On the Classification Problem for H-Spaces of Rank Two.

On the Classification Problem for H-Spaces of Rank Two

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§ 1. Introduction

We shall consider finite H-spaces of rank two. By definition, such a space X is an H-space of the homotopy type of a finite complex whose (reduced) rational cohomology $H^*(X; Q)$ is generated (as a ring) by two elements $a \in H^q(X; Q)$, $b \in H^n(X; Q)$, $q \leqslant n$. A classical theorem of Hopf asserts that q and n are odd, and that a and b are primitive classes. The pair (q, n) is called the type of X.

One aspect of the classification problem is to determine which pairs (q, n) of odd integers can be realized as the type of some finite H-space. This problem was considered and partially solved by Adams [1]. A complete solution in the case that X has no homological 2-torsion has been given recently by Douglas-Sigrist [6] and Hubbuck [8], in independent work. The result is the following:

1.1 THEOREM. *Let X be a finite H-space of rank* 2 *such that $H_*(X; Z)$ has no* 2-*torsion. Then the type of X is either* (1, 1), (1, 3), (1, 7), (3, 3), (3, 5), (3, 7) *or* (7, 7).

Moreover, each of these pairs does actually occur as a type, as can easily be seen.

In the present paper, using the above information, we shall consider the problem of enumerating all the homotopy types which occur as rank 2 H-spaces. Actually, for the most part, we restrict our attention to rank 2 H-spaces which have torsion-free integral homology. For such H-spaces, we give a complete enumeration of homotopy types, except for a single ambiguity in the case $(q, n)=(3, 7)$. For example, we show that any homologically torsion-free H-space of type (3, 5) is homotopy equivalent to $SU(3)$. We also point out here that our results yield the qualitative fact that there are only a finite number of possible homotopy types for this restricted class of rank 2 H-spaces. In fact, this number is 11 or 13.

The rest of the paper is organized as follows. In § 2, we state and prove our main results on the homotopy classification of homologically torsion-free rank 2 H-spaces. It turns out that the only homotopy types occurring are those of manifolds which are principal sphere bundles over spheres. In § 3, we sharpen the classification of § 2 by showing that, within the class of homologically torsion-free rank 2 H-manifolds (with one possible exception), the concepts of homotopy equivalence and PL-equivalence coincide, i.e. the Hurewicz conjecture for such manifolds is valid. In § 4, we make some remarks about general rank 2 H-spaces. In particular, we show how certain results of

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Browder enable us to reduce the 2-torsion-free, simply-connected case, modulo a certain hypothesis, to the case considered in § 2. We also discuss the exceptional Lie group G_2 in § 4. Finally, in § 5, we discuss the non-simply-connected case and make some remarks about projective groups, of a somewhat speculative nature.

We mention here that Browder [4] has given a complete homotopy enumeration of the rank 1 H-spaces: they are simply S^1, S^3, S^7, P^3 and P^7.

In an appendix we show that the methods used in proving the main results (Theorems 2.9 and 2.10) actually yield a classification of a much wider class of manifolds.

There is an overlap between the results of § 2 and independent unpublished work by Curtis, Mislin and Thomas, and by Zabrodsky.[1)]

§ 2. Homologically Torsion-free Rank Two H-spaces

We first dispose of the cases $(q, n)=(1, 1), (1, 3), (1, 7)$ by means of the following theorem whose proof we omit; the observation was made to us by G. Mislin.

2.1. THEOREM. *Let X be an H-space with $\pi_1(X)$ free abelian of rank k. Then we have a fibration $\tilde{X}\to X\to(S^1)^k$ in which the maps are H-maps, and this fibration has a cross-section. Thus $X\simeq\tilde{X}\times(S^1)^k$.*

2.2. COROLLARY. *If X is a torsion-free H-space of type $(1, n)$, then $X\simeq S^1\times S^n$.*

Having excluded the cases $(q, n)=(1, 1), (1, 3), (1, 7)$, we have to discuss the cases $(q, n)=(3, 3), (3, 5), (3, 7), (7, 7)$. In such a case, X is 1-connected and consequently we have a cellular decomposition for the homotopy type of X,

$$X\simeq S^q\cup_\alpha e^n\cup_\beta e^{n+q}. \tag{2.3}$$

2.4. THEOREM. *If $q=n$, then $X\simeq S^n\times S^n$.*

Proof: In this case $X\simeq(S^n\vee S^n)\cup_\beta e^{2n}$. Now the Whitehead product $[\iota_1, \iota_2]$ generates a cyclic infinite direct summand in $\pi_{2n-1}(S^n\vee S^n)$. Since Whitehead products vanish in $\pi_{2n-1}(X)$, $[\iota_1, \iota_2]$ must be a multiple of β. This implies that $\beta=\pm[\iota_1, \iota_2]$, whence $X\simeq S^n\times S^n$.

It remains to discuss the cases $(q, n)=(3, 5), (3, 7)$. We first quote a classical theorem of homotopy theory.

2.5. LEMMA (James [9]; cf. also [7]). *Let $n-1>q\geqslant 2$ and let $\sigma\in\pi_n(C_\alpha, S^q)$ denote the characteristic map for the n-cell of $C_\alpha\cup_\beta e^{n+q}=S^q\cup_\alpha e^n\cup_\beta e^{n+q}$ (so that $\alpha=\partial(\sigma)$).*

1) Zabrodsky has announced that the principal S^3-bundles $E_{2\omega}$, $E_{6\omega}$ and $E_{10\omega}$ (see § 2 below) are not H-spaces. He has also announced the homotopy classification of simply-connected, torsion-free rank 2 H-spaces.

Then $\sigma_*:\pi_{n+q-1}(D^n, S^{n-1})\to\pi_{n+q-1}(C_\alpha, S^q)$ *is monic and* $\pi_{n+q-1}(C_\alpha, S^q)$ *is isomorphic to the direct sum of* im σ_* *and an infinite cyclic group generated by the (relative) Whitehead product* $[\sigma, \iota_q]$, $\iota_q\in\pi_q(S^q)$ *the generator. Thus, if* $j: C_\alpha\to(C_\alpha, S^q)$ *denotes the inclusion, we have* $j_*\beta=m[\sigma, \iota_q]\oplus\sigma\circ\varrho$, $m\in Z$, $\varrho\in\pi_{n+q-1}(D^n, S^{n-1})$.

We now combine the information given in Lemma 2.5 with general results on the homology structure of finite H-spaces to get a grasp on the behavior of β. We prove

2.6. THEOREM. *Let X be an H-space having the form given in* (2.3), *with* $n>q>1$. *Then* $j_*\beta=\pm[\sigma, \iota_q]$.

Proof: We use the following two facts. First, by a theorem of Browder [2], X obeys Poincaré duality. Secondly, by a theorem of Browder-Spanier [5], X is stably reducible, i.e. $\Sigma^N\beta=0$ where Σ^N is the N-fold iterated suspension, N sufficiently large.

The proof uses the formula for $j_*\beta$ given in Lemma 2.5. Let $a\in H^q(X; Z)$, $b\in H^n(X; Z)$ and $c\in H^{n+q}(X; Z)$ be cohomology generators. Then, by Theorem 3.3 of James [9], $a\cup b=mc$, and by Poincaré duality, $a\cup b=\pm c$. Thus $m=\pm1$ and $j_*\beta=\pm[\sigma, \iota_q]\oplus\sigma\circ\varrho$, $\varrho\in\pi_{n+q-1}(D^n, S^{n-1})$. Now, $\Sigma^N\beta=0$ so that $\Sigma^N(j_*\beta)=0$ and therefore $\Sigma^N(\sigma\circ\varrho)=0$. We shall show that this latter equation implies that $\varrho=0$, thereby completing the proof. To see this, first note that $q=3$ in our case. Next, consider the commutative diagram

$$\begin{array}{ccc}
\pi_{n+2}(D^n, S^{n-1}) & \xrightarrow{\sigma_*} & \pi_{n+2}(C_\alpha, S^3)\\
\downarrow{\scriptstyle\Sigma'} & & \downarrow{\scriptstyle\Sigma}\\
\pi_{n+3}(D^{n+1}, S^n) & \xrightarrow{(\Sigma\sigma)_*} & \pi_{n+3}(\Sigma C_\alpha, S^4)
\end{array}$$

Then $\Sigma(\sigma\circ\varrho)=\Sigma\circ\sigma_*\varrho=(\Sigma\sigma)_*\circ\Sigma'\varrho$. But, since $n\geqslant5$, Σ' and $(\Sigma\sigma)_*$ are both isomorphisms, the latter by the homotopy excision theorem. Thus, if $\varrho\neq0$, we would also have $\Sigma(\sigma\circ\varrho)\neq0$. Iterating, we see that $\varrho\neq0$ implies $\Sigma^N(\sigma\circ\varrho)\neq0$. This establishes our contention and completes the proof of Theorem 2.6.

Of course, this theorem allows us to take $j_*\beta=[\sigma, \iota_q]$, since we do not change the homotopy type of X by replacing β by $-\beta$.

Remark. Theorem 2.6 provides an alternate proof of Theorem 3.2 of [7], as follows. Even though the space E_α of [7] is not, in general, an H-space, it is nevertheless true that E_α satisfies Poincaré duality and is stably reducible. (This follows from the fact, proved as Theorem 4.1 in [7], that E_α is a parallelizable smooth manifold.[2])) These are the only properties of E_α needed to prove the desired result.

The proof of Theorem 3.2 of [7] given in [7] relied on the fact that there is a map $p:(E_\alpha, S^3)\to(S^n, \text{point})$ (the bundle projection) inducing an isomorphism p_*: $\pi_*(E_\alpha, S^3)\xrightarrow{\approx}\pi_*(S^n)$, i.e. that p is a quasifibration. As we shall note directly below, the fact that $j_*\beta=\pm[\sigma, \iota_q]$ implies the converse of this statement.

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[2]) Although it is only proved in [7] that E_α is parallelizable for $n\neq4$ $(\alpha\in\pi_{n-1}(S^3))$, the result is also true for $n=4$. In fact, looking at the decomposition of the stable tangent bundle of E_α given on p. 105 of [7], it is easily seen that $p^*\alpha\{\hat{\alpha}\}$ is always 0 even though $\{\alpha\}$ need not be 0 for $n=4$.

2.7. COROLLARY. *If X is a homologically torsion-free rank 2 H-space, then there exists a map* $p:(X, S^q)\to(S^n, \text{point})$ *which is a quasifibration.*

Proof: In case the type of X is $(1, n)$ or (n, n), this follows from Corollary 2.2 and Theorem 2.4. If X is as in Theorem 2.6, the result follows by combining the conclusion of Theorem 2.6 and the work of Sasao [11; pp. 624–626].

This corollary suggests the conjecture that any homologically torsion-free rank 2 H-space has the homotopy type of a sphere bundle over a sphere. This is obviously true for the types covered by Corollary 2.2 and Theorem 2.4 and turns out to be correct in general, as we shall soon see. We first have the following general result.

2.8 THEOREM. *Let $X\simeq S^q\cup_\alpha e^n\cup_\beta e^{n+q}$ be an H-space with $\alpha=0$, i.e. $X\simeq(S^q\vee S^n)\cup_\beta e^{n+q}$. Then $X\simeq S^q\times S^n$.*

The proof is exactly as in the case $q=n$.

We now deal systematically with the cases $(q, n)=(3, 5)$, $(3, 7)$.

2.9. THEOREM. *If X is of type* $(3, 5)$, *then $X\simeq SU(3)$.*

Proof: We have $X\simeq S^3\cup_\alpha e^5\cup_\beta e^8$ with $\alpha\neq 0$. Indeed, if $\alpha=0$, then Theorem 2.8 implies $X\simeq S^3\times S^5$ and this would mean S^5 is an H-space. Now, by Theorem 2.6 and the subsequent remark, we may assume, without loss of generality, that $j_*\beta=[\sigma, \iota_3]$. If we now write $SU(3)\simeq S^3\cup_{\alpha'}e^5\cup_{\beta'}e^8$, we have $\alpha'=\alpha$ (since $\pi_4(S^3)=Z_2$) and, as above, we may assume $j_*\beta'=[\sigma, \iota_3]$. Thus $\beta-\beta'\in\ker j_*=\operatorname{im} i_*$, $i_*:\pi_7(S^3)\to\pi_7(S^3\cup_\alpha e^5)$. The proof will be completed by showing that $\operatorname{im} i_*=0$. To this end, recall that $\pi_7(S^3)=Z_2$ with generator $\alpha\circ\gamma$, $\gamma:S^7\to S^4$ the Hopf map. But $i_*(\alpha\circ\gamma)=i\circ\alpha\circ\gamma$ and plainly, $i\circ\alpha=0$. This completes the proof.

Before stating the next theorem, recall that $\pi_6(S^3)=Z_{12}$ with generator ω, the Blakers-Massey element. Thus, if X is of type $(3, 7)$, the map α has the form $k\omega$, $0\leqslant k<12$. Since we are only concerned with homotopy type we may indeed suppose $0\leqslant k\leqslant 6$.

2.10. THEOREM. *If X is of type* $(3, 7)$ *and $\alpha=k\omega$ then $X\simeq E_{k\omega}$, the principal S^3-bundle over S^7 with characteristic map $k\omega$.*

Remark. If $k=0$, $E_{k\omega}=S^3\times S^7$, an H-space; if $k=1$, $E_{k\omega}=Sp(2)$; Hilton-Roitberg show [7] that $E_{5\omega}$ $(=E_{7\omega})$ is an H-space; and, more generally, Stasheff [12] has shown that $E_{k\omega}$ is an H-space if $k\neq 2$ or 6; the same methods, due to Zabrodsky, show that $E_{2\omega}$ is an H-space iff $E_{6\omega}$ is an H-space. Zabrodsky has announced (unpublished) that $E_{2\omega}$ and $E_{6\omega}$ are not H-spaces.

Proof of 2.10: As in the proof of Theorem 2.9, we have

$$E_{k\omega}\simeq S^3\cup_{k\omega}e^7\cup_{\beta'}e^{10}=C_k\cup_{\beta'}e^{10},$$
$$X=X_k\simeq S^3\cup_{k\omega}e^7\cup_\beta e^{10}=C_k\cup_\beta e^{10}$$

and we will show that $\beta = \pm\beta'$. In fact, as before, we will choose β, β' so that $\beta=\beta'$. We have, by Theorem 2.6 and the subsequent remark, that $j_*\beta = j_*\beta' = [\sigma, \iota]$. Thus, if $k \neq 0, 3, 6$ we may immediately infer that $\beta=\beta'$. For consider the exact sequence

$$\pi_9(S^3) \xrightarrow{i_*} \pi_9(C_k) \xrightarrow{j_*} \pi_9(C_k, S^3). \tag{2.11}$$

Now $\pi_9(S^3) = Z_3$, generated by

$$S^9 \xrightarrow{\Sigma^3\omega} S^6 \xrightarrow{\omega} S^3.$$

It is thus generated by $k\omega \circ \Sigma^3\omega$, provided $3 \nmid k$. But $i_*(k\omega)=0$ so $i_*(k\omega \circ \Sigma^3\omega)=0$ and j_* is monic if $3 \nmid k$.

It remains to consider the cases $k=0, 3, 6$. The case $k=0$ is already disposed of in Theorem 2.8. We now take $k=3$. Then $E_{3\omega}$ and X_3 are both H-spaces. Consider

$$\begin{aligned} &\pi_9(S^3) \xrightarrow{i_*} \pi_9(C_3) \xrightarrow{j_*} \pi_9(C_3, S^3) \\ &\pi_7(S^3) \xrightarrow{i_*} \pi_7(C_3) \xrightarrow{j_*} \pi_7(C_3, S^3) \xrightarrow{\partial} \pi_6(S^3) \end{aligned} \tag{2.12}$$

Now $\partial(\sigma) = 3\omega$ so $\partial(4\sigma)=0$. Thus there is $\tau \in \pi_7(C_3)$ such that $j_*(\tau)=4\sigma$. For any such τ, $j_*[\tau, \iota] = 4[\sigma, \iota]$. Since attaching e^{10} kills $[\tau, \iota]$ in both $E_{3\omega}$ and X_3, we have $4\beta' = 4\beta = [\tau, \iota]$. But $\pi_9(S^3)=Z_3$ so $3(\beta-\beta')=0$. It follows immediately that $\beta=\beta'$.

Finally we consider the case $k=6$. Here X_6 is an H-space but we do not assume that $E_{6\omega}$ is an H-space.

For convenience we retain the symbols β, β' to refer to X_3, $E_{3\omega}$ and will use $\bar\beta$, $\bar\beta'$ to refer to the attaching maps for e^{10} in X_6, $E_{6\omega}$, and similarly with $\bar\sigma$, $\bar\tau$. Then by the same argument as above we know that $2\bar\beta = [\bar\tau, \iota]$ and it will be sufficient to show that

$$2\bar\beta' = [\bar\tau, \iota], \tag{2.13}$$

since we then complete the argument just as in the case $k=3$. We know that $2\bar\beta' = [\bar\tau, \iota] + i_*\lambda$, $\lambda \in \pi_9(S^3)$. We have the diagram

$$\begin{array}{ccccc} S^3 & \to & E_{6\omega} & \to & S^7 \\ \| & & \downarrow f & & \downarrow 2 \\ S^3 & \to & E_{3\omega} & \to & S^7 \end{array} \tag{2.14}$$

inducing

$$\begin{array}{ccccc} \pi_9(S^3) & \xrightarrow{i_*} & \pi_9(C_6) & \to & \pi_9(C_6, S^3) \\ \| & & \downarrow f_* & & \downarrow f_* \\ \pi_9(S^3) & \xrightarrow{i_*} & \pi_9(C_3) & \to & \pi_9(C_3, S^3), \\ \\ \pi_7(C_6) & \xrightarrow{j_*} & \pi_7(C_6, S^3) & \xrightarrow{\partial} & \pi_6(S^3) \\ \downarrow f_* & & \downarrow f_* & & \| \\ \pi_7(C_3) & \xrightarrow{j_*} & \pi_7(C_3, S^3) & \xrightarrow{\partial} & \pi_6(S^3). \end{array} \tag{2.15}$$

Now $\partial(\bar{\sigma})=6\omega$, $f_*(\bar{\sigma})=2\sigma$ and we may take $\tau=f_*(\bar{\tau})$. It follows that $f_*[\bar{\tau}, \iota]=[\tau, \iota]$. We claim that $f_*(\bar{\beta}')=m\beta'$ for some m. For we have the diagram

$$\begin{array}{ccc} \pi_{10}(E_{6\omega}, C_6) & \xrightarrow{\partial} & \pi_9(C_6) \\ \downarrow f_* & & \downarrow f_* \\ \pi_{10}(E_{3\omega}, C_3) & \xrightarrow{\partial} & \pi_9(C_3) \end{array} \tag{2.16}$$

and the groups on the left of (2.16) are cyclic infinite with their ∂-images generated by $\bar{\beta}'$, β' respectively. Thus, applying f_* to the equation $2\bar{\beta}'=[\bar{\tau}, \iota]+i_*\lambda$, we get $2m\beta'=[\tau, \iota]+i_*\lambda$. We know that $4\beta'=[\tau, \iota]$. Thus, applying j_*, we infer that $m=2$ (which was in any case clear on other grounds). Thus $i_*\lambda=0$ so that $\lambda=0$, whence (2.13) is proved and, with it, the proof of Theorem 2.10 is complete.

We thus have a complete list of homotopy types of torsion-free rank 2 H-spaces, modulo one ambiguity. The list reads $S^1\times S^1$; $S^1\times S^3$; $S^1\times S^7$; $S^3\times S^3$; $SU(3)$; $S^3\times S^7$, $Sp(2)$, $\mathbf{E}_{2\omega}$, $E_{3\omega}$, $E_{4\omega}$, $E_{5\omega}$, $\mathbf{E}_{6\omega}$; $S^7\times S^7$.

The one doubt concerns the bold examples; both or neither are H-spaces – and, as we have said, Zabrodsky has announced that neither is. This announcement does not, of course, render the arguments of Theorem 2.10, in the cases $k=2$, 6, superfluous.

§ 3. Topological Classification

Our purpose in this section is to prove

3.1. THEOREM. *Let X be an arbitrary homologically torsion-free rank 2 H-space, not of the homotopy type of $S^1\times S^3$. Then there is a closed PL manifold M, unique up to PL-equivalence, which is homotopy equivalent to X.*

Remark. This theorem is quite analogous to the situation in the case of rank 1 H-spaces. There, the only possible exception is, of course, S^3.

Proof of 3.1: Of course, the existence of M (indeed, of a smooth manifold M) was proved in § 2. For uniqueness, we first consider the case where the type of X is $(1, n)$. If $n=1$, the result is obvious. Since we are excluding $n=3$, there remains only $n=7$. But, as has been observed independently by several people, of M is any PL manifold homotopy equivalent to $S^1\times S^n$, $n\geqslant 5$,[3]) then a theorem of Browder-Levine implies that M must be PL-equivalent to $S^1\times S^n$.

We turn now to the case where the type of X is (q, n), $q>1$. Here, uniqueness will be deduced from simply-connected surgery considerations. A convenient way to see this is to use Sullivan's formulation of the surgery technique [13]. Thus, if N is another closed PL manifold homotopy equivalent to X, let $h:N\to M$ be a homotopy equivalence. According to [13], h is "classified" by a map $C_h:M_0\to F/PL$, M_0 denoting M

[3]) The result is actually true for $n\geqslant 4$ as is shown in J. Shaneson's thesis.

with a small open disc removed and F/PL denoting the fibre of $B_{PL} \to B_F$. But $M \simeq X \simeq S^q \cup_\alpha e^n \cup_\beta e^{n+q}$ so that $M_0 \simeq S^q \cup_\alpha e^n$. Applying the cohomology functor $[-, F/PL]$ to the cofibration $S^q \to S^q \cup_\alpha e^n \to S^n$ yields an exact sequence

$$\pi_n(F/PL) \to [M_0, F/PL] \to \pi_q(F/PL);$$

since q and n are odd, $\pi_q(F/PL) = \pi_n(F/PL) = 0$ by [13] and hence $[M_0, F/PL] = 0$. In particular, C_h is nullhomotopic and thus, again by [13], h can be deformed to a PL-equivalence $h': N \to M$. This completes the proof of Theorem 3.1.

Observe finally that Theorem 3.1, while giving a combinatorial classification, also contains a topological classification. For if M is a closed, topological (not a priori PL) manifold homotopy equivalent to X, X a homologically torsion-free rank 2 H-space, then the recent solution of the Triangulation Conjecture by Kirby-Siebenmann and Lashof-Rothenberg implies that M can be triangulated as a PL-manifold. In fact, the only possible obstruction to triangulating M lies in $H^4(M; Z_2)$, and the latter group is clearly 0 in our situation.

§ 4. General Rank Two H-spaces

We begin by quoting a general theorem of Browder on torsion in H-spaces. The main result of this section will then follow as an immediate corollary.

4.1. THEOREM (Browder [3, Th. 6.7 and succeeding remarks]). *Let X be a finite, connected H-space (not necessarily of rank 2) and suppose $H_*(\Omega X; Z)$ is torsion-free. Then*

(a) *if $H^*(X; Z)$ has p-torsion, then there exist integers $k \geqslant 1$ and m, with $m \equiv 1 \bmod p$, such that $H^*(X; Q)$ has a generator of dimension $2m\,p^k - 1$; and $2m$ is the smallest dimension where $H^*(X; Z)$ has p-torsion;*

(b) *if $H^*(X; Z)$ has higher p-torsion (i.e. elements of order p^r, $r > 1$), then there exist integers $l \geqslant 2$ and q, with $q \equiv 1 \bmod p$, such that $H^*(X; Q)$ has a generator of dimension $2q\,p^l - 1$; and $2q$ is the smallest dimension where $H^*(X; Z)$ has higher p-torsion.*

4.2. THEOREM. *If X is a connected, rank 2 H-space without 2-torsion, and if $H_*(\Omega X; Z)$ is torsion-free,*[4] *then X has no p-torsion in its integral cohomology (or equivalently, its integral homology) for $p \geqslant 5$, and has no higher 3-torsion in its integral cohomology.*

If, in addition, X is simply-connected, then X has torsion-free integral cohomology.

Proof: We consider the simply-connected case, the proof being similar in the general case. The existence of p-torsion in $H^*(X; Z)$ implies, by Browder's Theorem 4.1, that $H^*(X; Q)$ has a generator in dimension $2m\,p^k - 1$, with both $m, k \geqslant 1$.

[4]) J. Hubbuck has informed us that the restriction on ΩX can be removed.

Moreover, we cannot have $m=1$, since otherwise, again by Browder's theorem, we would have $H^2(X; Z)$ possessing p-torsion, which is impossible. (In fact, $H^2(X; Z)=0$ since, by another theorem of Browder [2], $\pi_2(X)=0$ for a finite, simply-connected H-space.) Thus, since $m \equiv 1 \bmod p$, we must have $m \geqslant 4$, and then $2m\,p^k - 1 \geqslant 23$. But this evidently contradicts Theorem 1.1 and completes the proof of the theorem.

Thus, Theorem 4.2 provides us, in view of the results in § 2, with a solution of the classification problem in the 2-torsion-free, simply-connected case, modulo the hypothesis on ΩX.

Concerning this hypothesis, we mention that a well known theorem of Bott asserts that $H_*(\Omega X; Z)$ is torsion-free if X is a Lie group. Moreover, the only known examples of finite H-spaces (cf. Hilton-Roitberg [7], Zabrodsky [14], Stasheff [12]) are "locally" either Lie groups or products of odd-dimensional spheres, i.e. for each prime p, X is H-equivalent, mod p, to such a space. Thus, for all known finite H-spaces, it is true that $H_*(\Omega X; Z)$ is torsion-free. It is therefore not inconceivable that $H_*(\Omega X; Z)$ is torsion-free for an arbitrary finite H-space X. Nevertheless, it would certainly be desirable to avoid using this hypothesis on ΩX in Theorem 4.2 (provided, of course, the result is true in that generality).

We consider next what occurs when we remove the restriction that there be no 2-torsion present. Then the conclusion of Theorem 1.1 is no longer necessarily valid and, in fact, the exceptional Lie group G_2 provides a specific counterexample.

Recall that G_2 may be considered as a bundle over S^6 with fibre $SU(3)$. This implies that G_2 has a cellular decomposition of the form

$$G_2 \simeq S^3 \cup_\alpha e^5 \cup e^6 \cup e^8 \cup e^9 \cup e^{11} \cup e^{14}. \tag{4.3}$$

Thus G_2 has rank 2 and type (3, 11). Moreover, the known homology structure of G_2 tells us that the 6-cell (resp. 9-cell) is attached to the 5-cell (resp. 8-cell) by a map of degree 2. (This also follows from Theorem 4.1, which implies that if X is any simply-connected finite H-space of type (3, 11), satisfying the condition on ΩX, then X has no p-torsion for $p \geqslant 3$ and no higher 2-torsion.)

In addition to G_2, there are three other H-spaces of type (3, 11) which can be constructed by Zabrodsky's method.

We state this as

4.4. THEOREM. *There are four homotopically distinct H-spaces, X_1, X_2, X_3 and X_4 of type* (3, 11), *which can be obtained by "mixing" G_2 with $S^3 \times S^{11}$ (X_1 being G_2).*

Proof: Following Zabrodsky, we take a decomposition $P = P_1 \cup P_2$ of the set of primes P into disjoint subsets and we mix $G_2(P_1)$ with $S^3 \times S^{11}(P_2)$. (In order that the result be an H-space, we must have $2 \in P_1$.) The four examples arise by taking (a) $P_2 = \emptyset$ (yielding G_2, of course), (b) $P_2 = \{3\}$, (c) $P_2 = \{5\}$, (d) $P_2 = \{3, 5\}$. The reason that these examples are homotopically distinct is that G_2 and $S^3 \times S^{11}$ are definitely not equivalent, either mod 3 or mod 5. In fact, by a theorem of Bott-Samelson, reproved by Kumpel in [10], $\pi_{10}(G_2)=0$, whereas $\pi_{10}(S^3\times S^{11})=Z_{15}$. Thus the four examples are distinguished by π_{10} and the theorem is proved.

Remarks. (1) The reason that only the primes 3 and 5 play a role in Theorem 4.4 is that, by a theorem of Kumpel [10], G_2 and $S^3\times S^{11}$ are equivalent mod p for all $p\geqslant 7$.

(2) The homotopy types X_i $(i=2, 3, 4)$ all have representatives which are smooth manifolds; this follows from the Browder-Novikov Theorem.

(3) Presumably, further examples can be obtained by mixing G_2 with $\bar{G}_2$, $\bar{G}_2$ being G_2 with the opposite multiplication. See Stasheff [12], where this operation is carried out with $Sp(2)$ in place of G_2.

§ 5. The Non-Simply-Connected Case

We conclude the paper with several remarks about non-simply-connected rank 2 H-spaces. For simplicity, we restrict attention to H-spaces whose type is one of those appearing in the conclusion of Theorem 1.1. (Of course, this is no real restriction if the space has no 2-torsion.)

First we note that Theorem 4.2 does not rule out the possibility of the existence of 3-torsion or higher 2-torsion in such a space. In fact, 3-torsion does occur: simply observe that $SU(3)$ has center Z_3 so that if $PSU(3)$ denotes the corresponding projective group, $H^2(PSU(3); Z)=Z_3$. Of course, 2-torsion also occurs: for example, reason as above with $PSp(2)$, the projective symplectic group, With regard to higher 2-torsion, it is easily seen from Theorem 4.1 that if it occurs, it appears already in $H^2(X; Z)$. This would imply that $\pi_1(X)$ has elements of order 4 and this seems unlikely. The known examples certainly have no 4-torsion in their fundamental groups.

We would now like to mention the existence of certain spaces which look like very strong candidates for the role of rank 2 H-spaces. These spaces, which we denote by $PE_{k\omega}$, are obtained as follows. There is, on the principal S^3-bundle $E_{k\omega}$, a certain smooth, fixed-point-free involution T_k; $PE_{k\omega}$ is the resulting quotient manifold. For $k=1$, $PE_{k\omega}$ is $PSp(2)$; indeed, it turns out that T_1 is just multiplication by $\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$. It seems quite likely that whenever $E_{k\omega}$ is an H-space, so is $PE_{k\omega}$. A detailed treatment of the spaces $PE_{k\omega}$ will be presented elsewhere.

Finally, we mention another possible way of obtaining rank 2 H-spaces with fundamental group Z_2. That is simply by applying Zabrodsky's mixing technique to the spaces $PSp(2)$, $\overline{PSp(2)}$, $P(S^3\times S^7)$. (Zabrodsky works only with simply-connected spaces, but this appears to be overly restrictive.) Since the principal S^3-bundles $E_{k\omega}$ $(k\not\equiv 2 \bmod 4)$ are obtained by mixing $Sp(2)$, $\overline{Sp(2)}$, $S^3\times S^7$, it is tempting to conjecture that the spaces $PE_{k\omega}$ of the preceding paragraph are obtained in a similar way from the corresponding projectivized spaces. We hope to return to this point on a future occasion.

§ 6. Appendix

We remarked in § 2 that the conclusion of Theorem 2.6 only requires that X satisfy Poincaré duality and be stably reducible. This remark enables us to prove the following theorem.

6.1. THEOREM. *If X is a closed smooth 1-connected torsion-free manifold of rank 2 and type (q, n) with $q=3$ and $n=5$ or 7, then X is homotopy equivalent (and therefore PL-equivalent) to an orthogonal S^q-bundle over S^n.*

Proof. We first show that X is a π-manifold. Since $\pi_i(BO)=0$, $i=3, 5, 7$, it follows immediately that $\tau \mid (X\text{-point})$ is trivial, where $\tau \in [X, BO]$ is the stable tangent bundle. Thus X is almost parallelizable. If $\dim X=8$, we complete this stage of the argument by observing that the obstruction to trivializing τ vanishes iff $p_2(X)=0$, which is true by the Hirzebruch Signature Theorem since $\sigma(X)=0$. If $\dim X=10$, we reason by the classical argument, invoking the injectivity of the J-homomorphism in dimension 10.

Now the argument of Theorem 2.6 allows us to infer, in the notation of § 2, that $j_*\beta = \pm[\sigma, i]$, so that X quasi-fibres over S^n. A close examination of the arguments proving Theorems 2.9 and 2.10, involving a study of the self-homotopy-equivalences of $C_\alpha = S^q \cup_\alpha e^n$, then shows that there are precisely 3 such quasifibrations if $n=5$ and 10 such if $n=7$. However, James and Whitehead, in their classical study of sphere-bundles over spheres, showed that there are precisely 3 orthogonal S^3-bundles over S^5, and Curtis and Mislin, basing themselves on the work of James and Whitehead, have recently shown (as yet unpublished) that there are precisely 10 orthogonal S^3-bundles over S^7. Thus each of our quasifibrations is homotopy equivalent to an orthogonal S^3-bundle and the proof of the theorem is complete.

Remark. The same conclusion holds if X is assumed only to be a closed, *PL* manifold. Indeed, by carefully applying smoothing theory to X, it can be seen that all the obstructions to smoothing X vanish, and we may then apply Theorem 6.1.

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