

# Pinching Implies Strong Pinching

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## Pinching Implies Strong Pinching

by HERMANN KARCHER<sup>1)</sup>

The author was led to look for bounds as in (2) by a preprint manuscript of the article by E. Ruh which precedes this paper. Denote by  $\delta$  the minimum, by  $\Delta$  the maximum of the sectional curvatures at  $p \in M$  ( $\dim M = n$ ) and by  $\lambda$  the smallest, by  $\Lambda$  the largest eigenvalue of the curvature operator  $\varrho: M_p \wedge M_p \rightarrow M_p \wedge M_p$  (defined by  $\varrho\omega(X, Y) := \text{trace}_Z \omega(R(X, Y)Z, Z)$  in terms of the curvature tensor  $R$  of the Riemannian metric on  $M$ ).

It is well known that  $\lambda \leq 2\delta$  and  $2\Delta \leq \Lambda$ . We prove:

$$\text{trace } \varrho = \sum_{i \neq k} \langle R(X_i, X_k) X_k, X_i \rangle = \text{scalar curvature.} \tag{1}$$

$$\left. \begin{aligned} \Delta + \delta - \frac{2}{3}(\Delta - \delta) [n(n-1)(n + \frac{1}{4})]^{1/2} &\leq \lambda \leq \Lambda \leq \\ \Delta + \delta + \frac{2}{3}(\Delta - \delta) [n(n-1)(n + \frac{1}{4})]^{1/2}. & \end{aligned} \right\} \tag{2}$$

From (2) and Ruh's theorem [3] we have the

**COROLLARY.** *A complete simply connected Riemannian manifold with  $\delta/\Delta \geq 1 - [\frac{1}{2} + \frac{5}{3}(n(n-1)(n + \frac{1}{4}))^{1/2}]^{-1}$  has  $\lambda/\Lambda \geq \frac{2}{3}$ , hence is diffeomorphic to the standard sphere*

*Remark.* The numerical values are worse than the ones obtained by Gromoll (explicitly computed up to  $n = 12$ ); for example for  $n = 7$  resp.  $n = 12$  we need  $\delta/\Delta \geq 0.966$  resp.  $0.985$ , Gromoll's figures are  $0.819$  resp.  $0.931$ .

*Proof.* Use a normal coordinate system at  $p \in M$  with associated basis  $X_i$  ( $i = 1, \dots, n$ ) of  $M_p$ . For  $\omega \in M_p \wedge M_p$  put  $\omega_{ik} = \omega(X_i, X_k)$ , hence  $\omega = \frac{1}{2} \sum_{i \neq k} \omega_{ik} X_i \wedge X_k$ . The scalar product in  $M_p \wedge M_p$  is given by  $\langle \omega, \bar{\omega} \rangle = \frac{1}{2} \sum_{i \neq k} \omega_{ik} \bar{\omega}_{ik}$ .

We have

$$\varrho(X_i \wedge X_j)(X_k, X_l) = 2R_{klji} = \langle \varrho(X_i \wedge X_j), X_k \wedge X_l \rangle, \tag{3}$$

hence

$$\langle \varrho\omega, \omega \rangle = \frac{1}{2} \sum' R_{klji} \omega_{ij} \omega_{kl}. \tag{4}$$

(  $\sum' = \sum_{k \neq l, i \neq j}$  here and below )

(3) and (4) imply  $\lambda \leq 2\delta$  and  $2\Delta \leq \Lambda$ , but (3) shows also that the eigenvectors  $\omega^v$  ( $v = 1, \dots, \frac{1}{2}n(n-1)$ ) of  $\varrho$  are in general not forms of rank 2 so that the reversed inequalities

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cannot be expected. (4) implies (1) immediately:

$$\begin{aligned} \text{trace } \rho &= \sum_{\nu} \langle \rho \omega^{\nu}, \omega^{\nu} \rangle = \frac{1}{2} \sum' R_{klji} \sum_{\nu} \omega_{ij}^{\nu} \omega_{kl}^{\nu} \\ &= \frac{1}{2} \sum' R_{klji} (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) = \sum_{k \neq l} R_{kllk}. \end{aligned}$$

Define  $R_0(X, Y)Z := \frac{1}{2}(\Delta + \delta)(\langle Y, Z \rangle X - \langle X, Z \rangle Y)$  and  $D := R - R_0$ . Since  $D$  and  $R$  have the same symmetries we have from (7), (8), (9) in [2] for unit vectors in  $M_p$ :

$$|\langle D(X, Y)Y, U \rangle| \leq 2^{-1/2}(\Delta - \delta), \quad (\leq \frac{1}{2}(\Delta - \delta) \text{ if } X \perp U) \tag{5}$$

$$\text{If } Z \perp X, Y \text{ then } |\langle D(X, Y)Z, U \rangle| \leq \frac{2}{3}(\Delta - \delta) \tag{6}$$

$$\|D\| = \max_{|X|=|Y|=|Z|=|U|=1} |\langle D(X, Y)Z, U \rangle| \leq (34/36)^{1/2}(\Delta - \delta). \tag{7}$$

*Remark.* In forthcoming papers on the differentiable pinching problem Sugimoto and Shiohama have used  $\|D\| \leq k$  (with normalization  $\Delta + \delta = 2$ ) as a pinching condition.  $\|D\| \leq k$  clearly implies for the sectional curvatures  $1 - k \leq K \leq 1 + k$ . The converse is not true since for the complex projective space  $\|D\| = \frac{4}{3}k$ . However  $1 - k \leq K \leq 1 + k$  and (7) imply  $\|D\| \leq (34/9)^{1/2} k$ .

We rewrite (4) as

$$\langle \rho \omega, \omega \rangle = \Delta + \delta + \frac{1}{2} \sum' D_{klji} \omega_{ij} \omega_{kl} \tag{8}$$

and apply Schwarz' inequality (note  $|\omega| = 1$ ):

$$|\frac{1}{2} \sum' D_{klji} \omega_{ij} \omega_{kl}| \leq (\sum' D_{klji}^2)^{1/2}. \tag{9}$$

(Schwarz' inequality can be applied in various ways to (8) leading to different expressions for the [ ]-bracket in (2); in our computations they were all of the order  $n^{3/2}$  or worse. If (2) is a poor estimate the loss probably occurs in (9) since the following estimates seem fairly sharp to us.)

$$\sum' D_{klji}^2 = \sum_{k, l=j, i} D_{klji}^2 + \sum_{k, l=i, j} D_{klji}^2 + \sum_{\substack{k, l, i, j \\ l \neq i, j}} D_{klji}^2 \tag{10}$$

We have from (5) with  $l, i$  fixed and  $a_k := D_{kjjl} (\sum_k D_{kjjl}^2)^{-1/2}$

$$2^{-1/2}(\Delta - \delta) \geq \left| \sum_k a_k D_{kjjl} \right| = \left( \sum_k D_{kjjl}^2 \right)^{1/2}. \tag{11}$$

If  $l \neq i, j$  we have from (6) with  $a_k := D_{klji} (\sum_k D_{klji}^2)^{-1/2}$

$$\frac{2}{3}(\Delta - \delta) \geq \left| \sum_k a_k D_{klji} \right| = \left( \sum_k D_{klji}^2 \right)^{1/2}. \tag{12}$$

Insert (11) and (12) in (10) to get

$$\sum' D_{klji}^2 \leq \sum_{j \neq i} (\Delta - \delta)^2 + \sum_{l \neq i \neq j \neq l} \frac{4}{9} (\Delta - \delta)^2 = \frac{4}{9} (\Delta - \delta)^2 n(n-1)(n + \frac{1}{4}). \quad (13)$$

(8), (9) and (13) prove (2).

*Remark.* The method can also be used to improve Berger's estimate [1] for the Eulercharacteristic:

$$\begin{aligned} \chi(M^{2m}) &= \frac{(-1)^m}{2^{3m} \pi^m m!} \int_M (\varepsilon_{i_1 \dots i_{2m}} \cdot \varepsilon_{j_1 \dots j_{2m}} \cdot R_{i_1 i_2 j_1 j_2} \cdots R_{i_{2m-1} i_{2m} j_{2m-1} j_{2m}}) dV \\ &\leq 2^{-m} (2m)! (\Delta/\delta)^m. \end{aligned}$$

Berger estimates the integrand by  $((2m)!)^2 \Delta^m$ . We use first  $m \cdot \prod_{i=1}^m a_i \leq \sum |a_i|^m \leq \max |a_i|^{m-2} \sum a_i^2$ , then Berger's  $|R_{kllj}| \leq \frac{1}{2} (\Delta - \delta)$ ,  $|R_{klji}| \leq \frac{2}{3} (\Delta - \delta)$  – see (5), (6) – and the analogues of (11), (12) for  $R$  instead of  $D$  to estimate the integrand by

$$((2m-2)!)^2 \Delta^m \cdot 2m(2m-1) \left[ 1 + \left( 1 - \frac{\delta}{\Delta} \right)^m \left( \left( \frac{1}{4} \right)^m + (m-1) \left( \frac{4}{9} \right)^m \right) \right].$$

We obtain ( $m \geq 3$ )

$$\chi(M^{2m}) \leq 2.4 \cdot 2^{-m} (2m-2)! (\Delta/\delta)^m.$$

In Dimension 4 Chern's coordinate choice [1] simplifies the integrand to  $32(R_{1212}R_{3434} + R_{1313}R_{2424} + R_{1414}R_{2323} + R_{1234}^2 + R_{2314}^2 + R_{3124}^2)$  which Berger estimates by  $32(3\Delta'^2 + \frac{8}{9}(\Delta' - \delta')^2)$ . Here  $\Delta'$  and  $\delta'$  are minimum and maximum at each point. However  $R_{1212} = \delta'$  in Chern's coordinates. This improves the estimate to  $32(\Delta'\delta' + 2\Delta'^2 + \frac{8}{9}(\Delta' - \delta')^2) \leq 32 \cdot 3\Delta'^2$  and gives  $\chi(M^4) \leq (\Delta/\delta)^2$  instead of  $\leq (\Delta/\delta)^2 + \frac{8}{27}((\Delta/\delta) - 1)^2$ . The same estimate improves known lower bounds for the volume in terms of the second Betti number:

$$2 + b_2 = \chi(M^4) \leq (4\pi^2)^{-1} \cdot 3\Delta^2 \cdot \text{vol}(M^4).$$

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