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Toda Brackets in Differential Topology[1]

A. Kosinski

It is well known-and indeed it is a basic link between differential topology and homotopy theory – that every map of a manifold W into a sphere S can be obtained by the Thom-Pontriagin construction on a framed submanifold W of M. The following two questions arise then:

A. Given a submanifold $W \subset M$, what maps $M \to S$ can be obtained by various framings of W?

B. Given a map $f: M \to S$, what manifolds W can be so framed as to give f?

This paper discusses a special case of the second question. More precisely, we deal here with the problem of realizing Toda brackets ("secondary compositions") by the Thom-Pontriagin construction. It turns out that under certain dimensional restrictions Toda brackets of elements in the image of Hopf-Whitehead homomorphism J can be realized by framings of sphere bundles over spheres. We determine the characteristic element of bundles which occur; in some of the cases this turns out to be a certain Samelson product. This provides an interesting relation between Toda brackets and Samelson products from which one can derive some results about Toda brackets and some computation of non-zero Samelson products.

The results have also applications in differential topology. Namely, given a framed sphere bundle over a sphere then a framed surgery "on a fibre" leads to a homotopy sphere Σ in the image of the Milnor pairing. Moreover, Σ is certainly non-trivial if the framing we started with yielded an element not in the image J. Hence the method of passing from Toda brackets not in the image J to non-trivial elements of θ^n.

The organization of the present paper is as follows.

§1. introduces notation and gives methods of constructing reduced join of two maps. A simple general construction of an element in a Toda bracket is also given there. Main theorems are stated and proved in section 2, the proofs however depend on a proposition proved in section 3. In section 4 we derive some corollaries for the stable case and give some applications. Section 5 discusses Samelson products and gives construction of sphere bundles over spheres with Samelson products as characteristic elements.

1. Notation and General Constructions

We will, in general, adopt the conventions of [4]. R^n will denote the n-dimensional

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Euclidean space; R_+, R_- will denote the non-negative, resp. non-positive, part of the real line. S^n will stand for the unit sphere and D^{n+1} for the unit ball in R^{n+1}, $z_0=(-1,0,\dots,0)$ will be the base point of S^n. S^n will be oriented by the map $\psi_n: R^n \to S^n - z_0$ given by

$$\psi_n(x_1,\dots,x_n)=\left(\frac{1-x^2}{1+x^2},\frac{2x_1}{1+x^2},\dots,\frac{2x_n}{1+x^2}\right), x^2=\sum x_i^2 .$$

We will also need the map $\tau_n:(D^n, S^{n-1}) \to (S^n, z_0)$ given by

$$\tau_n(x_1,\dots,x_n)=\psi_n\left(\frac{x_1}{1-x^2},\dots,\frac{x_n}{1-x^2}\right) \quad \text{for } x^2<1, \quad \tau_n(S^{n-1})=z_0 .$$

The upper (resp. lower) hemisphere of S^n will be denoted S^n_+ (resp. S^n_-). "Upper" will refer to the last coordinate.

If α and β are $k\times k$ and $n\times n$ matrices then $\|\alpha;\beta\|$ will be the notation for the $(n+k)\times(n+k)$ matrix $\|{\alpha\, 0 \atop 0\, \beta}\|$. I_k will denote the unit $k\times k$ matrix.

If W is a submanifold of dimension $n-p$ of R^n and F a framing of its normal bundle then the Thom-Pontriagin construction yields a map $(S^n, z_0)\to(S^p, z_0)$, we will denote this map $t(W,F)$.

Let $f: S^n \to S^p$, $g: S^m\to S^q$. Then $f\times g: S^{n+m}\to S^{p+q}$ is the reduced join of f and g as defined in [1].

Suppose we are given submanifolds W, V of R^n, R^m respectively with framings F, G. The submanifold $W\times V\subset R^{m+n}$ has then a natural framing $F\times G$.

1.1. PROPOSITION. $t(W\times V, F\times G)=t(W,F)\times t(V,G)$.

Proof. Let N be a tubular neighborhood of W in R^n, M a tubular neighborhood of V in R^m, and $K\subset N\times M$ a tubular neighborhood of $W\times V$ in $R^n\times R^m$. Consider a product structure on $S^{p+q}-z_0$ given by $\psi_{p+q}: R^p\times R^q=R^{p+q}\to S^{p+q}$. According to [1, 3.3], in terms of this product structure, $t(W,F)\times t(V,G)$ is given by

$$t(W,F)\times t(V,G)(x,y)=\begin{cases}(t(W,F)(x), t(V,G)(y)) & \text{for } (x,y)\in N\times M\\ z_0 & \text{for } (x,y)\notin N\times M .\end{cases}$$

On the other hand

$$t(W\times V, F\times G)(x,y)=\begin{cases}(t(W,F)(x), t(V,G)(y)) & \text{for } (x,y)\in K\subset N\times M\\ z_0 & \text{for } (x,y)\notin K .\end{cases}$$

Since both definitions yield obviously homotopic maps, the proposition follows.

Remark. A similar proposition has been proved by Kervaire in [4, 1.11]. There is however a wrong sign in his paper. See also [8, §6].

The following proposition is due to Toda [7, proof of 4.6]:

1.2 PROPOSITION. *Assume that* $\alpha\in\pi_{p+h}(S^h)$, $\beta\in\pi_{q+k}(S^k)$, $\gamma\in\pi_{r+l}(S^l)$ *satisfy* $\alpha\times\beta=0$, $\beta\times\gamma=0$ *and let* A_t *and* B_t *be null homotopies of* $\alpha\times\beta$ *and* $\beta\times\gamma$ *respectively. Then* $\alpha\times B_t$ *can be interpreted as a map of the upper hemisphere of* S^ε, $\varepsilon=p+h+q+k+r+l+1$, *and* $A_t\times\gamma$ *as the map of the lower hemisphere of* S^ε *so that together they yield a map of* S^ε *which is an element of* $(-1)^{(p+h)q+(p+h+q+k)r}\{E^{k+l}\alpha, E^{p+h+l}\beta, E^{p+h+q+k}\gamma\}$.

Suppose now that $\alpha=t(W^p, F_p)$, $\beta=t(W^q, F_q)$, $\gamma=t(W^r, F_r)$, $W^p\subset R_{p+h}$, $W^q\subset R_{q+k}$, $W^r\subset R_{r+l}$. Assume that $\alpha\times\beta=0$ and $\beta\times\gamma=0$. By Proposition 1.1 this implies that $(W^p\times W^q, F_p\times F_q)$ bounds (V^{p+q+l}, G), $V^{p+q+1}\subset R^{p+h}\times R^{q+k}\times R_+$, and $(W^q\times W^r, F_q\times F_r)$ bounds (V^{q+r+1}, H), $V^{q+r+1}\subset R^{q+k}\times R^{r+1}\times R_+$. Imbed now $R^{p+h}\times R^{q+k}\times R_+$ in $R^{p+h}\times R^{q+k}\times R^{r+l}\times R_+$ by a map $(x, y, t)\to(x, y, 0, -t)$ and $R^{q+k}\times R^{r+r}\times R_+$ in $R^{p+h}\times R^{q+k}\times R^{r+l}\times R_+$ by a map $(y, z, t)\to(0, y, z, t)$. These imbeddings exhibit $(V^{p+q+1}\times W^r, G\times F_r)$ and $(W^p\times V^{q+r+1}, F_p\times H)$ as framed submanifolds with boundary of $R^{p+h+q+k+r+l}\times R_-$, $R^{p+h+q+k+r+l}\times R_+$ respectively. Moreover, since $\partial(V^{p+q+1}\times W^r)=\partial(W^p\times V^{q+r+1})=W^p\times W^q\times W^r$, and both framings $G\times F_r$, $F_p\times H$ restrict on the boundary to $F_p\times F_q\times F_r$, we obtain naturally a manifold $W=V^{p+q+1}\times W^r\cup W^p\times V^{q+r+1}\subset R^{p+h+q+k+r+l}\times R$ with a framing F. By proposition 1.2 we have

1.3 THEOREM. $t(W, F)\in(-1)^\varepsilon\{E^{k+l}\alpha, E^{p+h+l}\beta, E^{p+h+q+k}\gamma\}$, $\varepsilon=(p+h)q+(p+h+q+k)r$.

2. Main Theorems

If α, β, γ are in the image of the Hopf-Whitehead homomorphism J then one can take spheres as W_p, W_q, W_r and it turns out that it is possible to determine what can be taken as W in the Theorem 1.3.

More precisely let $\alpha\in\pi_p(SO_h)$, $\beta\in\pi_q(SO_k)$, $\gamma\in\pi_r(SO_l)$ and set $\alpha'=t(S^p, F(\alpha))$, $\beta'=t(S^q, F(\beta))$, $\gamma'=t(S^r, F(\gamma))$ where vectors of $F(\alpha)$ at a point x are rows of the matrix $\alpha(x)$ and $F(\beta)$, $F(\gamma)$ are defined analogously.

The natural homomorphisms $\pi_i(SO_j)\to\pi_i(SO)$ will be denoted here (i, j) and $T(\alpha, \beta, \gamma)$ will stand for $(-1)^\varepsilon\{E^{k+l}\alpha', E^{p+h+l}\beta', E^{p+h+k+q}\gamma'\}$, $\varepsilon=(p+h)q+(p+h+q+k)r$. If $\lambda\in\pi_p(SO_{q+1})$, $\mu\in\pi_r(SO_{q+1})$ then $\langle\lambda, \mu\rangle$ will denote the Samelson product of λ and μ. Notice that according to [4, 1.8 and the footnote on p. 346]

$$\begin{aligned} E^{k+l}\alpha' &= (-1)^p E^{k+l}(((-1)^{h-1}\iota_h)\circ J\alpha) = (-1)^{p+h-1} E^{k+l}J\alpha \\ &= (-1)^{p+h+k+l-1} J s_*^{k+l}\alpha \end{aligned}$$

and similarly for suspensions of β' and γ', where s_*^k is the k-times iterated suspension in the homotopy groups of the orthogonal group. Therefore $T(\alpha, \beta, \gamma)$ is a Toda bracket of certain elements in Im J.

In the following three theorems it will be assumed that

2.1 $\quad h + k \geqslant p + q + 2, \quad k + l \geqslant q + r + 2$

2.2 $\quad \alpha' \times \beta' = 0, \beta' \times \gamma' = 0.$ (i.e. $J\alpha \times J\beta = 0, J\beta \times J\gamma = 0$)

$E(\xi)$ will denote the total space of the sphere bundle with characteristic map ξ.

2.3 THEOREM. *If (p, k), (p, q), (r, k), (r, q) are surjective then there exist elements $\lambda' \in \pi_p(SO_q)$, $\mu' \in \pi_r(SO_q)$ satisfying*

$$s_*^{q+k}\alpha + s_*^{h+k}\lambda' = 0, \; s_*^{q+k}\gamma + s_*^{l+k}\mu' = 0, \qquad 2.31$$

and such that for a certain framing F

$$t(E(\langle\mu, \lambda\rangle), F) \in T(\alpha, \beta, \gamma), \text{ where } \lambda = s_*\lambda', \; \mu = s_*\mu'.$$

2.4 THEOREM. *If (q, h), (q, p), (r, k), (r, q) are surjective then there exist elements $\lambda' \in \pi_q(SO_p)$, $\mu' \in \pi_r(SO)$ satisfying*

$$s_*^{p+h}\beta + s_*^{k+h}\lambda' = 0, \; s_*^{q+k}\gamma + s_*^{l+k}\mu' = 0 \qquad 2.41$$

and such that for a certain framing F

$$t(E(\lambda^{-1} \circ J\mu'), F) \in T(\alpha, \beta, \gamma).$$

2.5 THEOREM. *If (q, p), (q, r), (q, l), (q, h), (p, r), (p, l) are surjective and $J\alpha \times J\gamma = 0$ then there are elements $\lambda' \in \pi_q(SO_r)$, $\mu' \in \pi_p(SO_r)$ satisfying*

$$s_*^{r+l}\beta + s_*^{k+l}\lambda' = 0, \; s_*^{r+l}\alpha + s_*^{h+l}\mu' = 0 \qquad 2.51$$

and such that for a certain framing F

$$t(E(\langle\lambda, \mu\rangle), F) \in T(\alpha, \beta, \gamma), \text{ where } \lambda = s_*\lambda', \; \mu = s_*\mu'.$$

Proof of 2.3 By Theorem 1.3 an element of $T(\alpha, \beta, \gamma)$ can be realized by a framing of $W = V^{p+q+1} \times S^r \cup S^p \times V^{q+r+1}$. By 3.1 we can take as V^{p+q+1} the manifold $S^p \times S^q \cup_{f_\lambda} D^{p+1} \times S^q$ where λ satisfies 2.31. Similarly, as V^{q+r+1} we can take $S^q \times S^r \cup_{f_\mu} S^q \times D^{r+1}$. Recall that $f_\lambda(x, y) = (x, \lambda(x) \cdot y)$ and $f_\mu(y, z) = (\mu(z) \cdot y, z)$. Therefore $W = D^{p+1} \times S^q \times S^r \cup_{f_{\lambda,\mu}} S^p \times S^q \times D^{r+1}$ where $f_{\lambda,\mu}(x, y, z) = (x, (\mu^{-1}(z) \cdot \lambda(x)) \cdot y, z)$.

Applying 5.2 we see that W is an S^q-bundle over S^{p+r+1} with characteristic element $\langle\mu, \lambda\rangle$.

Proof of 2.4 We proceed as in the proof above. As V^{p+q+1} we take $S^p \times S^q \cup_{f_\lambda} S^p \times D^{q+1}$ and as V^{q+r+1} we take $S^q \times S^r \cup_{f_\mu} S^q \times D^{r+1}$ where λ, μ satisfy 2.41 and $f_\lambda(x, y) = (\lambda(y) \cdot x, y), f_\mu(y, z) = (\mu(z) \cdot y, z)$. Thus $W' = S^p \times D^{q+1} \times S^r \cup_{f'_{\lambda,\mu}} S^p \times S^q \times$

$\times D^{r+1}$ where $f'_{\lambda,\mu}(x,y,z)=(\lambda^{-1}(\mu(z)\cdot y)\cdot x,\ \mu(z)\cdot y,\ z)$. Set $W=S^p\times D^{q+1}\times S^r\cup_{f_{\lambda,\mu}}$ $S^p\times S^q\times D^{r+1}$ where $f_{\lambda,\mu}(x,y,z)=(\lambda^{-1}(\mu(z)\cdot y)\cdot x, y, z)$.

We claim that W and W' are diffeomorphic. For consider the identity map of the "right hand half", $S^p\times S^q\times D^{r+1}$, of W onto the right hand half of W'. This map induces on the boundary of $S^p\times D^{q+1}\times S^r$ the diffeomorphism $f'_{\lambda,\mu}\circ f^{-1}_{\lambda,\mu}$. This diffeomorphism however sends (x, y, z) to $(x, \mu(z)\cdot y, z)$ and therefore extends over $S^p\times D^{q+1}\times S^r$. Hence W and W' are diffeomorphic.

Now, W is an S^p-sphere bundle over S^{p+r+1} and by 3.2 its characteristic element is $\lambda^{-1}\circ J\mu'$.

Proof of 2.5 Again, proceeding as above we obtain

$$W=S^p\times D^{q+1}\times S^r\cup_{f_{\lambda,\kappa}} S^p\times D^{q+1}\times S^r,$$

where $f_{\kappa,\lambda}(x, y, z)=(\kappa(y)\cdot x, y, \lambda^{-1}(y)\cdot z)$ and $s_*^{r+l}\beta+s_*^{k+l}\lambda'=0,\ s_*^{p+h}\beta+s_*^{k+h}\kappa=0$
However, now W is not a sphere bundle over a sphere. We will thus construct a framed cobordism between W and a desired bundle.

We can consider W as one of the boundaries of $W\times[0,1]$ and we attach to $S^p\times D^{q+1}\times S^r$ the product $D^{p+1}\times D^{q+1}\times S^r$ by a map $(x,y,z)\to(x,y,\mu(x)\cdot z)$, where $s_*^{r+l}\alpha+s_*^{h+l}\mu'=0$, $\mu=s_*\mu'$. It is not difficult to check that our assumptions allow an application of 3.1 to conclude that the framing of $S^p\times 0\times S^r$ can be extended over $D^{p+1}\times 0\times S^r$. This last set is a deformation retract of $D^{p+1}\times D^{q+1}\times S^r$, hence we eventually obtain a framed cobordism between W and the other boundary of the cobordism. This other boundary is $E=D^{p+1}\times S^q\times S^r\cup_{g'} S^p\times D^{q+1}\times S^r$ where $g'(x,y,z)=(\kappa(y)\cdot x, y, \lambda^{-1}(y)\cdot\mu(x)\cdot z)$. The same construction as in the proof of 2.4 shows that E is diffeomorphic to $D^{p+1}\times S^q\times S^r\cup_g S^p\times D^{q+1}\times S^r$ where $g(x,y,z)=(x,y,(\lambda^{-1}(y)\cdot\mu(x))\cdot z)$. By 5.2 E is an r-sphere bundle over S^{p+q+1} with characteristic element $\langle\lambda,\mu\rangle$. This concludes the proof of 2.5.

3. Framings of products of spheres

Let $S^p\subset R^{p+1}\subset R^{p+h}$, $S^q\subset R^{q+1}\subset R^{q+k}$ be the standard imbeddings. We will coordinatize the normal bundle to S^p by vectors $v, v_2,\dots, v_h$ where v is the unit outwards normal to S^p in R^{p+1} and $v_2,\dots, v_h$ are coordinate vectors in R^{h-1}, $R^{p+h}=R^{p+1}\times R^{h-1}$. Similarily, the coordinates in the normal bundle to S^q will be given by $w, w_2,\dots, w_k$.

Let $\alpha\in\pi_p(SO_h)$, $\beta\in\pi_q(SO_k)$, we have a framing $F(\alpha,\beta)$ of $S^p\times S^q\subset R^{p+h}\times R^{q+k}$ given in terms of $v-w$ coordinateds by rows of the matrix $\|\alpha;\beta\|$.

Let $\gamma'\in\pi_q(SO_p)$ and $f_\gamma:\partial(S^p\times D^{q+1})\to S^p\times S^q$ be given by $f_\gamma(x,y)=(\gamma(y)\cdot x, y)$ where $\gamma=s_*\gamma'$. If $p+q+2\leqslant h+k$ then f_γ extends to an imbedding of $S^p\times D^{q+1}$ in $R^{p+h}\times R^{q+k}\times R_+$. Let V_γ be the resulting sub-manifold, i.e. $V_\gamma=S^p\times S^q\cup_{f_\gamma} S^p\times\times D^{q+1}\subset R^{p+h}\times R^{q+k}\times R_+$. Can we choose γ' so that $F(\alpha,\beta)$ extends over V_γ? The purpose of this section is to prove the following

3.1 PROPOSITION. *Assume that*

(a) $h+k \geqslant p+q+2$,

(b) $\pi_q(SO_h) \to \pi_q(SO)$ *and* $\pi_q(SO_p) \to \pi_q(SO)$ *are surjective,*

(c) $J\alpha \times J\beta = 0$.

Then there exists $\gamma' \in \pi_q(SO_p)$ *satisfying*

(d) $s_*^{p+h}\beta + s_*^{k+h}\gamma' = 0$

and if γ' *is so chosen then* $F(\alpha, \beta)$ *extends over* V_γ.

Remarks.

3.11 By [1, 3.2] $J\alpha \times J\beta = \pm (Js_*^k\alpha) \circ (Js_*^{p+h}\beta)$. Therefore assuming, say, $p \geqslant q$ the condition (c) is certainly satisfied unless either $q=1$ and $p=8s$ or $8s-1$, or $p=q=1, 3, 7$ [6, 7.5].

3.12 Assume $p \geqslant q$, but not $p=q=1, 3, 7$, and $h \geqslant p+1$, $k \geqslant q+1$. This implies that both (a) and (b) are satisfied and (d) simplifies to $s_*^p\beta + s_*^k\gamma' = 0$.

If, in addition, $q>1$ then by 3.11 the condition (c) will be satisfied too.

Proof. Consider first the framing $F(\bar{\gamma})$ of $S^p \times S^q$ given at (x, y) by the rows of the matrix $\|\bar{\gamma}(y), I_k\|$ where $\bar{\gamma} \in \pi_q(SO_h)$ is such that

(i) $$s_*^{p+k}\bar{\gamma} + s_*^{h+k}\gamma' = 0.$$

We claim that $F(\bar{\gamma})$ extends over V_γ. An extension is found in two steps. First we extend $F(\bar{\gamma})$ over a neighborhood of $(x_0) \times D^{q+1}$. To achieve this, notice that a tangent framing of $S^p \times S^q$ given by twisting the standard framing by $\|\gamma(y), I_{q+1}\|$ extends over V_γ: the extension is precisely the image of the standard framing of $S^p \times D^{q+1}$ under the imbedding map. Now, notice that the normal framing $F(\bar{\gamma})$ together with the restriction of the just extended tangential framing yields a map $w: S^q \to \to SO_{p+h+k+q+1}$ which is null homotopic by (i). The tangential framing above gives a map $S^q \to V_{p+h+q+k+1, p+q+1}$ which equals $\pi \circ w$ where $\pi: SO_{p+h+k+q+1} \to \to V_{p+h+q+k+1, p+q+1}$ is the projection of the fibration $SO_{p+h+k+q+1}/SO_{h+k}$. To extend $F(\bar{\gamma})$ over a neighborhood of $x_0 \times D^{q+1}$ we have then to find a null-homotopy of w which covers a null homotopy of $\pi \circ w$ given by the extension of the tangential framing over $x_0 \times D^{q+1}$. But this is possible since, by (a), $q<h+k$ and so $\pi_q(SO_{p+h+q+k+1}, SO_{h+k}) = 0$.

The second step consists in completing the extension over the complement of $S^p \times S^q \cup_{f_\gamma} (x_0) \times D^{q+1}$ in V_γ. To achieve this notice first that $S^p \times S^q \subset R^{p+h} \times R^{q+k}$ is a boundary of a standardly imbedded $D^{p+1} \times S^q$ in $R^{p+h} \times R^{q+k} \times R_-$ and that $F(\bar{\gamma})$ certainly extends over $D^{p+1} \times S^q$. However $V_\gamma \cup D^{p+1} \times S^q$ is an imbedded sphere and by [5, proof of 3.1] every framing of a complement of an open disc in an imbedded sphere extends over that disc, provided that $h+k \geqslant p+q+2$, which we assumed in (a). This concludes the extension.

Now, at the point $(x, y) \in S^p \times \partial D^{q+1}$ which corresponds to $(\gamma(y) \cdot x, y) \in S^p \times S^q$ we have also the frame of $F(\alpha, \beta)$ given by rows of $\|\alpha(\gamma(y) \cdot x); \beta(y)\|$. Therefore extending $F(\alpha, \beta)$ amounts to extending the map $\phi:\partial(S^p \times D^{q+1}) \to SO_{h+k}$ given by

$$(x, y) \mapsto \|\bar{\gamma}(y); I_k\|^{-1} \|\alpha(\gamma(y)\cdot x); \beta(y)\| = \|\bar{\gamma}^{-1}(y); \beta(y)\| \cdot \|\alpha(\gamma(y)\cdot x); I_k\|.$$

over $S^p \times D^{q+1}$.

First, define maps $\phi_1, \phi_2:\partial(S^p \times D^{q+1}) \to SO_{h+k}$ by $\phi_1(x, y) = \|\bar{\gamma}^{-1}(y); \beta(y)\|$, $\phi_2(x, y) = \|\alpha(\gamma(y)\cdot x); I_k\|$. Then ϕ_1 will extend over $S^p \times D^{q+1}$ if $\bar{\gamma}$ satisfies

(ii) $\quad s_*^k \bar{\gamma} = s_*^h \beta.$

On the other hand since γ is a suspension there is $x_0 \in S^p$ such that for all $y \in S^q$ $\gamma(y)\cdot x_0 = x_0$. Thus $\phi_2(x_0, y) = \|\alpha(x_0), I_k\| = I_{k+h}$ for we can assume $\alpha(x_0) = I_h$. Hence ϕ_2 extends trivially over $x_0 \times D^{q+1}$ and to complete the extension of ϕ we have to extend ϕ_2 over the complement of $\partial(S^p \times D^{q+1}) \cup x_0 \times D^{q+1}$.

Let $g: D^p \times D^{q+1} \to S^p \times D^{q+1}$ be given by $g(x, y) = (\tau_p(x), y)$ where $\tau_p: D^p \to S^p$ is a relative homeomorphism shrinking the boundary of D^p to a point. Let $\omega: S^{p+q} \to SO_{h+k}$ be a composition of $g \mid \partial(D^p \times D^{q+1})$ with ϕ_2. Then ϕ_2 will extend if ω is null-homotopic.

This is certainly the case if $p+q = 2, 4, 5, 6$, mod 8. To deal with the case $p+q = 3$ mod 4 notice that $\omega = \omega_1 \circ \omega_2$ where $\omega_1: S^p \to SO_{h+k}$ equals $s_*^k \alpha$ and $\omega_2: S^{p+q} \to S^p$ is given by

(iii) $$\omega_2(x, y) = \begin{cases} x_0 & \text{if } (x, y) \in \partial D^p \times D^{q+1} \\ \gamma(y) \circ \tau_p(x) & \text{if } (x, y) \in D^p \times \partial D^{q+1} \end{cases}$$

Now, ω_2 is of finite order except, possibly, when p is even. However for p even ω_1 is in the (stable) group $\pi_p(SO_{h+k})$ of finite order. Therefore ω is always of finite order which implies that it is null homotopic since $\pi_{p+q}(SO_{h+k})$ is cyclic infinite for $p+q = 3$ mod 4.

There remains the case $p+q = 0$, 1 mod 8. One sees easily using (iii) and [4, 1.8] that $\omega_2 = \pm J\gamma'$ where $\gamma' \in \pi_q(SO_p)$ is such that $s_*\gamma' = \gamma$. Hence $\omega = \pm s_*^k \alpha \circ J\gamma'$ and, by [6, 7.5]

$$J\omega = +(Js_*^k \alpha) \circ E^{h+k} J\gamma' = \pm (Js_*^k \alpha) \circ (Js_*^{h+k} \gamma').$$

However, by (i) and (ii) $s_*^{h+k}\gamma' = s_*^{p+h}\beta$ and we have

$$J\omega = +(Js_*^k \alpha) \circ (Js_*^{p+h} \beta) = \pm J\alpha \times J\beta$$

by [1, 3.2]. Thus $J\omega = 0$ and, since J is a monomorphism in the case under consideration, we have $\omega = 0$. This shows that if $\gamma', \bar{\gamma}$ satisfy (i) and (ii) then $F(\alpha, \beta)$ can be extended over V_γ.

To show that they can be so found notice, first, that since $\pi_q(SO_p) \to \pi_q(SO_{p+k+h})$ is surjective, in view of (a) and (b), it is possible to find $\gamma' \in \pi_q(SO_p)$ so that (d) is satisfied. Now, it follows from (a) and (b) that it is possible to find $\bar{\gamma}$ satisfying (i). Then

(i) and (d) together imply

$$s_*^{p+h}\beta = s_*^{p+k}\bar{\gamma}$$

which is equivalent to (ii) by (a). This concludes the proof of 3.1.

The following useful remark can be extracted from the above proof.

Let $\psi:\partial(S^p\times D^{q+1})\to SO_h$ be given by $\psi(x,y)=\|\alpha(\gamma(y)\cdot x)\|$, where $\alpha:S^p\to SO_h$, $\gamma:S^q\to SO_{p+1}$, $\gamma=s_*\gamma'$, $x\in S^p$, $y\in S^q$. We have then

3.2 *The only obstruction to extending ψ over $S^p\times D^{q+1}$ is the homotopy class of $\alpha\circ J\gamma'\in\pi_{p+q}(SO_h)$.*

4. Applications

First, we will restate theorems 2.3–2.5 in a less general but much simpler form.

Let $\alpha\in\pi_p(SO_{p+h})$, $\beta\in\pi_q(SO_{q+k})$, $\gamma\in\pi_r(SO_{r+l})$ and assume

4.1 $h, k, l\geqslant 1$, it is neither $p=q=1, 3, 7$ nor $r=q=1, 3, 7$.

4.2 $J\alpha\times J\beta=0=J\beta\times J\gamma$.

Under those assumptions we have

4.3 THEOREM. *An element of $\{J\alpha, J\beta, J\gamma\}$ can be realized by a framing of a sphere bundle over a sphere where*

(a) *if $p\leqslant q\geqslant r$ then we have a q-sphere bundle over S^{p+r+1} with characteristic element $\langle s_*^{q-r-l+1}\gamma, s_*^{q-p-h+1}\alpha\rangle$*

(b) *if $p\geqslant q\geqslant r$ then we have a p-sphere bundle over S^{q+r+1} with characteristic element $(s_*^{p-q-k+1}\beta)\circ Js_*^{q-r-l+1}\gamma$*

(c) *if $q\leqslant p\leqslant r$ but not $p=r=1, 3, 7$ then we have an r-sphere bundle over S^{p+q+1} with characteristic element $\langle s_*^{r-p-h+1}\alpha, s_*^{r-q-k+1}\beta\rangle$.*

This leaves out cases $p\leqslant q\leqslant r$ and $q\leqslant r\leqslant p$ which can however be obtained from (b) and (c), respectively, by interchanging p and r.

The theorem follows immediately from 2.3–2.5; for case (c) use also 3.11. (Notice that by [2] (i, j) is surjective if $i<2j-2$ provided $j\geqslant 13$. This fact allows to use 2.3–2.5 to derive information about Toda brackets of elements in the meta-stable range).

Now, let $x=\max(p, q, r)$ and let y, z be the remaining two numbers.

4.4 THEOREM. *If $x>y+z+2$ and $y, z>7$ then $\{J\alpha, J\beta, J\gamma\}\equiv 0$* mod *the indeterminacy*

Proof. The proof is based on the following facts:

(i) Every framing of $S^p\times S^q$ yields an element of Im J except, perhaps, when $p=q=1, 3, 7$. (This is due to Novikov.)

(ii) $\{\alpha, \beta, \gamma\circ\delta\}\supset\{\alpha, \beta, \gamma\}\circ\delta$ and similarily for compositions with other elements in the bracket (Toda).

(iii) If ε_i is a generator of $\pi_i(SO)$ and η_j generates $\pi_{j+1}(S^j)$ then $\varepsilon_{8s-1}\circ\eta_{8s-1}=\varepsilon_{8s}$, $\varepsilon_{8s-1}\circ\eta_{8s-1}\circ\eta_{8s}=\varepsilon_{8s+1}$ (Kervaire).

We can restrict ourselves to consideration of Toda brackets of generators of Im J and to the case when p, q, r are congruent to 0, 1, 3, 7 mod 8. Now (ii) and (iii) show that $\{J\alpha, J\beta, J\gamma\}=T$ contains a composition with $\{J\alpha', J\beta', J\gamma'\}=T'$ where dimensions of α', β', γ' are 3 or 7 mod 8. Moreover, T' will satisfy the relation $x>y+z$. By 4.3 an element of T' can be realized by a framing of an x-sphere bundle over an $(y+z+1)$-sphere, hence of a stable bundle. But a stable bundle which is a π-manifold must be a product. We conclude that an element of T' can be realized by a framing of $S^x\times S^{y+z+1}$. Since $y+z>10$ (i) implies that an element of T' is in Im J, more precisely in $J_{x+y+z+1}$ where the numbers x, y, z are 3 or 7 mod 8. However $x+y+z+1=2, 6$ mod 8 in such a case and $J_{8s+2}=J_{8s+6}=0$. Hence T' will always contain zero and so will T which either equals T' or contains a composition of T' with either η or η^2. This concludes the proof of 4.4.

Remark. The dimensional restrictions in 4.4 can be somewhat relaxed, at the expense of more detailed analysis, however things do go wrong in low dimensions: $\{\eta, \nu, \eta\}=\nu^2\neq 0$.

To give an example of computations we begin with the following

4.5 COROLLARY. *Let α, β, γ be as in* 4.3(a) *and assume that it is not $q=p+r+1=3, 7$. If $\{J\alpha, J\beta, J\gamma\}\cap J_{p+q+r+1}=\emptyset$ then $\langle s_*^{q-r-l+1}\gamma, s_*^{q-p-h+1}\alpha\rangle\neq 0$.*

Proof. It follows from 4.3(a) that an element ζ of $\{J\alpha, J\beta, J\gamma\}$ can be represented by a framing of a q-sphere bundle over S^{p+r+1} with the above product as the characteristic element. If this bundle is trivial then by 4.4(i) $\zeta\in J_{p+q+r+1}$

4.6 EXAMPLE. Let $\alpha\in\pi_7(SO_9)$, $\beta\in\pi_8(SO_{10})$, $\gamma\in\pi_3(SO_5)$ be chosen so that $J\alpha=\sigma$, $J\beta=\sigma\eta$, $J\gamma=\eta$. Since $\{\sigma, \sigma\eta, \nu\}$ is a single element and not in J_{19}, 4.5 implies that $\langle s_*^4\gamma, \alpha\rangle\neq 0$.

Set $E=E(\langle s_*^4\gamma, \alpha\rangle)$, E is an 8-sphere bundle over S^{11}. Let $t(E)\subset\Pi_{19}$ denote the set of elements obtainable by all possible framings of E. Then $t(E)=\Pi_{19}$. For Π_{19} is the direct sum of J_{19} and of the subgroup generated by $\{\sigma, \sigma\eta, \nu\}$. Since E bounds a parallelelizable manifold, $t(E)\supset J_{19}$. But 4.3(a) implies that $\{\sigma, \sigma\eta, \nu\}$ can be realized by a framing of E and the remark follows. I do not know whether there does always exist such a "universal manifold" (for the part of Π_k with trivial Kervaire invariant).

Let E be so framed as to realize $\{\sigma, \sigma\eta, \nu\}$. By a framed surgery on a fibre we obtain a homotopy sphere Σ and it is not difficult to show that $\Sigma\in I(E)$, the inertia group of E. Since $\{\sigma, \sigma\eta, \nu\}$ is not in J_{19}, Σ does not bound a parallelizable manifold. Hence we have another counter-example to a conjecture of Novikov (that $t(I(W))\subset \subset \mathrm{Im}\, J$ if W is a π-manifold). A theorem of W. Browder implies that $I(E)=\Sigma$.

5. Samelson products

We give here, first, a representation of Samelson-James product [3] of two maps $\alpha: S^p \to G$, $\beta: S^q \to G$, where G is a group, as a certain map $\langle\alpha, \beta\rangle: S^{p+q} \to G$ and then use it to give a simple construction of sphere bundles over spheres with the characteristic element a Samelson product.

Let $\alpha: (S^p, z_0) \to (G, e)$, $\beta: (D^q, \partial D^q) \to (G, e)$ be two maps. Define a map $\phi_{\alpha\beta}: S^p \times D^q \to G$ by $\phi_{\alpha\beta}(x, y) = \alpha^{-1}(x)\,\beta(y)\,\alpha(x)$. Notice that $\phi_{\alpha\beta}(x, y) = e$ if $(x, y) \in \partial(S^p \times D^q)$. Therefore for an arbitrary imbedding $S^p \times D^q \to S^{p+q}$ the map $\phi_{\alpha\beta}$ induces a map $S^{p+q} \to G$. We choose the standard imbedding and denote the induced map $\langle\alpha, \beta\rangle$. Clearly, the homotopy class of $\langle\alpha, \beta\rangle$ depends only on the homotopy classes of α and β.

5.1 PROPOSITION. *$\langle\alpha, \beta\rangle$ is the Samelson product of α and β.*

Proof. We can assume that $\alpha(x) = e$ for $x \in S^p_-$. Let $|_p: S^p \to G$ be the trivial map: $|_p(x) = e$. Obviously $\langle |_p, \beta^{-1}\rangle$ is null-homotopic. On the other hand $\langle |_p, \beta^{-1}\rangle + \langle\alpha, \beta\rangle$ is the homotopy class of the map $\phi: S^{p+q} \to G$ given by

$$\phi(z) = \begin{cases} \beta^{-1}(y)\,\alpha^{-1}(x)\,\beta(y)\,\alpha(x) & \text{if } z = (x, y) \in S^p_+ \times D^q \\ e & \text{if } z \notin S^p_+ \times D^q \end{cases}$$

Since ϕ is the Samelson product of α and β, the proof is complete.

Remark. Let $i: S^p \times D^q \to S^{p+q}$ and let $\langle\alpha, \beta\rangle_i: S^{p+q} \to G$ be the map constructed as above using i. One can show that

$$\langle\alpha, \beta\rangle_i = \langle\alpha, \beta\rangle + \beta^{-1} \circ t(i)$$

where $t(i): S^{p+q} \to S^q$ is the map corresponding to i via the Thom-Pontriagin construction.

In particular, if $i \mid S^p \times 0$ is the standard imbedding then

$$\langle\alpha, \beta\rangle_i = \langle\alpha, \beta\rangle + \beta^{-1} \circ (\pm J\gamma)$$

where $\gamma \in \pi_p(SO_q)$ gives the twist of the normal bundle of $i(S^p \times 0)$.

Now, let $S^n = S^p \times D^{q+1} \cup D^{p+1} \times S^q$, $p + q + 1 = n$ be the standard decomposition of S^n.

Let $\alpha: S^p \to SO_{r+1}$, $\beta: S^q \to SO_{r+1}$. We will construct an S^r bundle over S^n by glueing $S^p \times D^{q+1} \times S^r$ to $S^p \times S^q \times S^r$ by the map

$$(x, y, z) \mapsto (x, y, \beta(y) \cdot z), (x, y) \in \partial(S^p \times D^{q+1}).$$

and $D^{p+1} \times S^q \times S^r$ to $S^p \times S^q \times S^r$ by the map

$$(x, y, z) \mapsto (x, y, \alpha(x) \cdot z), (x, y) \in \partial(D^{p+1} \times S^q).$$

The same bundle E is also obtained by glueing

$S^p \times D^{q+1} \times S^r$ to $D^{p+1} \times S^q \times S^r$ by the map of the boundaries

$$(x, y, z) \mapsto (x, y, \alpha^{-1}(x)\cdot\beta(y)\cdot z).$$

The projection $\pi: E \to S^n$ is given by $\pi(x, y, z) = (x, y)$.

5.2 PROPOSITION. *The characteristic element of the bundle E is $\langle\alpha, \beta\rangle$.*

Proof. We will assume that in the coordinates in $S^n - z_0$ given by ψ_n, $S^p \times D^{q+1}$ corresponds to a tubular neighborhood of the unit p-sphere in the first $p+1$ coordinates. Let $A = S^n_- \cup S^p \times D^{q+1}$. A is then an n-disc and the boundary of A contains "one-half" of the boundary of $S^p \times D^{q+1}$, precisely $S^p \times S^q_+$.

We will assume that β is concentrated on S^q_+, i.e. $\beta(y) = e$, $x \in S^q_-$. We can then trivialize $\pi^{-1}(A)$ by the map $t_1: A \times S^r \to \pi^{-1}(A)$ given by

$$t_1(u, z) = \begin{cases} (u, z) & \text{if } u \in A - S^p \times D^{q+1}, \\ (x, y, \alpha(x)\cdot z) & \text{if } u = (x, y) \in S^p \times D^{q+1}. \end{cases}$$

Let $B = Cl(S^n - A)$. Then $B \subset D^{p+1} \times S^q$ and $\pi^{-1}(B)$ is trivialized by $t_2: B \times S^r \to$ $\to \pi^{-1}(B)$, $t_2(x, y, z) = (x, y, z)$. Hence $t_2^{-1} t_1: \partial A \times S^r \to \partial B \times S^r$ is given by

$$t_2^{-1} t_1(u, z) = \begin{cases} (u, z) & \text{if } u \in \partial A - \partial(S^p \times D^{q+1}) \\ (u, \alpha^{-1}(x)\, \beta(y)\, \alpha(x)\cdot z) & \text{if } u = (x, y) \in \partial(S^p \times D^{q+1}) \cap \cap\, \partial A \\ & \qquad = S^p \times S^q_+. \end{cases}$$

Using 5.1 we see that the characteristic element of E is $\langle\alpha, \beta\rangle$.

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