Embeddings of Open Riemann Surfaces

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1. Embedded Surfaces

1.1. In the final section of his famous thesis Riemann states that in his investigations the branched covering surfaces of the plane could be replaced by smooth orientable surfaces embedded in Euclidean space. For the metric structure induced by the surrounding space can be used to define a complex structure in the following natural way: The admissible local parameters are those which preserve angles and orientation.

1.2. We will call C^∞-embedded surfaces *classical surfaces* if they are viewed as Riemann surfaces in this way. The existence of the admissible local parameters is highly non-trivial, it means solving the Beltrami differential equation. This was done for analytic embeddings by Gauss, for differentiable embeddings by Korn-Lichtenstein. Because of the fundamental importance of this problem in the theory of quasiconformal mappings, it was investigated more thoroughly in recent years. For a most elegant treatment see [3], for an elementary one see [10].

2. The Embedding Problem

2.1. In his lectures Felix Klein emphasized the concept of viewing classical surfaces as Riemann surfaces, i.e., domains of analytic functions and integrals. It was also he who asked in 1882 if every Riemann surface were conformally equivalent to a classical surface. [F. Klein, *Gesammelte mathematische Abhandlungen*, Bd. 3, (Springer 1923), p. 502 and p. 635.]

2.2. For a long time the only results in this direction were that every compact Riemann surface of genus zero is conformally equivalent to the sphere, every noncompact planar (schlichtartig) surface is conformally equivalent to a subregion of the plane, and a compact Riemann surface of genus 1 is conformally equivalent to a ring surface provided its modulus is purely imaginary (see [16]).

2.3. The first result beyond these facts was obtained by Teichmüller in [15], where he applied his theory of spaces of Riemann surfaces to the embedding problem. He

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could show that not all compact embedded surfaces of genus 1 are conformally equivalent to ring surfaces. More important than this result was the method by which he obtained it: He deformed an embedded surface by moving each point along the normal line and studied the dependence of the modulus of the deformed surface on the deformation.

2.4. Around 1960 Garsia constructed a surprisingly large class of compact Riemann surfaces whose moduli could be determined ([5], [6]). But he succeeded in answering Klein's question in the affirmative for all compact Riemann surfaces only when he abandoned his beautiful models and embarked on Teichmüller's road. His proof in [7] and [8] is an ingenious combination of Teichmüller's ideas and results, constructions inspired by Nash's isometric embeddings, and Brouwer's fixed point theorem.

2.5. We will see in this paper that his methods are even strong enough to prove this theorem for noncompact surfaces too. Because we will use the modifications described in [13] we may formulate our theorem as follows:

EMBEDDING THEOREM. *Every Riemann surface R is conformally equivalent to a complete classical surface. A model can be constructed by deforming any topologically equivalent complete classical surface X in the direction of the normals.* (See also 6.12.) X is complete, if X is a closed subset of the Euclidean space.

2.6. A nontrivial corollary (due to R. Osserman) follows, if R is the unit disc and $X=\mathbf{C}$:

For a suitable real-valued C^∞-function f the classical surface represented by

$$(x, y) \to (x, y, f(x, y)), \quad x + iy \in \mathbf{C},$$

is hyperbolic.

2.7. Comparing the proofs in this paper and in [14] it is evident that every Riemann surface is conformally equivalent to an embedded polyhedral surface.

3. Outline of the Proof

3.1. We may assume that R is noncompact because the Embedding Theorem is known to be true otherwise ([13]).

3.2. The first step is to find a topologically equivalent complete classical surface X and a topological mapping $f' : R \to X$. We choose on R a regular exhaustion (see [4]), i.e., a sequence $\{R_i\}$ of relatively compact regular subregions, such that $\bar{R}_i \subset\subset R_{i+1}$, $\bigcup R_i = R$, and ∂R_i consists of analytic arcs. It is easy to show that R_i can be mapped by f_i' topologically onto a classical surface X_i such that ∂X_i consists of circles contained in ∂B_i where $B_i = \{(x, y, z) \mid x^2 + y^2 + z^2 \leqslant i^2\}$, $X_i \subset B_i$, $X_{i+1} \cap B_i = \bar{X}_i$ and $f'_{i+1}|_{R_i} = f_i'$. $f' = \lim f_i'$ and $X = \bigcup X_i$ satisfy the above conditions.

3.3. We may assume that R_1 is a disc. Let $p \in R_1$ and $q \in \partial R_1$ be distinguished points and put $p' = f'(p)$, $q' = f'(q)$ and $f'(R_i) = X_i$. If R is simply connected, we introduce four distinguished points.

3.4. We will deform X in successive steps such that the i-th deformation ($i \geqslant 2$) takes place on $X_i - X_{i-1}$ only, and we will denote the resulting surface by X'. Let X_i' be the part of X' corresponding to X_i. We will show that R_i can be mapped conformally onto X_i' by a mapping f_i with the additional properties $f_i(p) = p'$, $f_i(q) = q'$, $i \geqslant 1$. The existence of f_1 follows by Riemann's mapping theorem, the existence of f_i, $i \geqslant 2$, will be proved by induction.

3.5. If this is accomplished, our theorem is implied by the following

LEMMA 1. *Let $\{R_i\}$ and $\{X_i\}$ be exhaustions of the noncompact Riemann surfaces R and X, and let p and q be fixed points in R, p' and q' be fixed points in X. If the mappings $f_i: R_i \to X_i$ are conformal and if $f_i(p) = p'$, $f_i(q) = q'$, $i \geqslant i_0$, then R and X are conformally equivalent.*

3.6. This is a generalization of a theorem which is well-known, if R and X are planar surfaces (see e.g. [11], p. 76). In order to reduce it to this case, we look at the (standard) universal coverings of R and X. Their elements are homotopy classes $[\gamma]$ of arcs γ whose initial points are p and p' respectively. We define

$$\tilde{R}_i = \{[\gamma] \mid \exists \gamma_1 \in [\gamma] \quad \text{with} \quad |\gamma_1| \subset R_i\}$$

where $|\gamma|$ denotes the point set corresponding to γ. $\tilde{X}_i$ is defined in the same way. $\tilde{R}$ and $\tilde{X}$ are conformally equivalent to subregions U_R and U_X of the plane, and $\tilde{R}_i$ and $\tilde{X}_i$ may be viewed as subregions of U_R and U_X. The functions $\tilde{f}_i: \tilde{R}_i \to \tilde{X}_i$ defined by $\tilde{f}_i([\gamma]) = [f_i \circ \gamma]$ are conformal liftings of f_i and form a normal family, because the distances between $\tilde{f}_i(\tilde{p}) = \tilde{p}'$ and the points $\tilde{f}_i(\tilde{q})$ on U_X are bounded away from zero. Therefore a suitable subsequence converges uniformly on compact sets to a conformal mapping $\tilde{f}: U_R \to U_X$ and so do their projections f_i. The limit function $f: R \to X$ is onto because $\tilde{f}$ is and it is univalent because of the uniform convergence of the f_i on compact subsets and the fact that each f_i is univalent. Our lemma is proved.

4. Teichmüller Spaces of Bordered Riemann Surfaces

4.1. In this section we recall the results about Teichmüller spaces which are necessary for the construction of the deformed surfaces X_i'. For details and proofs see [1], [2] and [8].

4.2. Let S be a bordered Riemann surface with two distinguished points p and q (four if S is simply connected) and a finite number of handles and a positive finite number of boundary curves. Double the surface with respect to all boundary curves and finally go over to the two-sheeted covering surface S^* of this double with branch points at the distinguished points and their doubles.

4.3. S^* is a compact Riemann surface whose universal covering surface is the upper halfplane H. The group G of decktransformations with respect to S^* is a subgroup of index 2 of the group Γ of decktransformations with respect to the double of S. Denote by $j: H \to H$ an anticonformal involution whose projection on S^* maps each point onto its double. $\gamma \to j \circ \gamma \circ j$ is an automorphism of both G and Γ.

4.4. A quasiconformal mapping $f: S \to S_1$ can be extended in an obvious way to a mapping $f^*: S^* \to S_1^*$ and lifted to H in such a way that the lifting $\tilde{f}: H \to H$ satisfies

$$\tilde{f} \circ j = j_1 \circ \tilde{f} \tag{1}$$

and such that

$$\tilde{f} \circ \gamma = \gamma_f \circ \tilde{f} \tag{2}$$

defines isomorphisms $\theta_f(\gamma) = \gamma_f$ of G onto G_1 and Γ onto Γ_1.

4.5. For the complex dilatation $\mu = \tilde{f}_{\bar{z}} / \tilde{f}_z$ the above equations imply that

$$\mu(j(z)) \frac{\overline{j_{\bar{z}}}}{j_{\bar{z}}} = \overline{\mu(z)} \tag{3}$$

and

$$\mu(\gamma(z)) \frac{\overline{\gamma'(z)}}{\gamma'(z)} = \mu(z) \qquad \forall \gamma \in \Gamma. \tag{4}$$

In particular μ is a Beltrami differential with respect to G (i.e., on S^*).

4.6. Conversely if $\tilde{f}$ is a global solution of the Beltrami differential equation $w_{\bar{z}} = \mu w_z$ mapping H onto itself and if μ satisfies (3) and (4), then $\tilde{f}$ satisfies (1) and

(2) for suitable groups G_1 and Γ_1 and induces a quasiconformal mapping $f[\mu]$ of S onto a bordered Riemann surface S_1.

4.7. DEFINITION. *The Teichmüller space $T(S)$ is the set of equivalence classes $[f]$ of quasiconformal orientation-preserving mappings $f: S \to f(S)$ with respect to the following equivalence relation: $f_1 \sim f_2$ if and only if $f_2^* \circ f_1^{*-1}$ is homotopic to a conformal mapping g^* of $f_1^*(S^*)$ onto $f_2^*(S^*)$.*

Remark. g^* automatically induces a conformal mapping $g: f_1(S) \to f_2(S)$ which is homotopic to $f_2 \circ f_1^{-1}$ and maps $f_1(p)$ onto $f_2(p)$ and $f_1(q)$ onto $f_2(q)$.

4.8. $T(S)$ is a metric space with respect to the Teichmüller distance d defined by

$$d([f_1],[f_2]) = \inf_{g \simeq f_2^* \circ f_1^{*-1}} (\sup D_g(z),$$

$$D_g(z) = \frac{|\tilde{g}_z(z)| + |\tilde{g}_{\bar{z}}(z)|}{|\tilde{g}_z(z)| - |\tilde{g}_{\bar{z}}(z)|} \quad (= \text{dilatation of } \tilde{g} \text{ at } z),$$

and at the same time it is a manifold. Local parameters may be defined using the spaces of Beltrami differentials $B(S)$ and holomorphic quadratic differentials $Q(S)$. But in order to do so, some preparations are necessary.

4.9. We introduce the following notations where φ represents holomorphic and μ measurable and essentially bounded functions in H:

$$Q(S^*) = \{\varphi \mid (\varphi \circ \gamma) \cdot (\gamma')^2 = \varphi \quad \forall \gamma \in G\},$$

$$B(S^*) = \{\mu \mid (\mu \circ \gamma) \cdot \bar{\gamma}'/\gamma' = \mu \quad \forall \gamma \in G\},$$

$$N(S^*) = \{\mu \in B(S^*) | \iint_{H/G} \mu(z)\, \varphi(z)\, dx\, dy = 0 \quad \forall \varphi \in Q(S^*)\}.$$

The elements of $N(S^*)$ are called trivial Beltrami differentials. Analoguously we define

$$Q(S) = \{\varphi \mid (\varphi \circ \gamma) \cdot (\gamma')^2 = \varphi \quad \forall \gamma \in \Gamma, (\varphi \circ j) \cdot (j_{\bar{z}})^2 = \bar{\varphi}\},$$

$$B(S) = \{\mu \mid (\mu \circ \gamma) \cdot \overline{\gamma'}/\gamma' = \mu \quad \forall \gamma \in \Gamma, (\mu \circ j) \cdot \overline{j_{\bar{z}}}/j_{\bar{z}} = \bar{\mu}\},$$

$$N(S) = B(S) \cap N(S^*).$$

4.10. The mapping

$$\Lambda^*: B(S^*)/N(S^*) \to Q(S^*)$$

defined by

$$\mu \mapsto -\frac{6}{\pi i} \iint\limits_H \frac{(\bar{\mu} z)}{(\bar{z}-\zeta)^4}\, dz \wedge d\bar{z} = \varphi(\zeta)$$

is an **R**-linear isomorphism, its inverse is given by

$$\varphi \mapsto -\bar{\varphi} \cdot (\operatorname{Im} z)^2 .$$

4.11. LEMMA 2. *Λ^* induces an isomorphism*

$$\Lambda : B(S)/N(S) \to Q(S).$$

4.12. *Proof.* If $\mu \in B(S)$ and $\varphi = \Lambda^*(\mu)$, then

$$\varphi(j(\zeta)) = -\frac{6}{\pi i} \iint\limits_H \frac{\bar{\mu}(z)\, dz \wedge d\bar{z}}{(\bar{z}-j(\zeta))^4} = \frac{6}{\pi i} \iint\limits_H \frac{\bar{\mu}(j(z))\, dj \wedge \overline{dj}}{(\overline{j(z)}-j(\zeta))^4},$$

because j is an anticonformal mapping of H onto itself. Applying the identity

$$(\gamma(z_1)-\gamma(z_2))^2 = (z_1-z_2)^2\, \gamma'(z_1)\, \gamma'(z_2)$$

to the Möbius transformation $\gamma = -\bar{j}$, we obtain

$$\varphi(j(\zeta)) = \frac{6}{\pi i} \iint\limits_H \frac{(\mu(z)\, \bar{j}_{\bar{z}}/j_{\bar{z}})\, (-j_{\bar{z}} \bar{j}_{\bar{z}}\, dz \wedge d\bar{z})}{(z-\bar{\zeta})^4\, \bar{j}_{\bar{z}}^2 \cdot j_{\bar{z}}(\zeta)^2} = \overline{\varphi(\zeta)} \cdot j_{\bar{z}}(\zeta)^{-2} .$$

$$\varphi(\gamma(z))\, \gamma'(z)^2 = \varphi(z) \quad \forall \gamma \in \Gamma$$

is proved by similar computations.

4.13. If $\varphi \in Q(S)$ and $\mu = \Lambda^{*-1}(\varphi)$, then

$$\mu(j(z)) = -\bar{\varphi}(j(z)) \cdot (\operatorname{Im} j(z))^2 = -\varphi(z) \cdot \overline{j_{\bar{z}}^{-2}} \cdot (\operatorname{Im} z)^2 \cdot j_{\bar{z}} \cdot \bar{j}_{\bar{z}} = \bar{\mu}(z) \cdot j_{\bar{z}} \cdot \overline{j_{\bar{z}}^{-1}}$$

and

$$\mu(\gamma(z)) = \mu(z)\, \gamma'(z)/\overline{\gamma'(z)}$$

by a similar computation, and Lemma 2 is an immediate consequence.

4.14. Now we are ready to define local parameters in $T(S)$. Let $\mu_1, \ldots, \mu_N$ be a basis of $B(S)/N(S)$. The mapping defined by

$$(t_1, \ldots, t_N) \to \left[f\left[\sum_{i=1}^N t_i \mu_i \right] \right]$$

maps a ball in $\mathbf{R}^N$ topologically onto a neighborhood of $[f_0 = \mathrm{id}]$ in $T(S)$, and $(t_1, \ldots, t_N)$ shall be our local parameters (see [2], pp. 137–145).

4.15. LEMMA 3. *If* $\mu \in B(S)$ and $\varphi \in Q(S)$ *then* $(\mu, \varphi) = \iint_H \mu\varphi \, dx \, dy$ *is real.*

Proof.

$$(\mu, \varphi) = \iint\limits_{H/G} \mu\varphi \frac{dz \wedge d\bar{z}}{2i} = \iint\limits_{j(H)/G} \mu\varphi \frac{dz \wedge d\bar{z}}{2i} = -\iint\limits_{H/G} \mu(j(z))\, \varphi(j(z)) \frac{d\bar{z} \wedge \overline{dj}}{2i}$$

$$= -\iint\limits_{H/G} \overline{\mu(z)} \frac{\overline{j_{\bar{z}}}}{j_{\bar{z}}} \bar{\varphi}(z)\, j_{\bar{z}}^{-2} j_{\bar{z}} \bar{j}_{\bar{z}} \frac{d\bar{z} \wedge dz}{2i} = \overline{(\mu, \varphi)} \qquad \text{q.e.d.}$$

4.16. Let U be any parametric disc in S with $p \notin \bar{U}$, $q \notin \bar{U}$, and denote by U^* the corresponding open set in S^* which consists of four discs. Using Lemma 3 and the definition of $N(S)$, we obtain easily the following

COROLLARY. *There is a basis* $\{\mu_i\}$ *of* $B(S)/N(S)$, *where the* μ_i *are* C^∞*-functions whose support is contained in* U^*.

4.17. *Proof.* We have to find a basis $\{\varphi_i\}$ of $Q(S)$ and $\{\mu_i\}$ as above such that $(\mu_i, \varphi_j) = \delta_{ij}$ for $i \leqslant j$, which can easily be done by induction. The construction of a similar basis was described more closely by T. Klotz in [9].

4.18. Good estimates for the distance of two points in $T(S)$ will be crucial. The following lemma which serves these purposes is due to Garsia. In order to formulate it, we have to fix in H a fundamental domain D_S for the group generated by Γ and j. Denote the restriction of the projection to D_S by π_S. Assume that $(t_1, \ldots, t_N) = t$ are local parameters for a neighborhood of [id] in $T(S)$ provided

$$\|t\| = \sqrt{\sum_{i=1}^{N} t_i^2} \leqslant 2r.$$

Let B_r be the set of elements in $T(S)$ corresponding to $\|t\| \leqslant r$ and write $\varphi[f] = t$, if $[f]$ corresponds to t.

4.19. DEFORMATION LEMMA. *If* $[f] \in B_r$ *and if there is a quasiconformal mapping*

$$\chi : f(S) \to g(S), \quad \chi^* \simeq g^* \circ f^{*-1},$$

whose dilatation D_χ *satisfies*

1) $D_\chi \leqslant K_0$,
2) $D_\chi \leqslant 1+\delta$ *except on* $A \subset f(S)$,
3) $|\pi_S^{-1} \circ f^{-1}(A)| \leqslant \eta$

($|\dots|$ *denotes the areal measure*), *then*

$$\|\varphi[g] - \varphi[f]\| \leqslant b(K_0, \delta, \eta),$$

where $b(K_0, \delta, \eta) \to 0$ *if* K_0 *is bounded while* $(\delta, \eta) \to (0, 0)$. The proof is given in [8], pp. 100–102.

5. The Existence of the Functions f_i

5.1. Let $\{R_i\}$ be the exhaustion of R, $\{X_i\}$ the corresponding exhaustion of X, mentioned in the outline, and let us assume that X_{i-1} is deformed into a surface X'_{i-1} such that a conformal mapping $f_{i-1}: R_{i-1} \to X'_{i-1}$ with $f_{i-1}(p) = p'$ and $f_{i-1}(q) = q'$ exists. We are going to construct X'_i and f_i.

5.2. Fix global uniformizing parameters for R_i and $X''_i = (X_i - X_{i-1}) \cup X'_{i-1}$ as in the deformation lemma. We extend f_{i-1} to R_i such that the extended mapping $g: R_i \to X''_i$ is K-quasiconformal for a suitable K, C^∞ except perhaps on ∂R_{i-1} and such that

$$g_{\bar{z}}/g_z = \frac{i}{2}$$

on a disc U in $R_i - R_{i-1}$. Such an extension is certainly possible (see [11], p. 89).

5.3. We identify X''_i and R_i (as topological surfaces) by g, in order to use the same parameter $z \in D_{R_i}$. We may also define on X''_i the conformal structure for which z is admissible. While the natural structure of X''_i is induced by the metric $(dX''_i)^2 = |dg|^2 = \lambda^2 |dz + \mu d\bar{z}|^2$ the second structure may be viewed as induced by the metric $ds_0^2 = \lambda^2 |dz|^2$. Therefore we denote X''_i together with the second structure by $X''_i(ds_0^2)$.

5.4. It is clear that $g: R_i \to X''_i(ds_0^2)$ is conformal. On the other hand, for the natural structure we have

$$|\mu| \leqslant \frac{K-1}{K+1}, \qquad \mu = 0 \quad \text{on} \quad R_{i-1}, \quad \mu = \frac{i}{2} \quad \text{on} \quad U.$$

Here of course, we identified D_{R_i} and R_i.

5.5. As in the corollary, we choose a basis $\{\mu_k\}_{k=1}^N$ of $B(R_i)/N(R_i)$, where μ_k is

C^∞ and $\mu_k=0$ outside U^* for all k. Finally we determine a ball $B_{2r}\subset\mathbf{R}^N$ such that

$$\varphi^{-1}: t\to[\mathrm{id}_t\circ g: R_i\to X_i''(ds_t^2)]$$

is a topological mapping of B_{2r} into $T(R_i)$. Here

$$ds_t^2=\lambda^2(z)\left|dz+\left(\sum_{i=1}^N t_i\mu_i(z)\right)d\bar{z}\right|^2$$

and id_t represents the identity projection.

5.6. In the next section we will construct real-valued C^∞-functions h_t with support in X_i-X_{i-1} such that the deformed surfaces X_t described by

$$X_t(z)=g(z)+h_t(z)\cdot N(z),\qquad z\in D_{R_i},$$

where $N(z)$ is the positive unit normal vector of X_i'' at $g(z)$, satisfy the following conditions: If we view $X_t=X_t(z)$ as a mapping defined on R_i, then

a) $t\to\varphi[X_t]$ is continuous in B_r,

b) $\|\varphi[X_t]-\varphi[\mathrm{id}_t\circ g]\|\leqslant r\quad \forall t\in B_r$.

In addition, the functions h_t will be so small, that all the surfaces X_t are embedded surfaces.

5.7. Brouwer's fixed point theorem applied to the mapping $t\to t-\varphi[X_t]\equiv\varphi[\mathrm{id}_t\circ g]-\varphi[X_t]$ implies that there exists a point $t_0\in B_r$ such that $t_0=t_0-\varphi[X_{t_0}]$, i.e., $\varphi[X_{t_0}]=0$, which means that the embedded surface $X_i'=X_{t_0}$ can be mapped conformally onto R_i by a mapping f_i which is homotopic to g and satisfies the conditions $f_i(p)=p'$, $f_i(q)=q'$ (see the Remark in 4.7.).

6. The Construction of the Family X_t

6.1. Garsia's Deformation Lemma implies that the family X_t satisfies condition a), if the coefficients of $(dX_t)^2$ depend continuously on $(z,t)\in R_i\times B_r$. That this condition is satisfied can be seen by direct inspection, as soon as we have written down an explicit formula for the functions h_t.

6.2. In order to check condition b), we will again apply the Deformation Lemma 4.19 putting $\chi=\mathrm{id}_t\circ g\circ X_t^{-1}$. Its dilatation D_t satisfies

$$D_t^2=\sup\frac{dX_t^2}{ds_t^2}\Big/\inf\frac{dX_t^2}{ds_t^2}\quad\text{(sup and inf taken over all directions).}$$

6.3. We have

$$dX_t^2 = (dX_i'')^2 + (dh_t)^2 + O(|h_t|\cdot|dz|^2)$$
$$= \lambda^2 |dz + \mu\, d\bar{z}|^2 + \left(\left(\frac{\partial}{\partial z} h_t\right) dz + \left(\frac{\partial}{\partial \bar{z}} h_t\right) d\bar{z}\right)^2 + O(|h_t|\cdot|dz|^2),$$

and

$$ds_t^2 = \lambda^2 |dz + \mu_t\, d\bar{z}|^2 = \lambda^2 \left|dz + \sum_{i=1}^{N} t_i \mu_i\, d\bar{z}\right|^2, \qquad t \in B_r.$$

6.4. Assume that r is taken so small that $|\mu_t| < \frac{1}{5}$ if $t \in B_r$. Then there is also a positive number $\varrho^* < 1$ such that

$$|\mu| \leqslant \sup \left|\frac{\mu_t - \mu}{1 - \mu_t \bar{\mu}}\right| \leqslant \varrho^*.$$

Finally put $K_0 = 4(1-\varrho^*)^{-3}$ and choose $\delta_0 > 0$ and $\eta_0 > 0$ such that in the Deformation Lemma $b(K_0, \delta_0, \eta_0) \leqslant r$.

6.5. For the definition of h_t we have to solve the equation

$$|dz + \mu_t\, d\bar{z}|^2 = c_t\left(|dz + \mu\, d\bar{z}|^2 + (a_t\, dz + \bar{a}_t\, d\bar{z})^2\right).$$

The explicit solutions are

$$c_t = \frac{|1 - \mu_t \bar{\mu}|^2}{(1 - |\mu|^2)^2} (1 - |[\mu_t, \mu]|)^2,$$

$$a_t = \frac{\sqrt{|[\mu_t, \mu]| + \operatorname{Re}[\mu_t, \mu]}}{\sqrt{2}(1 - |[\mu_t, \mu]|)} \left(1 + \bar{\mu} - i \frac{(1 - \bar{\mu}) \operatorname{Im}[\mu_t, \mu]}{|[\mu_t, \mu]| + \operatorname{Re}[\mu_t, \mu]}\right),$$

where

$$[\mu_t, \mu] = \frac{\mu_t - \mu}{1 - \mu_t \bar{\mu}}.$$

The following regularity properties and estimates will be essential: a_t and c_t are continuous in $B_r \times D_R$, c_t is C^∞ in D_R for all $t \in B_r$ and a_t is C^∞ in D_R for all $t \in B_r$ at least if $\mu = i/2$, $\operatorname{Im} a_t < 0$ if $\mu = i/2$, $|a_t| < 4(1-\varrho^*)^{-1}$, $(1-\varrho^*)^4 < |c_t| < 16(1-\varrho^{*2})^{-2}$. In addition, we have $\mu_t = 0$ outside U and $c_t = 1$ and $a_t = 0$ in R_{i-1}.

6.6. In U we may multiply the differential $\omega = \bar{a}_t\, dz + \bar{a}_t\, d\bar{z}$ by a positive function $\varrho < 1$ which is continuous in $U \times B_r$ and C^∞ in U for all $t \in B_r$ (see [13], pp. 435–436) such that $\varrho \cdot \omega$ is exact. Then $\varrho\omega = dk(z, t)$, where k is again continuous in $U \times B_r$ and C^∞ in U.

6.7. Triangulate the bordered surface $R_i - R_{i-1} - U$ in such a way that in each triangle Δ_j

$$|a_t(z_1) - a_t(z_2)| \leqslant \varepsilon \quad \text{if} \quad z_1, z_2 \in \Delta_j, \quad t \in B_r,$$

where $\varepsilon > 0$ is to be determined later. Choose in each Δ_j a point z_j, put $L = \bigcup \partial \Delta_j$, and denote by v_η a non-negative C^∞-function which vanishes in a neighborhood of $L \cup R_{i-1}$ and is equal to 1 on $R_i - R_{i-1} - A_1$, where $|A_1| < \eta_0/2$.

6.8. We extend the functions k and ϱ defined on U in the following way

$$k = \begin{cases} 2\,\mathrm{Re}[z a_t(z_j)] & \text{if} \quad z \in \Delta_j \\ 0 & \text{if} \quad z \in L \cup R_{i-1} \end{cases}$$

$$\varrho = 1 \quad \text{in} \quad R_i - U.$$

6.9. Finally denote by s_η a saw-shaped C^∞-function defined on $\mathbf{R}$ with the following properties:

$$-1 \leqslant s_\eta(x) \leqslant 1, \quad |\dot{s}_\eta| = \left|\frac{ds_\eta}{dx}\right| \leqslant 1,$$

$$s_\eta(x+2) = -s_\eta(x), \quad s_\eta(x) = x \quad \text{if} \quad |x| < 1 - \eta.$$

6.10. Now we define h_t by

$$h_t(z, M) = \lambda(z)\, v_\eta(z) \frac{1}{\varrho(z)} \frac{1}{M} s_\eta(Mk(z, t)),$$

where M is a natural number. For each M, h_t is continuous in $R_i \times B_r$ and is C^∞ in R_i, and we have

$$(dh_t)^2 = \lambda^2 v_\eta^2 \frac{1}{\varrho^2} \dot{s}_\eta^2(Mk) \cdot (dk)^2 + O\left(\frac{1}{M}|dz|^2\right).$$

Except on a small part $A_1 \cup A_2$ of D_{R_i}, this reduces to

$$(dh_t)^2 = \lambda^2 (a_t\, dz + \bar{a}_t\, d\bar{z})^2 + O\left(\left(\varepsilon + \frac{1}{M}\right)|dz|^2\right).$$

We have $|A_2| < \eta_0/2$ and therefore $|A_1 \cup A_2| < \eta_0$, if η is sufficiently small and M sufficiently large (see [13] p. 437 for details).

6.11. Now it is easy to check that the dilatations D_t satisfy all the conditions in the assumption of the Deformation Lemma, as soon as we have chosen ε sufficiently small and then M sufficiently large. Therefore we can conclude that the family X_t

satisfies conditions a) and b) which completes the proof of the Embedding Theorem.

6.12. Incidentally we proved a result which is somewhat stronger than what is actually needed for the Embedding Theorem. Therefore we will state it as a corollary.

Let S_0 be a classical properly embedded surface, $X_0 : S_0 \to \mathbf{R}^3$ an embedding function (C^∞) and $N_0(p)$ the positive unit normal vector of S_0 at $p \in S_0$. If $h : S_0 \to \mathbf{R}$ is a sufficiently small differentiable function, then π_h defined by

$$\pi_h(p) = X_0(p) + h(p) N_0(p),$$

maps S_0 onto an embedded surface. Denote by S_h the corresponding classical surface.

COROLLARY. *To any positive continuous function* $\varepsilon : S_0 \to \mathbf{R}$ *and any topological orientation-preserving mapping* f *of* S_0 *onto a Riemann surface* S, *there is a differentiable function* $h : S_0 \to \mathbf{R}$, $|h| < \varepsilon$, *such that* $\pi_h \circ f^{-1}$ *is homotopic to a conformal mapping of* S *onto* S_h.

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