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Curvature and Differentiable Structure on Spheres

by ERNST A. RUH[1])

1. Introduction

An important problem in differential geometry is to characterize the global behaviour of a manifold in terms of local invariants. A result in this direction is given by the following isomorphism theorem: A simply connected, complete, riemannian manifold whose curvature tensor is close to the curvature tensor of the sphere is isomorphic to the sphere. Several versions of this theorem have been obtained in the past. The difference between these versions is a result of the difference in the meaning attached to the terms "close" and "isomorphic".

Traditionally, the proximity of the curvature tensors R and R_0 of the manifold M and the sphere S respectively has been measured in terms of sectional curvature as follows: A riemannian manifold whose sectional curvature K satisfies the condition $\delta < K \leq 1$ is called δ-pinched. In a series of papers Rauch [10], Berger [1, 2] and Klingenberg [6, 7] proved that a complete, simply connected, $\frac{1}{4}$-pinched riemannian manifold is homeomorphic to a sphere. With the discovery of exotic differentiable structures on spheres by Milnor [9], the question arose whether the homeomorphism theorem could be sharpened to a diffeomorphism theorem. Gromoll [5] and Calabi proved that this can be done if, at the same time, the sectional curvature is more severely restricted. Gromoll showed that there exists a sequence δ_n with $\lim \delta_n = 1$ as n tends to infinity such that a simply connected, complete, δ_n-pinched riemannian manifold of dimension n is diffeomorphic to the standard sphere S^n.

Calabi and Gromoll's diffeomorphism theorem leaves the following question open: Do there exist riemannian metrics on exotic spheres with curvature tensors R arbitrarily close to the curvature tensor R_0 of the standard sphere? To make this question more precise, we introduce a different measure for the proximity of R and R_0. For this purpose, we think of the curvature tensor as a linear, self adjoint map $R: V \wedge V \rightarrow V \wedge V$, where $V \wedge V$ denotes the exterior product of the tangent space with itself. So, if the eigenvalues λ of the map R at every point of M satisfy the condition $\delta < \lambda \leqslant 1$, then the manifold is called *strongly δ-pinched.* Now, the purpose of this paper is to answer the question left open by Calabi and Gromoll's diffeomorphism theorem: There are no riemannian metrics on exotic spheres with curvature tensors close to the curvature tensor of the standard sphere.

Here we might add that Bochner and Yano [3] in their version of the isomorphism

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theorem measured the proximity of R and R_0 in terms of strong δ-pinching. Bochner and Yano obtained the following result: A compact, orientable, strongly $\frac{1}{2}$-pinched riemannian manifold is a homology sphere.

2. The Main Result

In previous studies of the diffeomorphism theorem the pinching constant depended on the dimension of the manifold. However, the introduction of strong δ-pinching has the following advantage: The constant δ in the theorem below is independent of the dimension of the manifold.

THEOREM. *A complete, simply connected, strongly δ-pinched riemannian manifold of dimension n with $\delta = 0.66$ is diffeomorphic to the standard sphere S^n.*

The main idea of the following proof is new. However, methods similar to those employed by Rauch [10], Berger [1, 2], Klingenberg [6, 7], Gromoll [5], and Cheeger [4] have been adapted to obtain the necessary estimates. The pinching constant δ enters in several of these estimates. The constant $\delta = 0.66$ could be improved somewhat, but to keep nonessential complications at a minimum, no attempt has been made to obtain the optimal constant possible with our method.

3. Outline of Proof

We prove the theorem by constructing an explicit C^∞-diffeomorphism $f: M \to S^n$. In case M is a strictly convex hyper surface in euclidian space E^{n+1}, a diffeomorphism is provided by the Gauss map $g: M \to S^n$. The idea now is to pattern the construction of f after the Gauss map g. To carry this idea out we first recall what makes the Gauss map possible and why it is a diffeomorphism in this special case. The map g sending $x \in M$ into the unit normal vector at x translated to a fixed point x_0 is well defined because parallel translation in $E = M \times E^{n+1} = \tau(M) \oplus \nu(M)$, where $\tau(M)$ and $\nu(M)$ denote tangent and normal bundle respectively, is independent of the path. In addition, g is a diffeomorphism because in the special case under consideration the derivative $D_X n$ of the unit normal vector field n in any direction $X \neq 0$ is non zero.

In the general case the normal bundle is not available; however, we replace it by a trivial line bundle $\varepsilon(M)$ and define a flat connection ∇' on $E = \tau(M) \oplus \varepsilon(M)$. At this point, the map $f: M \to S^n$ is defined analogous to the Gauss map by replacing the normal vector field by a cross section e of length one in $\varepsilon(M)$; i.e., the image $f(x) \in S^n \subset E^{n+1}$ is defined by parallel translation of $e(x)$ to the fibre E^{n+1} over a fixed point x_0. Again, $f: M \to S^n$ is a local; and since M is simply connected, a global diffeomorphism as long as $\nabla'_X e \neq 0$. Therefore, the proof consists of defining a flat connection ∇' on $\tau(M) \oplus \varepsilon(M)$ and checking $\nabla'_X e \neq 0$.

The first step in the construction of ∇' is to define a connection ∇'' in E with small curvature as follows:

$$\nabla''_X e_i = \nabla_X e_i - c < X, e_i > e$$

$$\nabla''_X e = cX,$$

where ∇ denotes the riemannian connection in the tangent bundle $\tau(M)$; and e_i, $i = 1, 2, \ldots, n$ denotes a moving orthonormal frame in $\tau(M)$; while e is a section of length one in $\varepsilon(M)$; and c is a constant close to one to be determined later. The curvature of ∇'' will be estimated in section 4. We might add that the idea for the definition of ∇'' originates from the following observation: In case M is the standard sphere embedded in E^{n+1}, the covariant derivative defined above is nothing but the ordinary derivative in E^{n+1}.

In the next step, ∇'' is used to construct a cross section u' in the principal bundle of $(n+1)$-frames with structure group $0(n+1)$ associated to E. The results necessary for this construction are compiled in the first four chapters of [5]. The proofs in [5] are based on the Alexandrov-Rauch-Toponogov comparison theorem and the Morse critical point theory. In particular, we use the following properties: Let q_0 and q_1 be a pair of points with maximal distance $\varrho(q_0, q_1)$ on M, where ϱ denotes the distance function induced by the riemannian metric. Set $\chi(p) = \varrho(q_0, p) - \varrho(q_1, p)$ and define $C = \chi^{-1}(0)$, $M_0 = \chi^{-1}((-\infty, 0])$, $M_1 = \chi^{-1}([0, \infty))$. The exponential maps $\exp_0$ and $\exp_1$ with centers at q_0 and q_1 respectively are bijective maps if restricted to a ball of radius π. Finally, C is diffeomorphic to S^{n-1} and takes the place of the equator while M_0 and M_1 take the place of upper and lower hemisphere respectively.

At this point we are in a position to indicate the definition of the section u'. First, we define a section u_0 on M_0 by moving an $(n+1)$-frame $u_0(q_0)$ chosen over the center q_0 of M_0 by parallel translation with respect to ∇'' along geodesic rays to points in M_0. Second, we define $u_1(q_1)$ by parallel translation of $u_0(q_0)$ along a shortest geodesic to q_1, the center of M_1, Now u_1 is defined analogous to u_0. On $C = M_0 \cap M_1$ the cross sections u_0 and u_1 may not coincide, but the distance in the fibre can be estimated in terms of the pinching constant δ. Therefore, for δ close enough to 1, the sections u_0 and u_1 can be modified to yield a differentiable cross section u' on M. Finally, let ∇' denote the flat covariant derivative in $E = \tau(M) \oplus \varepsilon(M)$ that corresponds to the section u' in the associated principal bundle.

It remains to be shown that $\nabla'_X e \neq 0$. The result follows because for δ close to 1, the difference of ∇' and ∇'' is small; and $\|\nabla''_X e\| = c\|X\| \sim \|X\|$. The details, as well as the estimate $\delta = 0.66$ will be furnished in the subsequent sections.

4. The Connection ∇''

The purpose of this section is first, to give an estimate for the curvature of ∇'', second, to use this estimate to obtain an upper bound for the effect of parallel translation along null homotopic closed paths. We recall the definition of ∇'' introduced in the outline:

$$\nabla''_X e_i = \nabla_X e_i - c < X, e_i > e$$

$$\nabla''_X e = cX.$$

Instead of dealing with the connection ∇'', we prefer to make the computations in terms of its connection form ω''. In order to give a formula for ω'', let (A_{ij}) denote the standard basis of the Lie algebra $o(n+1)$ of the orthogonal group $0(n+1)$; and let ω denote the connection form of the given riemannian connection on M. In terms of a frame field $(e_1, \ldots, e_n, e)$, where $e_1, \ldots, e_n$ form an orthonormal basis in the tangent space $\tau(M)$, and e is a section of length one in $\varepsilon(M)$, ω'' can be expressed as follows:

$$\omega'' = \omega + c \sum e_i^* \otimes A_{in+1} = \omega + c\alpha,$$

where e_i^* denotes the dual of e_i and α is defined by the above equation. To simplify the computation of the curvature, we assume further that $e_1, \ldots, e_n$ arises from a canonical coordinate system around a point $p \in M$. In addition, the connection form ω'', as well as the curvature form Ω'', will be considered as forms on M, rather than on the principal bundle P; i.e., we will deal with the pullback via the section $(e_1, \ldots, e_n, e)$.

At this point, we compute the curvature form Ω'' at $p \in M$. Because of the choise of a canonical coordinate system at p we have $d\omega'' = d\omega$. Therefore, the Cartan identity yields the following equation:

$$d\omega'' = -\omega'' \wedge \omega'' + \Omega'' = -\omega \wedge \omega + \Omega = d\omega.$$

Again, because of the choice of the section $(e_1, \ldots, e_n)$, ω is zero at $p \in M$; therefore, the above equation yields:

$$\Omega'' = \Omega + c^2 \alpha \wedge \alpha.$$

Now, both Ω'' and Ω may be considered maps from $V \wedge V$ into $o(n+1)$, since $o(n)$ is a a subspace of $o(n+1)$. Note that in the case where $M = S^n$, and $c = 1$, the map $\Omega'': V \wedge V \to o(n+1)$ is identically zero. This explains why the pinching condition implies that Ω'' is close to zero.

We arrive at an estimate of Ω'' by letting $\|\Omega''\|$ denote the maximum of $\|\Omega''(\beta)\|$ where β ranges over the unit sphere in $V \wedge V$. In addition, we identify $V \wedge V$ with $o(n)$ by means of the map $e_i \wedge e_j \to A_{ij}$, where A_{ij}, $i < j \leqslant n$ denotes the standard basis of $o(n)$. Under this identification, $\alpha \wedge \alpha$ corresponds to the identity map $id: V \wedge V \to$ $\to V \wedge V$ as indicated in the following:

$$\alpha \wedge \alpha(e_i, e_j) = [A_{in+1}, A_{jn+1}] = A_{ij},$$

where [,] denotes the Lie bracket in $o(n+1)$, and A_{ij} for $i<j\leq n+1$ denotes the standard basis of $o(n+1)$, while A_{ij} for $j<i$ stands for $-A_{ji}$. Note that, under the above identification of $V\wedge V$ with $o(n)$, the curvature form Ω of the riemannian connection coincides with $-R$, the negative of the curvature transformation $R\colon V\wedge V\to V\wedge V$. Therefore, the eigenvalues λ'' of the map $\Omega''=\Omega+c^2\alpha\wedge\alpha$ can be estimated as follows:

$$-1+c^2\leqslant\lambda''\leqslant-\delta+c^2,$$

where δ is the pinching constant. The best estimate of $\max|\lambda''|=\|\Omega''\|$ is obtained by setting $c^2=\frac{1}{2}(1+\delta)$. The result is: $\|\Omega''\|\leqslant\frac{1}{2}(1-\delta)$.

In the second half of this section, we apply this result to obtain an estimate for the effect of parallel translation along null homotopic, piecewise differentiable, closed paths. Let $H\colon I\times I\to M$ be a piecewise differentiable homotopy of the closed path γ defined by $s\to H(s,0)$ into the constant path $s\to H(s,1)$. Parallel translation along γ transforms the frame u into the frame ua, where ua is the image under right multiplication of the frame u by the element $a\in 0(n+1)$. We achieve an estimate for the distance $\varrho(e,a)$ of the element $a\in 0(n+1)$ from the identity element $e\in 0(n+1)$ in terms of the homotopy H with the following lemma:

LEMMA. *$\varrho(e,a)\leqslant\max\|\Omega''\|A\leqslant\frac{1}{2}(1-\delta)A$, where the maximum is taken over points in $H(I\times I)\subset M$; and A denotes the area of $H(I\times I)$.*

Of course, the distance $\varrho(e,a)$ depends on the metric in $0(n+1)$. However, this metric has been normalized such that the space $0(n+1)/0(n)$ is isometric to the unit sphere.

The proof is straightforward. The idea is the same as in the proof of the factorization lemma [8, p. 285]. Namely, we subdivide $I\times I$ into m^2 squares s_{ij} of equal size, and write parallel translation along γ in terms of parallel translation along the boundaries $\gamma_{ij}=\partial H(s_{ij})$ of the images $H(s_{ij})$ of the squares s_{ij}. Now let a_{ij} be the orthogonal map defined by parallel translation along γ_{ij}. Neglecting terms of higher order in A_{ij}=area of $H(s_{ij})$, we obtain the estimate $\varrho(e,a_{ij})\leqslant\|\Omega''\|\cdot A_{ij}$ for the distance $\varrho(e,a_{ij})$ of a_{ij} from the identity. Therefore, the factorization lemma yields $\varrho(e,a)\leqslant\leqslant\max\|\Omega''\|A$. The correction terms of higher order in A_{ij} can be neglected if the number of subdivisions of $I\times I$ is increased to achieve $A_{ij}\to 0$. Finally, the estimate $\|\Omega''\|\leqslant\frac{1}{2}(1-\delta)$ completes the proof.

5. A Preliminary Estimate

In section 3, the manifolds M_0 and M_1 were introduced and compared to the upper and lower hemispheres respectively of the standard sphere. In this section, the metric aspects of this comparison will be studied. The estimate obtained here is based on results compiled in the first four chapters of [5]. This estimate is necessary for the construction of ∇', which will be carried out in the next section.

In case $M=S^n$, we have the following isometry between the tangent spaces at the north and south poles; Source and image are related by lying on the same geodesic joining north and south poles. In case M is δ-pinched with $\delta>\frac{4}{9}$, Gromoll [5] proved the existence of a diffeomorphism h similar to the above map. To define h, let S_0 and S_1 denote unit spheres in the tangent space of the points q_0 and q_1 that were introduced in section 3. The map $h:S_0\to S_1$ is now defined by requiring $\exp_0 tx$ and $\exp_1 th(x)$ to coincide for some $t=t(x)$ satisfying $\pi/2<t(x)<\pi/(2\sqrt{\delta})$. Note that the point of intersection lies on the "equator" C defined in section 3.

At this point, we prove that the map h is close to an isometry; i.e., we give an estimate for the ratio $\|h_*Y\|:\|Y\|$ for $Y\in\tau(S_0)$, the tangent bundle of the unit sphere S_0 in the tangent space at $q_0\in M_0$. Here we recall that the definition of the map h implies $\|\exp_{0*}t(x)\,Y\|=\|\exp_{1*}t(x)\,h_*Y\|$, where $\exp_{0*}$ is the differential of the exponential map $\exp_0$ evaluated at a point $t(x)\,x$ for some $x\in S_0$. Now, the above equation, together with the Rauch comparison theorem, yields the following estimate:

$$\left(\sqrt{\delta}\sin\frac{\pi}{(2\sqrt{\delta})}\right)^{-1}\geqslant\|h_*Y\|:\|Y\|\geqslant\sqrt{\delta}\sin\frac{\pi}{(2\sqrt{\delta})}\cdot$$

We arrive at the estimate when we observe that the extreme ratio would occur if the sectional curvatures of M_0 and M_1 would be equal to δ and 1 respectively. Therefore, a comparison of the exponential maps $\exp_0$ and $\exp_1$ to exponential maps on spheres of radius $1/\sqrt{\delta}$ and 1 respectively, yields the estimate.

6. The Connection ∇'

The purpose of this section is to construct a flat connection ∇' on the bundle $E=\tau(M)\oplus\varepsilon(M)$ with the property $\nabla'_X e\neq 0$, where e denotes a section of length one in $\varepsilon(M)$; and X denotes a non zero tangent vector. As stated in the outline, ∇' is obtained by constructing a cross section $u':M\to P$ in the principal bundle P associated to E. Again, we follow the outline and define $u_0:M_0\to P$ and $u_1:M_1\to P$ as in section 3. The point now is to modify u_0 and u_1 to obtain a smooth section $u':M\to P$.

The sections u_0 and u_1, restricted to $M_0\cap M_1$ in general do not coincide. The idea is to replace u_0 and u_1 by their average. Since the average is defined only if $\varrho(u_0,u_1)<\pi$ we estimate the distance between $u_0(p)$ and $u_1(p)$ for $p\in C$. We recall that the frame $u_1(p)$ is obtained form $u_0(p)$ by parallel translation with respect to ∇'' of $u_0(p)$ along the closed path γ consisting of the shortest geodesic segments (p,q_0), (q_0,q_1), and (q_1,p). In order to obtain an estimate for the distance $\varrho(u_0,u_1)$ by means of the lemma of section 4, it is necessary to define a homotopy of the broken geodesic

(q_0, p, q_1) into the shortest geodesic (q_0, q_1) that was used in the definition of u_1, In addition, we need an estimate fo the area of this homotopy.

Instead of defining the homotopy on M_0 and M_1 directly, we define a homotopy in their inverse images, T_0 and T_1, under $\exp_0$ and $\exp_1$ respectively. In T_0 we define the homotopy by rotating the line corresponding to the geodesic (q_0, p) into the line corresponding to the geodesic (q_0, q_1). On T_1, we define the homotopy again in terms of a family of lines originating from $0 \in T_1$, The choice of the family is determined since we require the images of the homotopies under $\exp_0$ and $\exp_1$ respectively to match on $C = M_0 \cap M_1$. We estimate the area swept out by the homotopy on M_0 by means of the Rauch comparison theorem. For an estimate of the corresponding area on M_1 we need, in addition, the estimate on $\|h_* Y\| : \|Y\|$ which has been obtained in the preceeding section.

These considerations lead to the following estimate of the area A of the above homotopy:

$$A \leqslant \frac{\pi}{\delta}\left(1 + \left(\sqrt{\delta}\sin\frac{\pi}{2\sqrt{\delta}}\right)^{-1}\right).$$

With this upper bound for A, the lemma of section 4 provides an estimate for the distance $\varrho(u_0, u_1)$ of u_0 and $u_1 = u_0 a$, where $a \in 0(n+1)$; and the distance is measured in the fibre over points in $C = M_0 \cap M_1$. The result is:

$$\varrho(u_0, u_1) = \varrho(e, a) \leqslant \pi\,\frac{1-\delta}{2\delta}\left(1 + \left(\sqrt{\delta}\sin\frac{\pi}{2\sqrt{\delta}}\right)^{-1}\right).$$

If we choose δ close enough to 1 to make sure that $\varrho(e, a) < \pi$, then there is a unique shortest geodesic joining e and a in $0(n+1)$. Consequently, the average of u_0 and u_1 exists. In the next paragraph, we illustrate how this average leads to an approximation $u^* : M \to P$ of the section $u' : M \to P$.

We begin with the definition of u^* restricted to $C = M_0 \cap M_1$, by sending a point $q \in C$ into the midpoint of the shortest geodesic joining $u_0(q)$ and $u_1(q)$ in the fibre over q. Subsequently, we extend the definition of u^* to M. One might attempt to extend the definition of u^* to M_0 and M_1 by parallel translation of $u^*(q)$, $q \in C$, with respect to ∇'' along geodesic rays originating from q_0 and q_1 respectively. However, this extension of u^* results in singularities at the points q_0 and q_1. To avoid this difficulty, we modify parallel translation as follows: Instead of parallel translation of $u^*(q)$ along geodesics $\exp_0 sx$, we translate $u^* \exp d^{\#}(p)\, B$ along the same geodesic to the point $p = \exp_0 d(q)\, x$, where $d(p)$ is the distance from q_0 to p, while $B = B(q) \in o(n+1)$ is defined by the equation $u_0(q) \exp d(q)\, B = u^*(q)$; and $d^{\#} : M_0 \to \mathbf{R}$ is obtained by smoothing d at q_0 while keeping $d^{\#}(q_0)=0$. The above completes the definition of $u_0^*=u^*:M_0\to P$. Now, to extend u^* to a cross section $u_1^*=u^*:M_1\to P$ we utilize the same method. The section $u^*:M\to P$, as constructed satisfies all but one of the requirements of the section u' listed in the outline, i.e., u^* is not differentiable on C. In the next paragraph, we smooth u^* to obtain a section $u':M\to P$.

In order to define u' we first extend the definitions of u_0^* and u_1^*, so far defined on M_0 and M_1 respectively, to a tubular neighborhood $N_\varepsilon(C)$ of $C=M_0\cap M_1$. Subsequently, the following symbolic formula indicates the definition of u' on $N_\varepsilon(C)$:

$$u' = (1-t)\,u_0^* + ty_1^*,$$

where the function $t:N_\varepsilon(C)\to\mathbf{R}$ will be defined later. Here we define u' as the point on the geodesic joining u_0^* and u_1^* whose respective distances from u_0^* and u_1^* we determine by the ratio $t:(1-t)$. Outside $N_\varepsilon(C)$, the sections u^* and u' are identical. For the proper choice of the function t, the section u' is differentiable. It remains to be shown that the connection ∇' associated to u' satisfies the property $\nabla'_X e\neq 0$ discussed in the outline.

7. Some Estimates Concerning ∇'

The purpose of this section is to prove that $\nabla'_X e\neq 0$, where e is a section of length one in $\varepsilon(M)$; and X is any non zero vector in the tangent bundle $\tau(M)$. Since $\|\nabla''_X e\|\sim\|X\|$ for δ close to 1, it suffices to show that the difference of ∇'' and ∇' is small, provided that δ is close enough to 1.

In order to simplify the computations involved in the estimate of the difference between ∇'' and ∇' in a neighborhood U of a point $p\in M$, we identify the bundle P restricted to U with $U\times 0(n+1)$. We accomplish this by identifying $U\times\{e\}$ with the following section s'' adapted to ∇'': The section s'' is defined by parallel translation with respect to ∇'' along geodesic rays of an $(n+1)$-frame over $p\in M$. Subsequently, we identify the cross section u' that defines ∇' with the corresponding map $u':U\to 0(n+1)$. Because s'' is adapted to ∇'', the following estimate for the difference between ∇'' and ∇' at $p\in M$ holds: $\|\nabla''_X s-\nabla'_X e\|\leqslant\|Du'X\|$, where Du' denotes the differential of the map u'.

Instead of dealing with the estimate of Du' directly, we begin by showing that $\|Du^*X\|$ and $\|Du'X\|$ satisfy the same inequality and subsequently estimate $\|Du^*X\|$. To compare Du' and Du^*, we differentiate the symbolic formula $u'=(1-t)\,u_0^*+tu_1^*$ by means of the product rule as if it would be an actual formula. Since we are only interested in the first derivative this is acceptable because it is possible to give the above formula a precise meaning by replacing the orthogonal group by its tangent space $o(n+1)$ at $e\in 0(n+1)$. This being understood, we gain the following expression for the derivative Du' of u':

$$Du' = Dt(u_1^* - u_0^*) + (1-t)\, Du_0^* + tDu_1^*,$$

and therefore

$$\|Du'X\| \leqslant \|DtX\|\, \varrho(u_0^*, u_1^*) + \max(\|Du_0^*X\|, \|Du_1^*X\|).$$

To complete the argument, it suffices to show that $\|DtX\|\, \varrho(u_0^*, u_1^*)$ is arbitrarily small if we choose the function $t: N_\varepsilon(C) \to \mathbf{R}$ as follows: We begin by identifying $N_\varepsilon(C)$ with $[-\varepsilon, \varepsilon] \times C$ and we define t to be constant on $\{z\} \times C$. On the z-axis we define t to be the integral of the function $A(z)$, where $A(z)$ is basically the function $A/|z|$ but modified such that $A(z)$ is differentiable on $[\varepsilon, \varepsilon]$; $A(z)$ is tangent to the z-axis at $-\varepsilon$ and ε, and satisfies the property $\int_{-\varepsilon}^{\varepsilon} A(z)\, dz = 1$. Since we choose the constant A small, and since $\varrho(u_0^*, u_1^*)$ on $\{z\} \times \{q\}$ with $q \in C$ is bounded by $c\,|z|$ for some constant c, the quantity $\|Dt\, X\|\, \varrho(u_0^*, u_1^*)$ is negligeable.

Given the preceding remark, we reduced the problem of estimating $\|\nabla''_X e - \nabla'_X e\|$ to estimating $\|Du^*X\|$, where $u^*: M \to P$ in a neighborhood U of a point $p \in M$ is identified with $u^*: U \to 0(n+1)$ be means of the section s'' as previously defined. Now, in order to estimate $\|Du^*X\|$ we estimate the larger quantity $\|Dv^*X\|$, where v^* is the map $v^* = (\exp)^{-1} u^*: M \to o(n+1)$; i.e., the composition of u^* with the inverse of the exponential map $\exp: o(n+1) \to 0(n+1)$. Of course, since exp decreases distances, $\|Du^*X\|$ is smaller than $\|Dv^*X\|$.

Now we are prepared to estimate $\|Dv{*}X\|$ in terms of a canonical coordinate system with center $q_0 \in M_0$. The following estimate holds for points in M_0 only; however, the method works for M_1 as well. We begin with an estimate for $\|Dv^*X\|$, where X points in radial direction. Subsequently, we estimate $\|Dv^*X\|$ for vectors X pointing in angular direction by a similar method. To simplify the computations, we now substitute $\delta = 0.66$. This is not the best possible value for δ, but the will observe later that $\delta = 0.66$ cannot be improved by much.

With the substitution of $\delta = 0.66$ in the estimate for $\varrho(e, a)$ of section 6, we obtain the following numerical value: $\varrho(e, a) < 0.60\pi$. Keeping in mind that the distance of C from q_0 is at least $\pi/2$, and that u^* was defined on C by taking the average of u_0 and u_1, we conclude that $\|Dv^*X\| < 0.60\|X\|$ if X points in radial direction. A similar estimate shows that $\|Dv^*X\| < 0.67\ \|X\|$ for X in angular direction. We obtain the above numerical value by estimating the derivative of the composition $v^* \exp: T_0 \to$ $\to o(n+1)$, where T_0 denotes the tangent space of M at q_0. Of course, we have to take into account that $\exp_0$ decreases distances. However, the decrease is bounded from below by the factor $(2\sqrt{\delta}/\pi)\, \sin({}^{\pi}/_{2\sqrt{\delta}})$. Now we combine the above two inequalities and obtain: $\|Dv^*X\| < 0.90\ \|X\|$ for arbitrary direction of X.

Here, we recall that the above estimate of Dv^* yields $\|\nabla''_X e - \nabla'_X e\| < 0.90\ \|X\|$. On the other hand, we obtain from the definition of ∇'' that $\|\nabla''_X e\| \sim 0.91\ \|X\|$. Finally, the triangle inequality implies $\|\nabla'_X e\| \neq 0$ for non zero vectors X.

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