imbeddings, and $\phi\dot{B}^k\cap\phi\dot{B}^l=\phi$. Let $\tilde{\phi}:\partial(B^k\times B^l)\to S^{q-1}$ be defined by $\tilde{\phi}(x,y)=\dfrac{\phi x-\phi y}{\|\phi x-\phi y\|}$. Then an element $I(\phi\mid B^k, \phi\mid B^l)\,\varepsilon\,\pi_{k-1}(S^{q-l-1})$ is defined by WEBER such that the following propositions are true [6].

WEBER'S PROPOSITION 1. Let E denote the FREUDENTHAL suspension map. Then $(-1)^{q-l}E^l I(\phi\mid B^k, \phi\mid B^l)=$ the class of $\tilde{\phi}$ in $\pi_{k+l-1}(S^{q-1})$.

WEBER'S PROPOSITION 2. Suppose $I(\phi\mid B^k, \phi\mid B^l)=0$. By Proposition 1, $\tilde{\phi}\simeq 0$. Let Φ be an extension of $\tilde{\phi}$. Then if k, $l\leqslant n$ and $2q\geqslant 3(n+1)$ there is a proper isotopy $h_t: B^k\to B^q$ such that $h_0=\phi\mid B^k$, h_t is fixed on $\dot{B}^k$, $h_1B^k\cap\phi B^l=\phi$ and the map $\Phi_h:\partial(B^k\times B^l\times I)\to S^{q-1}$ defined as follows, is homotopic to 0.

$$\Phi_h\mid B^k\times B^l\times 0=\Phi$$

$$\Phi_h\mid \partial(B^k\times B^l)\times t=\frac{\phi x-h_t y}{\|\phi x-h_t y\|}$$

$$\Phi_h\mid B^k\times B^l\times 1=\frac{\phi x-h_1 y}{\|\phi x-h_1 y\|}.$$

4. Further Technical Results and the Proof of Proposition 2.

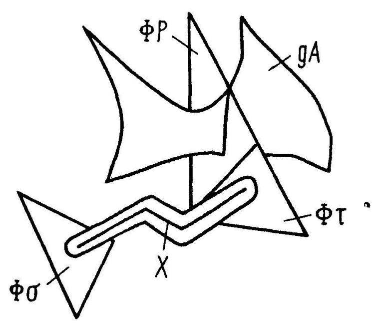
We must now separate the images of the double points of a locally flat map of the open manifold $B(1, \gamma)$ in E^q. We first establish the technical Proposition 3 and its corollary, which are based on the results of WEBER. Recall that in [6], WEBER distinguishes between "near" and "far" double points of a map of a given complex in E^q.

In the present case, the map of $B(1, \gamma)$ in E^q is locally an imbedding, thus the double points of this map are "far" points. Here, however, when we pull apart the images of these "far" points, we must not meet the images of other coordinate neighborhoods, which are only topologically imbedded.

As in [6], this is done by a succession of moves, each of which is the identity outside a ball in E^q. The effect of a move is to pull apart the images of two simplexes which lie in different coordinate neighborhoods; the corresponding ball is constructed so as to meet neither the images of simplexes which have already been pulled apart, nor the images of other topologically imbedded coordinate neighborhoods.

Suppose δ, γ and g are given with $0<\delta<\gamma$ and $g: B(s, \gamma)\to E^q$ a locally flat imbedding. Let H be a subcomplex of $C(\gamma)$, and let ψ be a coordinate neighborhood on M with $\psi H \subset B(s, \gamma)$. Write $A=\bar{A}(s, \delta)\cup \psi H$.

PROPOSITION 3. Let K^n be a finite complex whose simplexes are ordered so that a simplex follows all its faces. Let σ be the last simplex of K, suppose $St(\sigma, K)$ is full in K and let $K=L\cup\sigma$.



Proposition 3.

Let $\Phi: \tilde{K}\to S^{q-1}$ be an equivariant map and suppose there is a p.l. map $\phi: K\to E^q$ such that

1. $\phi \mid L$ is an imbedding.
2. $\phi \mid St(\sigma, K)$ is an imbedding.

Let $2q>3(n+1)$, $n\geqslant 2$.

(A) Let $\tau\in L-St(\sigma, K)$, $P=$ all simplexes of L preceding τ.
Suppose $\phi\sigma\cap[\phi P\cup gA]=\phi$ and $\phi\mathring{\tau}\cap gA=\phi$. Let $J=\tilde{L}\cup(\sigma\times P)\cup(P\times\sigma)$ and

suppose $\tilde{\phi}: J \to S^{q-1}$ is given by $\tilde{\phi}(x, y) = \frac{\phi x - \phi y}{\|\phi x - \phi y\|}$ with $\tilde{\phi} = \Phi \mid J$. Then there is an ambient p.l. isotopy ϕ_t of $\phi \mid St(\sigma, K)$ fixed outside $\mathring{\sigma}$ with $\phi_t \sigma \cap [\phi P \cup g A] = \phi$ and $\phi_1 \sigma \cap \phi \tau = \phi$. Let $\phi': K \to E^q$ be given by ϕ on L, ϕ_1 on σ.

If $J' = \tilde{L} \cup (\sigma \times (P \cup \tau)) \cup ((P \cup \tau) \times \sigma)$ and $\tilde{\phi}': J' \to S^{q-1}$ is given by $\tilde{\phi}'(x, y) = \frac{\phi' x - \phi' y}{\|\phi' x - \phi' y\|}$ then there is an equivariant homotopy $h_t: J' \to S^{q-1}$ with $h_0 = \Phi \mid J'$, $h_1 = \tilde{\phi}'$, h_t fixed on $\tilde{L}$ and for $(x, y) \in (P \cup \tau) \times \sigma$, $h_t(x, y) = \frac{\phi x - \phi_t y}{\|\phi x - \phi_t y\|}$.

(B) Let K_0 be a subcomplex of K and suppose $\phi \sigma \cap g A = \phi$, $\phi \mathring{\sigma} \cap \phi K_0 = \phi$ and $\phi L \cap g A \subset \phi K_0$.

Let $J = \tilde{L} \cup (\sigma \times K_0) \cup (K_0 \times \sigma)$ and let $\tilde{\phi}: J \to S^{q-1}$ be given by $\tilde{\phi}(x, y) = \frac{\phi x - \phi y}{\|\phi x - \phi y\|}$ with $\tilde{\phi} = \Phi \mid J$.

Then there is an ambient p.l. isotopy ϕ_t of $\phi \mid St(\sigma, K)$ fixed outside $\mathring{\sigma}$ with $\phi_t \sigma \cap g A = \phi$, $\phi_t \mathring{\sigma} \cap \phi K_0 = \phi$ and $\phi_1 \sigma \cap \phi L = \phi_1 \dot{\sigma}$. Thus $\phi': K \to E^q$ given by ϕ on L and ϕ_1 on σ is an imbedding.

If $\tilde{\phi}': \tilde{K} \to S^{q-1}$ is given by $\tilde{\phi}'(x, y) = \frac{\phi' x - \phi' y}{\|\phi' x - \phi' y\|}$, then there is an equivariant homotopy $h_t: \tilde{K} \to S^{q-1}$ with $h_0 = \Phi$, $h_1 = \tilde{\phi}'$, h_t fixed on $\tilde{L}$ and $h_t(x, y) = \frac{\phi x - \phi_t y}{\|\phi x - \phi_t y\|}$ on $K_0 \times \sigma$.

Proof of Proposition 3. Proposition 3. (B) follows from Proposition 3. (A) by an induction on the simplexes τ of $L - (K_0 \cup St(\sigma, K))$. The following is a proof of Proposition 3. (A).

3.1. There exists a p.l. ball $B^q \subset E^q$ which meets $\phi \sigma$, $\phi \tau$ in p.l. balls of dimension $\dim \sigma$, $\dim \tau$ respectively, properly imbedded in B and such that

$$\phi \sigma \cap \phi \tau \subset \mathring{B}, \; B \cap [\phi P \cup \phi(St(\sigma, K) - \mathring{\sigma}) \cup g A] = \phi.$$

Proof of 3.1. Note that the hypothesis of Proposition 3 implies $\phi \sigma \cap \phi \tau \subset \phi \mathring{\sigma} \cap \phi \mathring{\tau}$. By ZEEMAN's General Position, there is a p.l. homeomorphism $k: E^q \to E^q$ isotopic to the identity fixed on $\phi(P \cup (St(\sigma, K) - \mathring{\sigma}))$ such that $\dim k\phi \sigma \cap \phi \tau \leqslant \dim \sigma + \dim \tau - q \leqslant 2n - q$, $k\phi \sigma \cap \phi \tau \subset k\phi \mathring{\sigma} \cap \phi \mathring{\tau}$ and k so close to the identity that $k\phi \sigma \cap g A = \phi$. Without loss of generality, replace ϕ by the map ϕ on L, $k\phi$ on σ. Thus assume $\dim \phi \sigma \cap \phi \tau \leqslant 2n - q$.

Let $S = \sigma \cap \phi^{-1} \phi \tau$, $S' = \tau \cap \phi^{-1} \phi \sigma$ so $\dim S \cup S' \leqslant 2n - q$. By ZEEMAN's Engulfing Theorem, there are Q, Q' with $S \subset Q \subset \mathring{\sigma}$, $S' \subset Q' \subset \mathring{\tau}$, $Q \searrow 0$, $Q' \searrow 0$ and $\dim Q \cup Q' \leqslant 2n - q + 1$. Let $X = \phi(Q \cup Q')$ so $\dim X \leqslant 2n - q + 1$, and let B_k^q be a p.l. q-ball in E^q with $X \subset \mathring{B}_k$.

Now by Lemma 1, $g(\psi_{k-1} \mid C(\delta))$ is a topological imbedding of $C(\delta)$ in E^q, locally flat on every open simplex of $C(\delta)$. If $s \leqslant k-1$, $X \cap g\psi_{k-1} C(\delta) = \phi$. Also $2q > 3(n+1)$ and so $(2n-q+1)+n+1<q$. Thus by Lemma 5, there is a p.l. q-ball $B^q_{k-1} \subset E^q$ with $B_{k-1} \subset B_k$, $X \subset \mathring{B}_{k-1}$ and $B_{k-1} \cap g\psi_{k-1} C(\delta) = \phi$. If $s \leqslant k-2$, another application of Lemma 5 gives a p.l. q-ball $B^q_{k-2} \subset B_{k-1}$ with $X \subset \mathring{B}_{k-2}$, and $B_{k-2} \cap g\psi_{k-2} C(\delta) = \phi$. Continuing in this fashion, a sequence of p.l. q-balls $B_s \subset \cdots \subset B_{k-2} \subset B_{k-1}$ is obtained with $X \subset \mathring{B}_s$ and $B_s \cap g\bar{A}(s, \delta) = \phi$.

Again by Lemma 1, $g(\psi \mid H)$ is a topological imbedding of H in E^q, locally flat on every open simplex of H. Also $\phi Q \cap g\psi H = \phi$ and $\phi Q' \cap g\psi H = \phi$, hence $X \cap g\psi H = \phi$. Now apply Lemma 5 to find a p.l. q-ball $B^q_{s-1} \subset B_s$ with $X \subset \mathring{B}_{s-1}$ and $B_{s-1} \cap g\psi H = \phi$. Note $B_{s-1} \cap gA = \phi$.

Recall $\dim X \leqslant 2n-q+1$. Hence by ZEEMAN's Engulfing Theorem, there is a Q'', $X \subset Q'' \subset \mathring{B}_{s-1}$ with $Q'' \searrow 0$ and $\dim Q'' \leqslant 2n-q+2$. By ZEEMAN's General Position, there is a homeomorphism $h: E^q \to E^q$ of Q'' isotopic to the identity fixed on X such that hQ'' meets ϕK precisely in X. This is possible since $\dim Q'' + \dim K \leqslant 2n-q+2+n<q$. Further, h can be chosen so close to the identity that $hQ'' \subset \mathring{B}_{s-1}$. Thus assume without loss of generality that $Q'' \subset \mathring{B}_{s-1}$ and $Q'' \cap \phi K = X$.

Now let σ, τ and B_{s-1} be triangulated so that $\phi \mid \sigma \cup \tau$ is simplicial and so that if $B_\sigma = N(Q, \sigma'')$, $B_\tau = N(Q', \tau'')$ and $B = N(Q'', B''_{s-1})$, $B_\sigma \cap \dot{\sigma} = \phi$, $B_\tau \cap \dot{\tau} = \phi$ and $B \cap \phi(St(\sigma, K) - \mathring{\sigma}) = \phi$. Then B_σ, B_τ and B are proper p.l. balls and $\phi \mid B_\sigma$, $\phi \mid B_\tau$ are proper p.l. imbeddings of B_σ, B_τ in B. Since $B \subset B_{s-1}$, $B \cap gA = \phi$ as required. This completes the proof of 3.1.

Notice that $\phi\sigma \cap \phi\tau \subset \phi \mathring{B}_\sigma \cap \phi \mathring{B}_\tau$, and so $\tilde{\phi}: \sigma \times \tau - \mathring{B}_\sigma \times \mathring{B}_\tau \to S^{q-1}$ is well defined by $\tilde{\phi}(x, y) = \dfrac{\phi x - \phi y}{\|\phi x - \phi y\|}$.

3.2 Without loss of generality, assume $\tilde{\phi} = \Phi$ on $\sigma \times \tau - \mathring{B}_\sigma \times \mathring{B}_\tau$. For, $\Phi = \tilde{\phi}$ on $\partial(\sigma \times \tau)$ and by the annulus theorem of regular neighborhood theory [4], $\sigma \times \tau - \mathring{B}_\sigma \times \mathring{B}_\tau \approx \partial(\sigma \times \tau) \times I$ ($\approx$ denotes piecewise linear equivalence). Hence $\Phi \simeq \tilde{\phi}$ (Φ is homotopic to $\tilde{\phi}$) on $\sigma \times \tau - \mathring{B}_\sigma \times \mathring{B}_\tau$ rel $\partial(\sigma \times \tau)$. Now the homotopy extension property for $(\sigma \times \tau, \sigma \times \tau - \mathring{B}_\sigma \times \mathring{B}_\tau)$ shows that $\Phi \simeq \Phi'$ on $\sigma \times \tau$ rel $\partial(\sigma \times \tau)$ where $\Phi' \mid \sigma \times \tau - \mathring{B}_\sigma \times \mathring{B}_\tau = \tilde{\phi} \mid \sigma \times \tau - \mathring{B}_\sigma \times \mathring{B}_\tau$. Thus $\tilde{\phi} \mid \partial(B_\sigma \times B_\tau)$ extends over $B_\sigma \times B_\tau$, hence $\tilde{\phi} \mid \partial(B_\sigma \times B_\tau) \simeq 0$.

Now according to Proposition 1 of WEBER, $E^{\dim \tau} I(\phi \mid B_\sigma, \phi \mid B_\tau) = 0$, where $I(\phi \mid B_\sigma, \phi \mid B_\tau) \in \pi_{\dim \sigma - 1}(S^{q - \dim \tau - 1})$. By the classical suspension theorems, $E^{\dim \tau}$ is injective since $\dim \sigma - 1 \leqslant 2(q - \dim \tau - 1) - 2$. (Recall $2q > 3(n+1)$.) Hence $I(\phi \mid B_\sigma, \phi \mid B_\tau) = \phi$. Thus by WEBER's Proposition 2 there is a proper ambient isotopy $k_t: B_\sigma \to B$ such that $k_0 = \phi \mid B_\sigma$, k_t is fixed on $\dot{B}_\sigma$, $k_1 B_\sigma \cap B_\tau = \phi$ and the map $\Phi_h: \partial(B_\sigma \times B_\tau \times I) \to S^{q-1}$ defined as follows, is homotopic to 0.

$$\Phi_h \mid B_\sigma \times B_\tau \times 0 = \Phi$$

$$\Phi_h \mid \partial(B_\sigma \times B_\tau) \times t = \frac{\phi x - k_t y}{\|\phi x - k_t y\|}$$

$$\Phi_h \mid B_\sigma \times B_\tau \times 1 = \frac{\phi x - k_1 y}{\|\phi x - k_1 y\|}.$$

Extend k_t to an ambient isotopy ϕ_t of $St(\sigma, K)$ in E^q fixed outside $\mathring{\sigma}$ by $\phi_t = \phi$ outside B_σ. Further, the given extension of Φ_h to $B_\sigma \times B_\tau \times I$ extends to a homotopy $h_t : J' \to S^{q-1}$ with the required properties. This completes the proof of Proposition 3.

COROLLARY OF PROPOSITION 3. Let L^n, T^n be disjoint finite complexes ordered so that a simplex follows all its faces and let $\phi : L \cup T \to E^q$ be a p.l. map. Suppose $\phi \mid L$ and $\phi \mid T$ are imbeddings, $2q > 3(n+1)$ and $n \geq 2$.

Let $K \subset T$, $K' \subset L$ be subcomplexes and suppose $\phi L \cap \phi K = \phi$, $\phi L \cap gA \subset \phi K'$, $\phi T \cap \phi K' = \phi$ and no simplex of $T - K$ meets $\phi^{-1} gA$.

Let $\Phi : \overline{L \cup T} \to S^{q-1}$ be an equivariant map and suppose $\tilde{\phi} : \overline{T \cup K'} \to S^{q-1}$ is given by $\tilde{\phi}(x, y) = \dfrac{\phi x - \phi y}{\|\phi x - \phi y\|}$ with $\tilde{\phi} = \Phi \mid \overline{T \cup K'}$.

Then there is an ambient isotopy ϕ_t of $\phi \mid T$ fixed on K with $\phi_t T \cap \phi K' = \phi$, $\phi_t T \cap gA \subset \phi K$ and $\phi_1 T \cap \phi L = \phi$. If $\phi' : L \cup T \to E^q$ is the imbedding given by ϕ on L and ϕ_1 on T, let $\tilde{\phi}' : \overline{L \cup T} \to S^{q-1}$ be given by $\tilde{\phi}'(x, y) = \dfrac{\phi' x - \phi' y}{\|\phi' x - \phi' y\|}$. Then $\Phi \underset{\text{eq.}}{\simeq} \tilde{\phi}'$ by an equivariant homotopy h_t fixed on $\overline{L \cup K}$ with $h_t(x, y) = \dfrac{\phi x - \phi_t y}{\|\phi x - \phi_t y\|}$ on $K' \times T$.

Proof of the Corollary. The Corollary follows from Proposition 3. (B) by induction on the simplexes σ of $T - K$.

We are now ready to prove Proposition 2, which is restated here for convenience.

PROPOSITION 2. Let M^n be a topological manifold, $2q > 3(n+1)$, $n \geq 2$. Suppose $F : \tilde{M} \to S^{q-1}$ is an equivariant map. Let $\alpha < \gamma < 1$, $1 < s \leqslant k$ and suppose $g : B(1, \gamma) \to E^q$ is a locally flat map which is an imbedding on $A(1, \gamma)$ and $B(s, \gamma)$, with $F \mid \tilde{A}(1, \gamma) = \tilde{g} \mid \tilde{A}(1, \gamma)$ and $F \mid \tilde{B}(s, \gamma) \underset{\text{eq.}}{\simeq} \tilde{g} \mid \tilde{B}(s, \gamma)$. Then there are δ, g' with $\alpha < \delta < \gamma$ and g' a locally flat map of $B(1, \delta)$ in E^q with g' ambient isotopic to g on $A(1, \delta)$ and g' an imbedding of $B(s-1, \delta)$ in E^q such that $\tilde{g}' \mid \tilde{B}(s-1, \delta) \underset{\text{eq.}}{\simeq} F \mid \tilde{B}(s-1, \delta)$.

Proof of Proposition 2. Choose δ, $\alpha < \delta < \gamma$ and suppose T is a triangulation of $C(\delta)$ so fine that for each closed simplex σ of T

(1) No $\psi_k \sigma$ meets both $|\psi_i|_\delta$ and $|\psi_i|\gamma$ $\quad i = 1, \ldots (k-1)$.

Let K=all $\sigma\in T$ such that $\psi_k\sigma$ does not meet $|\psi_k|_\delta - A(1,\gamma)$. Let L=all $\sigma\in T$ such that $\psi_k\sigma$ does not meet $\bar{A}(1,\delta)$. Note $T=K\cup L$ and $g\psi_k L\cap g\bar{A}(s,\delta)=\phi$.

Now suppose T' is a triangulation of $C(\delta)$ so fine that for each closed simplex σ of T'

(2) No $\psi_{s-1}\sigma$ meets both $|\psi_j|_\delta$ and $|\psi_j|_\gamma$ $j=s,\dots k$.

Let H=all $\sigma\in T'$ such that $\psi_{s-1}\sigma$ does not meet $|\psi_{s-1}|_\delta - B(s,\gamma)$. Let L'=all $\sigma\in T'$ such that $\psi_{s-1}\sigma$ does not meet $\bar{B}(s,\delta)$. Note $T'=H\cup L'$.

2.1. Without loss of generality, assume that $g(\psi_k\,|\,T)$ and $g(\psi_{s-1}\,|\,L')$ are p.l. imbeddings of T, L' in E^q.

Proof of 2.1. Notice that Lemma 1 implies that $g(\psi_k\,|\,T)$ is a topological imbedding of T in E^q, locally flat on every open simplex of T. Hence by CERNAVSKII's Theorem there is a homeomorphism $h:E^q\to E^q$ isotopic to the identity such that $hg(\psi_k\,|\,T)$ is a p.l. imbedding of T in E^q. Thus, without loss of generality, relabel so that $g(\psi_k\,|\,T)$ is a p.l. imbedding of T in E^q. Observe $g\psi_k L\cap g\bar{A}(s,\delta)$ is still empty.

Again by Lemma 1, $g(\psi_{s-1}\,|\,L')$ is a topological imbedding of L' in E^q, locally flat on every open simplex of L'. Choose δ^+, $\delta<\delta^+<\gamma$ so that $\bar{A}(1,\delta^+)$ does not meet $\psi_k L$. By CERNAVSKII's Theorem there is a homeomorphism $h:E^q\to E^q$ isotopic to the identity by an arbitrarily small isotopy fixed outside a neighborhood of $g\psi_{s-1}L'$, in particular fixed on $g(A(1,\gamma)-\bar{A}(1,\delta^+))\cup g\psi_k K$, with $hg\psi_{s-1}\,|\,L'$ a p.l. imbedding.

Define $g_1:B(1,\gamma)\to E^q$ by $g_1=hg$ on $A(1,\gamma)$ and $g_1=g$ on $|\psi_k|_\gamma-\bar{A}(1,\delta^+)$. Now g_1 is locally flat and $g_1=g$ on $\psi_k T$. Thus $g_1(\psi_{s-1}\,|\,L')$ and $g_1(\psi_k\,|\,T)$ are p.l. imbeddings of L' and T in E^q, and h can be chosen so close to the identity that $g_1\psi_k L\cap g_1\bar{A}(s,\delta)$ is still empty. Thus by relabeling, assume $g(\psi_{s-1}\,|\,L')$ and $g(\psi_k\,|\,T)$ are p.l. imbeddings of L' and T in E^q with $g\psi_k L\cap g\bar{A}(s,\delta)=\phi$. This completes the proof of 2.1.

2.2. Order the simplexes of T and L' so that each simplex follows all its faces and let $\phi:T\cup L'\to E^q$ be given by $g\psi_{s-1}$ on L' and $g\psi_k$ on T. Let $K'=H\cap L'$ and let $A=\bar{A}(s,\delta)\cup\psi_{s-1}H$. Then $\phi L'\cap\phi K=\phi$, $\phi L'\cap gA\subset\phi K'$, $\phi T\cap\phi K'=\phi$ and for any simplex σ of $T-K$, $\phi\sigma$ does not meet gA.

Let $\psi:L'\cup T\to M^n$ be given by ψ_{s-1} on L' and ψ_k on T, and let $\Phi':\widetilde{L'\cup T}\to S^{q-1}$ be defined by $F(\psi\times\psi)$. Since $F\,|\,\tilde{B}(s,\gamma)\underset{\text{eq.}}{\simeq}\tilde{g}\,|\,\tilde{B}(s,\gamma)$ and $\psi(T\cup K')\subset B(s,\gamma)$, $\Phi'\,|\,\widetilde{T\cup K'}\underset{\text{eq.}}{\simeq}\tilde{\phi}\,|\,\widetilde{T\cup K'}$. By the homotopy extension property for $(\widetilde{T\cup L'},\widetilde{T\cup K'})$, $\Phi'\underset{\text{eq.}}{\simeq}\Phi$ where $\Phi:\widetilde{L'\cup T}\to S^{q-1}$ is an equivariant map with $\Phi=\tilde{\phi}$ on $\widetilde{T\cup K'}$.

Now by the Corollary of Proposition 3 there is an ambient isotopy ϕ_t of $\phi\,|\,T$ fixed on K with $\phi_t T\cap\phi K'=\phi$, $\phi_t T\cap gA\subset\phi K$ and $\phi_1 T\cap\phi L'=\phi$. Further, if $\phi':L'\cup T\to E^q$ is the imbedding given by ϕ on L' and ϕ_1 on T and $\tilde{\phi}':\widetilde{L'\cup T}\to S^{q-1}$ is defined by $\tilde{\phi}'(x,y)=\dfrac{\phi'x-\phi'y}{\|\phi'x-\phi'y\|}$ then $\Phi\underset{\text{eq.}}{\simeq}\tilde{\phi}'$ rel $\widetilde{L'\cup K}$ with the homotopy of the form $h_t(x, y) = \dfrac{\phi x - \phi_t y}{\|\phi x - \phi_t y\|}$ on $K' \times T$. Now define $g': \bar{B}(1, \delta) \to E^q$ by

$$g \text{ on } \bar{A}(1, \delta) \quad \text{and by} \quad \phi' \psi_k^{-1} \text{ on } |\psi_k|_\delta.$$

Then g' is well defined and is locally flat on $B(1, \delta)$.

2.3. g' is an imbedding on $B(s-1, \delta)$.

Proof of 2.3. g' has no double points on $\psi_k K$. For,

$$g' \psi_k \mid K = \phi' \psi_k^{-1} \psi_k \mid K = \phi_1 \mid K = \phi \mid K = g \psi_k \mid K.$$

Now if $g\psi_k x = g\psi_j y$ with $j<k$, $x \in K$, note $\psi_k x = \psi_l z$ with $l<k$. Then $g\psi_l z = g\psi_j y$ implies $\psi_l z = \psi_j y$, $\psi_k x = \psi_j y$.

Now g' has no double points on $\bar{A}(s, \delta)$. For by the above remark, such a double point x would have a correspondent $x' \in \psi_k T - K \subset \psi_k L$. However, $g\psi_k L \cap g\bar{A}(s, \delta) = \phi$ as remarked in 2.1. Note also that $g\psi_k L \cap g\psi_{s-1} H = \phi$ since g is an imbedding on $B(s, \gamma)$. Thus $\phi L \cap gA = \phi$.

Since ϕ_1 imbeds T, $\phi_1 T \cap gA \subset \phi K$ and $\phi_1 = \phi$ on K, $\phi_1 L \cap gA \subset \phi_1(K \cap L) = \phi(K \cap L)$ which does not meet gA. Thus $(\phi_1 L \cap gA) \cap gA = \phi$ and so $\phi_1 L \cap gA = \phi$. In particular, $g' \psi_k L \cap g' \bar{A}(s, \delta) = \phi$.

Further, g' has no double points on $|\psi_{s-1}|_\delta$. Again, such a double point x would have a correspondent $x' \in \psi_k L$. However, $\phi' L \cap \phi L' = \phi$ and by the above remarks $\phi' L \cap \phi H = \phi$, thus $g' \psi_k L \cap g\psi_{s-1} T' = \phi$.

Therefore g' has no double points on $A(s-1, \delta)$. Hence g' is an imbedding on $B(s-1, \delta)$.

2.4. $\tilde{g}' \mid \tilde{B}(s-1, \delta) \underset{\text{eq.}}{\simeq} F \mid \tilde{B}(s-1, \delta)$.

Proof of 2.4. Note that $\bar{B}(s-1, \delta) = \bar{A}(s-1, \delta) \cup |\psi_k|_\delta$, thus

$$\tilde{\bar{B}}(s-1, \delta) = \tilde{\bar{A}}(s-1, \delta) \cup [(|\psi_k|_\delta \times \bar{B}(s-1, \delta) \cup \bar{B}(s-1, \delta) \times |\psi_k|_\delta) - \Delta].$$

Let $g_t: \bar{B}(s, \delta) \to E^q$ be the ambient isotopy defined by $g_t = g$ on $\bar{A}(s, \delta)$ and $g_t = \phi_t \psi_k^{-1}$ on $|\psi_k|_\delta$.

Define an equivariant homotopy H_t: $\tilde{\bar{B}}(s-1, \delta) \to S^{q-1}$ as follows:

(a) $H_t(x, y) = \tilde{g}(x, y)$ for $(x, y) \in \tilde{\bar{A}}(s-1, \delta)$

(b) $H_t(x, y) = \dfrac{g_t x - g_t y}{\|g_t x - g_t y\|}$ for $(x, y) \in (|\psi_k|_\delta \times \bar{B}(s, \delta)) - \Delta$

(c) $H_t(x, y) = \dfrac{g_t x - g y}{\|g_t x - g y\|}$ for $(x, y) \in (|\psi_k|_\delta \times \psi_{s-1} H) - \Delta$

(d) $H_t(x, y) = h_t(\psi_k^{-1} x, \psi_{s-1}^{-1} y)$ for $(x, y) \in (|\psi_k|_\delta \times \psi_{s-1} L') - \Delta$

Now (b), (c) agree, since $g_t = g$ on $\bar{B}(s, \delta) \cap \psi_{s-1} H$. (c), (d) agree, since if $(u, v) \in T \times (H \cap L') = T \times K'$,

$$\frac{g_t \psi_k u - g \psi_{s-1} v}{\|g_t \psi_k u - g \psi_{s-1} v\|} = \frac{\phi_t u - \phi v}{\|\phi_t u - \phi v\|} = h_t(u, v)$$

and (b), (d) agree, since $\psi_{s-1} L \cap \bar{B}(s, \delta) = \phi$. Note $\bar{A}(s-1, \delta) \cap |\psi_k|_\delta \subset \psi_k K$ and $g_t = g$ on $\psi_k K$. Thus (a), (b) and (a), (c) agree. Also, $h_t = \tilde{\phi}' = \tilde{\phi}$ on $K \times L'$ so (a), (d) agree. Thus H_t is well defined.

Observe $H_0 = \tilde{g}$ on (a), (b), (c) and $H_0 = \Phi(\psi^{-1} \times \psi^{-1})$ on (d). Recall that the homotopy $F \mid \tilde{B}(s, \gamma) \underset{\text{eq.}}{\simeq} \tilde{g} \mid \tilde{B}(s, \gamma)$ was extended to obtain a homotopy of F and $\Phi(\psi^{-1} \times \psi^{-1})$. Note that $\bar{B}(s-1, \delta) \subset \tilde{B}(s, \gamma) \cup \psi_{s-1} L'$. Thus $F \mid \tilde{\bar{B}}(s-1, \delta) \underset{\text{eq.}}{\simeq} H_0$. Also, $\tilde{g}' = H_1$. Thus $F \mid \tilde{B}(s-1, \delta) \underset{\text{eq.}}{\simeq} \tilde{g}' \mid \tilde{B}(s-1, \delta)$. This completes the proof of 2.4 and of Proposition 2.

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