On the Property P_1 of Locally Compact Groups

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Introduction

A locally compact group G (with a left Haar measure dx and modular function Δ_G) is said to have property P_1 if for every $\varepsilon>0$ and every compact subset K of G there exists $s\in L^1(G)$ with $\|s\|_1=1$ and $\sup_{x\in K}\int_G |s(xy)-s(y)|\,dy<\varepsilon$. This suggests, for a general locally compact group G, studying the minimum ϱ_1 of all non-negative real numbers λ such that for every $\varepsilon>0$ and every compact subset K of G there exists $s\geqslant 0$ with $\|s\|_1=1$ and $\sup_{x\in K}\int_G |s(xy)-s(y)|\,dy<\lambda+\varepsilon$.

We prove (theorem 6) that $\varrho_1<1$ implies property P_1 (in fact a stronger result is obtained). In other words from $\varrho_1\neq 0$ it follows $\varrho_1\geqq 1$. An extension to the case of $L^1(G/H)$, with H satisfying property P_1, is given in section 2 (theorem 7).

The regular representation of G weakly contains the one dimensional identity representation i_G of G if and only if G has property P_1. This leads us to consider, for an arbitrary unitary continuous representation π of G acting on a Hilbert space $\mathscr{H}(\pi)$, the $\sup_{K\subset G}\inf_{\|\xi\|=1}\sup_{x\in K}|(\pi(x)\,\xi,\xi)-1|$ denoted $d(\pi)$.

We remark that π weakly contains i_G if and only if $d(\pi)=0$. It is therefore possible to consider $d(\pi)$ as the "distance" from i_G to π.

For a large class of π (including those obtained by inducing the identity from closed subgroups) a stronger result is obtained (theorem 13): $d(\pi)\geqslant 1$ if and only if π does not weakly contain i_G.

In the last part we prove a property similar to property P_1 but valid for arbitrary G (corollary 16): For every $\varepsilon>0$ and every compact subset K of G there exists $s\in L^1(G)$ with $s\geqslant 0$, $\int_G s(x)\,dx=1$ and $\sup_{x\in K}\|_x s-s\|_\Sigma<\varepsilon$ where $\|s\|_\Sigma$ is the norm of s as an element of the full C^∞-algebra of G.

The author would like to thank Professor K. Chandrasekharan and Professor G. W. Mackey for advice and many valuable discussions. Also many thanks are due to Professor P. Eymard for his helpful suggestions.

The main results of this paper had been announced in the Notices of the Amer. Math. Soc. *17* (1970), p. 822 and *17* (1970), p. 958.

1. Some Results on $L^1(G)$

In what follows, G is a locally compact group with unit element e. We use the following notations:

[1]) This work was supported by the Fonds National Suisse de la recherche scientifique.

$\mathscr{F}$ (resp.) $\mathscr{K}$) is the set of all finite (resp. compact) subsets of G,

$${}_a\phi_b(x) = \phi(a x b) \qquad \phi \in \mathbf{C}^G, \quad a, b \in G;$$

$$\check{f}(x) = f(x^{-1}), \qquad f^*(x) = \overline{f(x^{-1})}\, \Delta_G(x^{-1}),$$

$$A_x f = \Delta_G(x) f_x \quad \text{where} \quad f \in L^1(G).$$

$\mathscr{A}$ is the convex hull of $\{A_x \mid x \in G\}$. $\mathscr{M}$ is the set of all means on $L^\infty(G)$. Let M be an element of $\mathscr{M}$, then we set $\alpha(M) = \sup\{|M({}_x\phi) - M(\phi)| \mid x \in G, \|\phi\|_\infty \leqslant 1\}$.

Finally we define $L^+(G)$ to be the set of all $f \in L^1(G)$ which are non-negative and have L^1-norm equal to one.

PROPOSITION 1. *If there exists a non-negative λ such that, for every $\varepsilon > 0$ and every $F \in \mathscr{F}$, there is an $s \in L^+(G)$ with $\max_{x \in F} \|s - {}_x s\|_1 < \lambda + \varepsilon$, then the set $\{M \in \mathscr{M} \mid \alpha(M) \leqslant \lambda\}$ is non empty.*

Proof. It is possible to associate to every $s \in L^+(G)$ a linear functional L_s on $L^\infty(G)$ by setting $L_s(\phi) = \int_G \phi(x) s(x)\, dx$. We have $\{L_s \mid s \in L^+(G)\} \subset \mathscr{M}$. For $\varepsilon > 0$ and $F \in \mathscr{F}$, the set $\{L_s \mid s \in L^+(G), \sup_{z \in F} \|{}_{z^{-1}} s - s\|_1 < \varepsilon + \lambda\}$ is denoted $A_{F,\varepsilon}$. By assumption $A_{F,\varepsilon} \neq \emptyset$. The inequality $|L_s({}_z\phi) - L_s(\phi)| \leqslant \|\phi\| \, \|{}_{z^{-1}} s - s\|_1$ ($s \in L^+(G)$, $z \in G$ and $\phi \in L^\infty(G)$) implies that for every M in the $\sigma(L^{\infty\prime}, L^\infty)$-closure $B_{F,\varepsilon}$ of $A_{F,\varepsilon}$ we have $\max_{x \in F} |M({}_x\phi) - M(\phi)| \leqslant \|\phi\|_\infty (\lambda + \varepsilon)$ for every $\phi \in L^\infty(G)$.

It is easy to verify that $\{A_{F,\varepsilon} \mid F \in \mathscr{F}, \varepsilon > 0\}$ has the finite intersection property; *a fortiori* so does $\{B_{F,\varepsilon} \mid F \in \mathscr{F}, \varepsilon > 0\}$. Then from the $\sigma(L^{\infty\prime}, L^\infty)$-compactness of $\mathscr{M}$ it follows that $\bigcap \{B_{F,\varepsilon} \mid F \in \mathscr{F}, \varepsilon > 0\}$ is non empty. Let M be any element of this set and x an arbitrary element of G. We have $M \in B_{x,\varepsilon}$ for every $\varepsilon > 0$. This implies $|M({}_x\phi) - M(\phi)| \leqslant (\lambda + \varepsilon) \|\phi\|_\infty$ for every $\phi \in L^\infty(G)$. This inequality is satisfied for every $x \in G$ therefore we have $\alpha(M) \leqslant \lambda + \varepsilon$, i.e. $\alpha(M) \leqslant \lambda$.

Remark. See ([9] p. 179 and [10]) for the case $\lambda = 0$.

We observe that for $M \in \mathscr{M}$ and $g \in L^+(G)$ the map which associates to every $\phi \in L^\infty(G)$ the number $M(g^* * \phi)$ is an element M_g of $\mathscr{M}$.

LEMMA 2. *If $M \in \mathscr{M}$ and $g \in L^+(G)$, then*

$$|M_g(f^* * \phi) - \int_G f(x)\, dx\, M_g(\phi)| \leqslant \alpha(M) \|\phi\|_\infty \left(\|f\|_1 + \left| \int_G f(x)\, dx \right| \right).$$

Proof. We can assume that both f and ϕ are different from 0. Choose an arbitrary $\varepsilon > 0$. It is possible to find $h \in L^+(G)$ with $\|h * f * g - f * g\|_1 < \eta$ and

$$\|h * g - g\|_1 < \eta \quad \text{where} \quad \eta = \min\left(\frac{\varepsilon}{3 \|f\|_1 \|\phi\|_\infty}, \frac{\varepsilon}{3 \|\phi\|_\infty} \right).$$

This clearly implies

$$|M_g(\overline{f^*} * \phi) - M((h * \bar{f} * g)^* * \phi)| < \frac{\varepsilon}{3} \tag{1}$$

and

$$\left|\int_G f(x)\,dx\right| |M_{h*g}(\phi) - M_g(\phi)| \leqslant \frac{\varepsilon}{3} \tag{2}$$

If we take into account the fact that the mapping $x \mapsto {}_x(h^* * \phi)$ of G into $C^b(G)$ is continuous, we see that we can find a finite subset $\{x_j\}_{j=0}^n$ of G and disjoint Borel subsets $\{A_j\}_{j=0}^n$ of G such that $\bigcup_{j=0}^n A_j = G$ and

$$\left\|(\bar{f} * g)^* * h^* * \phi - \sum_{j=0}^n a_{j\,x_j}(h^* * \phi)\right\|_\infty < \frac{\varepsilon}{3}$$

where $a_j = \int_{A_j} (\bar{f} * g)^*(x)\,dx$. We therefore have

$$\left|M_{\bar{f}*g}(h^* * \phi) - \left(\int_G f * g(x)\,dx\right) M_h(\phi)\right| < \frac{\varepsilon}{3} + \alpha(M)\,\|f\|_1\,\|\phi\|_\infty \tag{3}$$

In the same way we get

$$\left|\int_G f(x)\,dx\, M_{h*g}(\phi) - \int_G f * g(x)\,dx\, M_h(\phi)\right| \leqslant \alpha(M)\,\|\phi\|_\infty \left|\int_G f(x)\,dx\right|. \tag{4}$$

From (1), (2), (3) and (4) it finally follows

$$\left|M_g(\overline{f^*} * \phi) - \int_G f(x)\,dx\, M_g(\phi)\right| < \alpha(M)\,\|\phi\|_\infty \left(\left|\int_G f(x)\,dx\right| + \|f\|_1\right) + \varepsilon.$$

For $f \in L^1(G)$ we denote by $d(f)$ the infimum of $\{\|Af\|_1 \mid A \in \mathscr{A}\}$.

PROPOSITION 3. *For arbitrary* $f \in L^1(G)$ *and* M *in* $\mathscr{M}$ *the following inequality holds:*

$$(1 - \alpha(M))\,d(f) \leqslant (1 + \alpha(M)) \left|\int_G f(x)\,dx\right|.$$

Proof. We can assume $d(f) > 0$. In this case there exists $\phi \in L^\infty(G)$ such that $\operatorname{Re} \int_G Af(x)\,\overline{\phi(x)}\,dx \geqslant 1$ for every $A \in \mathscr{A}$ and $\|\phi\|_\infty = 1/d(f)$.

From $(\overline{Af})^* * \bar{\phi}(z) = \int_G A_{z^{-1}} Af(y)\,\overline{\phi(y)}\,dy$ it follows that $1 \leqslant \operatorname{Re} M_g((\overline{Af})^* * \bar{\phi})$

for arbitrary $g \in L^+(G)$ and every $A \in \mathscr{A}$. By lemma 2 we therefore have

$$1 \leqslant \left|\int_G f(x)\,dx\right| |M_g(\bar{\phi})| + \alpha(M)\,\|\phi\|_\infty \left(\|Af\|_1 + \left|\int_G f(x)\,dx\right|\right)$$

for every $A \in \mathscr{A}$, i.e.

$$d(f) \leqslant \left|\int_G f(x)\,dx\right| + \alpha(M)\,d(f) + \alpha(M)\left|\int_G f(x)\,dx\right|.$$

LEMMA 4. *If for* $\{f_n\}_{n=1}^M \subset L^1(G)$ *we have*

$$\int_G f_n(x)\,dx = 0, \qquad 1 \leqslant n \leqslant M,$$

then

$$\inf_{A \in \mathscr{A}} \max_{1 \leqslant n \leqslant M} \|A_{f_n}\|_1 \leqslant \sup\left\{d(g) \mid \int_G g(x)\,dx = 0,\ \|g\|_1 \leqslant \max_{1 \leqslant n \leqslant M} \|f_n\|_1\right\}.$$

Proof. We denote by L_M the right hand side of the above inequality. For $M=1$ there is nothing to prove. Assume that for arbitrary $\varepsilon>0$ there exists $A' \in \mathscr{A}$ with $\max_{1 \leqslant m \leqslant M-1} \|A'f_m\|_1 < \varepsilon + L_{M-1}$. We can find $A'' \in \mathscr{A}$ such that $\|A''Af_M\|_1 < d(A'f_M) + \varepsilon$. $\int_G A'f_M\,dx = 0$ and $\|A'f_M\|_1 \leqslant \|f_M\|_1 \leqslant L_M$ imply $\|A''A'f_M\|_1 < \varepsilon + L_M$.

For $1 \leqslant n \leqslant M-1$ we have $\|A''A'f_n\|_1 \leqslant \|A'f_n\|_1 < \varepsilon + L_{M-1} \leqslant \varepsilon + L_M$. To conclude the proof of the lemma it is enough to take $A = A''A'$.

PROPOSITION 5. *Let G be an arbitrary locally compact group. For every* $f \in L^1(G)$ *and every* $K \in \mathscr{K}$ *we have*

$$\inf_{A \in \mathscr{A}} \sup_{x \in K} \|{}_x(Af) - Af\|_1 \leqslant 2 \sup\left\{d(g) \mid \int_G g(x)\,dx = 0,\ \|g\|_1 \leqslant \|f\|_1\right\}.$$

Proof. Let ε be an arbitrary positive real number. We can find U an open neighborhood of e such that $y \in U$ implies $\|{}_yf - f\|_1 < \varepsilon/2$. On the other hand we can choose a finite set $\{a_n\}_{n=1}^M \subset K$ with $\bigcup_{n=1}^M Ua_n \supset K$. By lemma 4 there exists $A \in \mathscr{A}$ such that

$$\max_{1 \leqslant n \leqslant M} \|A({}_{a_n}f - f)\|_1 < \varepsilon/2$$
$$+ \sup\left\{d(g) \mid \int_G g(x)\,dx = 0,\ \|g\|_1 \leqslant \max_{1 \leqslant n \leqslant M} \|{}_{a_n}f - f\|_1\right\}$$
$$\leqslant \varepsilon/2 + \sup\left\{d(g) \mid \int_G g(x)\,dx = 0,\ \|g\|_1 \leqslant 2\,\|f\|_1\right\}.$$

Therefore we obtain

$$\sup_{y\in K} \|_y(Af) - Af\|_1 < \varepsilon + 2 \sup \left\{ d(g) \mid \|g\|_1 \leqslant \|f\|_1, \int_G g(x)\, dx = 0 \right\}.$$

Remarks.

1) The idea of this proof comes from [9] p. 176–177. However our formulation is more general.
2) It follows from prop. 5 that if G is a locally compact group such that $f\in L^1(G)$ with $\int_G f(x)\, dx = 0$ implies $d(f)=0$, then G has property P_1. In fact, only functions f with $\int_G f(x)\, dx=0$ are used in the proof given in [9] p. 176–177.

THEOREM 6. *If there exists* λ *with* $0\leqslant\lambda<1$, *such that for every* $\varepsilon>0$ *and every* $F\in\mathscr{F}$ *one can find* $s\in L^+(G)$ *with* $\max_{y\in F} \|_y s - s\|_1 < \lambda+\varepsilon$, *then* G *has property* P_1.

Proof. By proposition 1 there exists $M\in\mathscr{M}$ with $\alpha(M)\leqslant\lambda$; from proposition 3 it follows that for every $f\in L^1(G)$ $(1-\lambda)\, d(f)\leqslant(1+\lambda)\, |\int_G f(x)\, dx|$. Finally proposition 5 and the assumption $0\leqslant\lambda<1$ imply that G has property P_1.

Remarks.

1) In fact we have proved a stronger result. Namely, for every $f\in L^1(G)$ and for every $K\in\mathscr{K}$, $\inf_{A\in\mathscr{A}} \sup_{x\in K} \|_x(Af)-Af\|_1=0$.
2) Let ϱ_1^* be the least non-negative real number λ such that for every $\varepsilon>0$ and every $F\in\mathscr{F}$ there exists an $s\in L^+(G)$ with $\max_{x\in F}\|_x s - s\|_1<\lambda+\varepsilon$. Replacing $\mathscr{F}$ by $\mathscr{K}$ we define ϱ_1 in the same way. By prop. 1, 3 and 5 we have

$$\sup \left\{ d(g) \mid \int_G g(x)\, dx = 0,\ \|g\|_1 \leqslant 1 \right\} \leqslant \varrho_1^* \leqslant \varrho_1$$

$$\leqslant 2 \sup \left\{ d(g) \mid \int_G g(x)\, dx = 0,\ \|g\|_1 \leqslant 1 \right\}.$$

Theorem 6 is equivalent to the following assertion: $\varrho_1^*<1$ implies $\varrho_1=0$.

2. Extension to the Case of $L^1(G/H)$

Let H be a closed subgroup of G with a left Haar measure $d\xi$, the modular function Δ_H and let q be a strictly positive continuous solution of the functional equation $q(x\xi)=q(x)\,\Delta_H(\xi)\,\Delta_G(\xi)^{-1}$ for $x\in G$ and $\xi\in H$; $d\dot{x}$ is the corresponding quasi-invariant measure on G/H. We set $\chi(y, \dot{x})=q(yx)\, q(x)^{-1}$ where $x, y\in G$ and $\dot{x}=xH=\pi_H(x)$. Define a map of $L^1(G)$ onto $L^1(G/H)$ by

$$T_H f(\dot{x}) = \int_H \frac{f(x\xi)}{q(x\xi)}\, d\xi$$

and denote by $L^+(G/H)$ the set $\{s\in L^1(G/H) \mid \|s\|_1=1, s\geqslant 0\}$.

THEOREM 7. *If H has property P_1 and if there exists $0 \leqslant \lambda < 1$ such that, for every $\varepsilon > 0$ and $F \in \mathscr{F}$ there is some $s \in L^+(G/H)$ with* $\max_{y \in F} \int_{G/H} |\chi(y^{-1}, \dot{x})\, s(y^{-1}\dot{x}) - s(\dot{x})|\, d\dot{x} < \varepsilon + \lambda$, *then G has property P_1.*

Proof. It is enough to prove that the assumptions of theorem 6 are satisfied. Choose $F \in \mathscr{F}$ and $\varepsilon > 0$ arbitrarily. Clearly there is some $s' \in L^+(G/H)$ which is continuous, has compact support, and satisfies

$$\int_{G/H} |\chi(y^{-1}, \dot{x})\, s'(y^{-1}, \dot{x}) - s'(\dot{x})|\, d\dot{x} < \lambda + \varepsilon/2 \quad \text{for every} \quad y \in F.$$

Let β be a Bruhat function for the closed subgroup H. Then by the definition of β, $f_1 = s' \pi_H \beta q$ is continuous and has compact support on G, and $\int_G f_1(x)\, dx = 1$. We verify that

$$\operatorname{supp}({}_{y^{-1}}f_1 - f_1) \subset (F \operatorname{supp} f_1) \cup \operatorname{supp} f_1 \quad \text{for every} \quad y \in F.$$

Then if we use a slight modification of the argument given in [9] p. 116, taking into account the definition of $d\dot{x}$, we can conclude that there exists an $s_1 \in L^+(H)$, continuous and with compact support on H, such that

$$\int_G \left| \int_H ({}_{y^{-1}}f_1 - f_1)(x\xi^{-1})\, \Delta_G(\xi^{-1})\, s_1(\xi)\, d\xi \right| dx$$
$$< \|T_H({}_{y^{-1}}f_1 - f_1)\|_1 + \varepsilon/2 \quad \text{for every} \quad y \in F.$$

Defining

$$s(x) = \int_H f_1(x\xi^{-1})\, \Delta_G(\xi^{-1})\, s_1(\xi)\, d\xi,$$

we have $s \geqslant 0$ and

$$\int_G s(x)\, dx = \int_H \left(\int_G f_1(x\xi^{-1})\, dx \right) \Delta_G(\xi^{-1})\, s_1(\xi)\, d\xi$$
$$= \int_G f_1(x)\, dx \int_H s_1(\xi)\, d\xi = 1, \quad \text{i.e.} \quad s \in L^+(G).$$

Observe that

$$T_H({}_{y^{-1}}f_1) = \chi(y^{-1}, \cdot)_{y^{-1}}(T_H f).$$

Therefore we have

$$\int_{G_1} |s(y^{-1}x) - s(x)|\, dx < \int_{G/H} |\chi(y^{-1}, \dot{x})\, s'(y^{-1}\dot{x}) - s'(\dot{x})|\, d\dot{x} + \varepsilon/2$$

for every $y \in F$ i.e.

$$\max_{y \in F} \|{}_{y^{-1}}s - s\|_1 < \lambda + \varepsilon .$$

Remarks.

1) This proof is a modification of the one for the case where H is normal and where P_1 holds for G/H and H ([9] p. 169).
2) For $\lambda = 0$ a different proof of theorem 7 has already been obtained ([3]).

As above, define $\mathcal{M}(G/H)$ as the set of all means on $L^\infty(G/H)$; and for $M \in \mathcal{M}(G/H)$, set

$$\alpha(M) = \sup \{|M({}_x\phi) - M(\phi)| \mid \|\phi\|_\infty \leqslant 1, \quad x \in G\} .$$

It is also possible ([3]) to formulate a version of property P_1, for G/H:

G/H is said to have property P_1, if for every $K \in \mathcal{K}$ and $\varepsilon > 0$ there is some $s \in L^+(G/H)$ with

$$\sup_{x \in K} \|\chi(x^{-1}, \cdot)_{x^{-1}} s - s\|_1 < \varepsilon .$$

PROPOSITION 8. *Let $\lambda \geqslant 0$. If for every $\varepsilon > 0$ and every $F \in \mathcal{F}$ there exists some $s \in L^+(G/H)$ with*

$$\max_{x \in F} \|\chi(x^{-1}, \cdot)_{x^{-1}} s - s\|_1 < \lambda + \varepsilon ,$$

then the set $\{M \in \mathcal{M}(G/H) \mid \alpha(M) \leqslant \lambda\}$ is non-empty.

The proof is exactly the same as for proposition 1.

$L^1(G)$ acts on $L^\infty(G/H)$ in the following way: if $f \in L^1(G)$ and $\phi \in L^\infty(G/H)$, then the function $\dot{t} \mapsto \int_G f(x)\, \phi(x\dot{t})\, dx$ is an element $\overline{f^* * \phi} \in L^\infty(G/H)$ (see [7] and [8]). For arbitrary $g \in L^+(G)$ and $M \in \mathcal{M}(G/H)$ the map which associates to every $\phi \in L^\infty(G/H)$ the number $M(g^* * \phi)$ is an element M_g of $\mathcal{M}(G/H)$. Similarly to lemma 2 one can prove that for $M \in \mathcal{M}(G/H)$, $f \in L^1(G)$, $\phi \in L^\infty(G/H)$ and $g \in L^+(G)$ the inequality

$$\left| M_g(\overline{f^*} * \phi) - \left(\int_G f(x)\, dx \right) M_g(\phi) \right| \leqslant \alpha(M)\, \|\phi\|_\infty \left(\left| \int_G f(x)\, dx \right| + \|T_H f\|_1 \right)$$

holds, provided that H is compact and ${}_h g = g$ for every $h \in H$ [2]). We were not able to drop the assumption on the compactness of H. In the case $\alpha(M) = 0$, but for an arbitrary closed subgroup H and arbitrary $g \in L^+(G)$, the preceding result is due to F. P. Greenleaf ([7] p. 303–304).

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[2]) The compactness of H is used only to assert the existence of such a g.

PROPOSITION 9. *For every* $M \in \mathcal{M}(G/H)$ *and every finite subset* $\{f_1, \dots, f_p\}$ *of* $L^1(G)$ *we have*

(a) *if* $\alpha(M) = 0$ *then*

$$\inf_{A \in \mathscr{A}} \max_{1 \leqslant j \leqslant p} \|T_H A f_j\|_1 = \max_{1 \leqslant j \leqslant p} \left| \int_G f_j(x)\, dx \right|,$$

(b) *if* H *is compact then*

$$(1 - \alpha(M)) \inf_{A \in \mathscr{A}} \max_{1 \leqslant j \leqslant p} \|T_H A\ {}_j\|_1 \leqslant (1 + \alpha(M)) \max_{1 \leqslant j \leqslant p} \left| \int_G f_j(x)\, dx \right|.$$

Proof. We denote by E the cartesian product of p copies of $L^1(G/H)$ and define on E a norm topology as follows: $\|v\| = \max_{1 \leqslant j \leqslant p} \|v_j\|_1$ where $v = (v_1, \dots, v_p)$. We can assume that $d = \inf_{A \in \mathscr{A}} \max_{1 \leqslant j \leqslant p} \|T_H A f_j\|_1$ is positive. Then we can find a continuous linear functional ϕ on E such that $\operatorname{Re} \phi((T_H A f_1, \dots, T_H A f_p)) \geqslant 1$ for every $A \in \mathscr{A}$ and $\|\phi\| = 1/d$. Clearly $\|\phi\| = \sum_{j=1}^{p} \|\phi_j\|_\infty$ where $\phi = (\phi_1, \dots, \phi_p)$ and $\phi_j \in L^\infty(G/H)$. For $z \in G$ and every $A \in \mathscr{A}$ we have

$$\phi((T_H A_{z^{-1}} A f_1, \dots, T_H A_{z^{-1}} A f_p)) = \sum_{j=1}^{p} \overline{(A f_j)^* * \bar{\phi}_j(\dot{z})}.$$

In case (a) we choose an arbitrary $g \in L^+(G)$. We have

$$M_g(\overline{(A f_j)^* * \bar{\phi}_j}) = \left(\int_G f_j(x)\, dx \right) M_g(\bar{\phi}_j)$$

for every $A \in \mathscr{A}$ and $1 \leqslant j \leqslant p$. We therefore obtain

$$1 \leqslant \sum_{j=1}^{p} \left| \int_G f_j(x)\, dx \right| \|\phi_j\|_\infty \leqslant \max_j \left| \int_G f_j(x)\, dx \right| 1/d.$$

This inequality implies (a).

It remains to prove case (b). We can find $g \in L^+(G)$ such that ${}_h g = g$ for every $h \in H$. This implies (see above comment)

$$1 \leqslant \left| M_g\left(\sum_{j=1}^{p} \overline{(A f_j)^* * \bar{\phi}_j} \right) \right| \leqslant \|\phi\| \max_j \left| \int_G f_j(x)\, dx \right|$$
$$+ \alpha(M) \|\phi\| \left\{ \max_j \|T_H A f_j\|_1 + \max_j \left| \int_G f_j(x)\, dx \right| \right\}$$

for each $A \in \mathscr{A}$. We therefore have

$$d \leqslant \max_j \left| \int_G f_j(x)\, dx \right| + \alpha(M) \left(d + \max_j \left| \int_G f_j(x)\, dx \right| \right).$$

Remarks.

1) For $\alpha(M)=0$ and $\int_G f_j\,dx=0$ $(1\leqslant j\leqslant p)$ prop. 9 is due to P. Eymard ([3] p. 8–9). Using it, he proves an analogue of theorem 6 for G/H in the case $\lambda=0$. Except for H compact, which is then a special case of theorem 7, we were not able to obtain a complete analogue of theorem 6 for G/H.
2) If H has property P_1 we have (by [9] p. 174) $\inf_{A\in\mathscr{A}}\|T_H A f\|_1=d(f)$ for every $f\in L^1(G)$.

3. Other Generalizations and Applications to the Study of $P(G)$

Let π be an arbitrary unitary continuous representation acting on the Hilbert space $\mathscr{H}(\pi)$. Directly related to $d(\pi)$ (defined in the introduction) is

$$\varrho(\pi)=\sup_{K\in\mathscr{K}}\ \inf_{\|\xi\|=1}\ \sup_{x\in K}\|\pi(x)\,\xi-\xi\|\,.$$

We have in fact $\frac{1}{2}\varrho(\pi)^2\leqslant d(\pi)\leqslant\varrho(\pi)$. If we replace $\mathscr{K}$ by $\mathscr{F}$ we define $\varrho^*(\pi)$ and $d^*(\pi)$, which satisfy the same types of inequalities.

Let $A(\pi)$ be the set of all continuous positive definite functions associated to π and $\sum A(\pi)$ the set of all finite sums of elements of $A(\pi)$. We recall ([4] p. 371) that π weakly contains π' if and only if $A(\pi')$ lies in the compact-open closure of $\sum A(\pi)$. The following proposition is just slightly different from theorem 1.5 ([4] p. 374) and lemma 2.2. ([5] p. 246). Nevertheless, we indicate a direct proof avoiding Banach-algebra techniques.

PROPOSITION 10. *π weakly contains an irreducible representation π' if and only if $A(\pi')$ is in the compact-open closure of $A(\pi)$.*

Proof. Let p be an arbitrary element of $A(\pi')$. We have to show that if p lies in the compact-open closure of $\sum A(\pi)$ then p is already contained in the compact-open closure of $A(\pi)$. It is easy to verify that $p/p(e)$ is in the compact-open closure of $\{q/q(e)\mid q\in\sum A(\pi),\,q(e)>0\}$ and that $\{q/q(e)\mid q\in\sum A(\pi),\,q(e)>0\}$ is contained in $P_0\cap\sum A(\pi)$, where P_0 denotes the set $\{u\in P(G)\mid u(e)=1\}$. The relation $P_0\cap\sum A(\pi)\subset\operatorname{co}(P_0\cap A(\pi))$ implies that $p(e)^{-1}p$ is in the compact-open closure of $\operatorname{co}(P_0\cap A(\pi))$. A fortiori $p(e)^{-1}p$ lies in the $\sigma(P(G),L^1(G))$-closed convex hull $\overline{\operatorname{co}}(P_0\cap A(\pi))$ of $P_0\cap A(\pi)$. The $\sigma(P(G),L^1(G))$-compactness of $\overline{\operatorname{co}}(P_0\cap A(\pi))$ and the irreducibility of π' imply ([2] p. 440) that $p(e)^{-1}p$ (an extremal point of $\overline{\operatorname{co}}(P_0\cap A(\pi))$) is in the $\sigma(P(G),L^1(G))$-closure of $P_0\cap A(\pi)$. By D. A. Raikov ([1] p. 260) for every $K\in\mathscr{K}$ and $\varepsilon>0$ there exists an $\eta>0$ and a finite set $\{f_j\}_{j=1}^n\subset L^1(G)$ such that $u\in P_0$ and $|\int_G f_j(x)\,(p(e)^{-1}p(x)-u(x))\,dx|<\eta$ $(1\leqslant j\leqslant n)$ imply $\sup_{x\in K}|p(e)^{-1}p(x)-u(x)|<$ $<p(e)^{-1}\,\varepsilon$. If we choose $q\in P_0\cap A(\pi)$ such that

$$\left|\int_G f_j(x)\,(p(e)^{-1}\,p(x)-q(x))\,dx\right|<\eta\quad 1\leqslant j\leqslant n,$$

we finally get

$$\sup_{x \in K} |p(x) - q'(x)| < \varepsilon \quad \text{where} \quad q' = p(e)\, q \in A(\pi).$$

COROLLARY 11. *A continuous unitary representation π of G weakly contains the one-dimensional identity representation i_G of G if and only if $d(\pi)=0$.*

PROPOSITION 12. *Let π be an unitary continuous representation of G such that $d(\pi)<1$. Then for every $f \in L^1(G)$ we have*

$$2\left|\int_G f(x)\, dx\right| \leqslant \|\pi(f)\| + \|\pi(\bar{f})\|.$$

Proof. a) For every $f \in C_{00}(G)$ (set of all complex-valued continuous functions with compact support) with $f \geqslant 0$ we have $(1-d(\pi)) \int_G f(x)\, dx \leqslant \|\pi(f)\|$.

We can assume $\int_G f(x)\, dx > 0$. For every $\varepsilon \in (0, (1-d(\pi)) \int_G f(x)\, dx)$ we can find $\xi \in \mathscr{H}(\pi)$ such that $\|\xi\|=1$ and

$$\sup_{x \in \operatorname{supp} f} |(\pi(x)\xi, \xi) - 1| < d(\pi) + \frac{\varepsilon}{1 + \int_G f(x)\, dx}.$$

This implies clearly that

$$\left|\left|\int_G f(x)(\pi(x)\xi, \xi)\, dx\right| - \int_G f\, dx\right| < \varepsilon + d(\pi) \int_G f(x)\, dx.$$

From

$$\left|\int_G (\pi(x)\xi, \xi) f(x)\, dx\right| = |(\pi(f)\xi, \xi)| \leqslant \|\pi(f)\|$$

it follows that

$$\|\pi(f)\| > (1 - d(\pi)) \int_G f(x)\, dx - \varepsilon$$

for every $\varepsilon \in (0, (1-d(\pi)) \int_G f(x)\, dx)$. This proves a).

b) For every $f \in C_{00}(G)$ with $f \geqslant 0$ we have $\|\pi(f)\| = \int_G f\, dx$. Let us assume that there exists $f_0 \in C_{00}(G)$ such that $f_0 \geqslant 0$ and $\|\pi(f_0) \neq \int_G f_0\, dx$. We clearly have $\|\pi(f_0)\| < \int_G f_0\, dx$ and therefore (by a)) $\|\pi(f_0)\| > 0$. Consider $f_1 = \|\pi(f_0)\|^{-1} f_0$. For arbitrary $n \in \mathbf{N}$ we have $\int_G f_1^{(n)}\, dx = (\int_G f_1)^n$ and $\|\pi(f_1^{(n)})\| \leqslant 1$ where $f_1^{(n)} = f_1 * \cdots * f_1$ (n-times). Assertion a) implies that $(1-d(\pi))(\int_G f_1\, dx)^n \leqslant 1$ for every $n \in \mathbf{N}$. But on the other hand this inequality is not satisfied for $n > -\log(1-d(\pi))/\log \int_G f_1\, dx$.

c) From b) it follows that for every real-valued $f \in C_{00}(G)$ we have $|\int_G f\,dx| \leqslant \|\pi(f)\|$. Let f be an arbitrary function in $C_{00}(G)$. We can write $\int_G f\,dx = |\int_G f\,dx|\,e^{i\theta}$. It follows that

$$\left|\int_G f\,dx\right| = \int_G \operatorname{Re}(e^{-i\theta} f)\,dx \leqslant \|\pi(\operatorname{Re} e^{-i\theta} f)\|.$$

Finally

$$\|\pi(\operatorname{Re} e^{-i\theta} f)\| = \left\|\pi\left(\frac{e^{-i\theta} f + e^{i\theta} \bar{f}}{2}\right)\right\|$$

implies

$$2\left|\int f\,dx\right| \leqslant \|\pi(f)\| + \|\pi(\bar{f})\|.$$

By continuity this inequality extends to $L^1(G)$.

THEOREM 13. *Assume that G acts continuously on a locally compact space X and that X admits a quasi-invariant Radon measure μ with modular function χ. Let π be the representation of G in $L^2(X, \mu)$ defined by $\pi(x)\,\varphi = \chi(x^{-1}, \cdot)^{1/2}\,\varphi$. If $d(\pi) < 1$, then π weakly contains i_G.*

Proof. By definition of π, $\|\pi(\bar{f})\| = \|\pi(f)\|$ for every $f \in L^1(G)$. Then by prop. 12 we have $|\int_G f\,dx| \leqslant \|\pi(f)\|$.

This inequality permits us to finish the proof (by [11] theoreme 1).

Remark. An important exemple of a representation of the above kind is the unitary representation U^{i_H} induced on G by the one dimensional identity representation i_H of an arbitrary closed subgroup H of G.

Let π be a representation of G of the type described in theorem 13. It makes sense to define

$$\varrho_1(\pi) = \sup_{K \in \mathscr{K}} \inf_{s \in L^1(X, \mu),\, s \geqslant 0} \sup_{x \in K} \|\chi(x^{-1}, \cdot)_{x^{-1}} s - s\|_1$$

and $\varrho_1^*(\pi)$. It is straightforward to verify that

$$\varrho(\pi)^2 \leqslant \varrho_1(\pi) \leqslant 4\varrho(\pi)$$

$$\varrho^*(\pi)^2 \leqslant \varrho_1^*(\pi) \leqslant 4\varrho^*(\pi).$$

Taking into account these inequalities, theorem 6, corollary 11 and theorem 13, we can deduce the following:

COROLLARY 14. *Let π be an unitary representation of G obtained as in theorem 13. Then the following statements are equivalent:*

(i) *π does not weakly contain i_G,*
(ii) $d(\pi)\geqslant 1$
(iii) $\varrho(\pi)\geqslant 1$
(iv) $\varrho_1(\pi)\geqslant 1$

Moreover for $\pi = U^{i\{e\}}$ the preceding assertions are equivalent to:
(v) $\varrho_1^*(\pi)\geqslant 1$
(vi) $d^*(\pi)\geqslant \frac{1}{32}$.

Remark. H. Leptin introduced (see Bull. Amer. Math. Soc. *72* (1966), p. 870 and Proc. Math. Soc. *19* (1968), p. 489) the following invariant:

$$I(G) = \sup_{K\in\mathscr{K}} \inf \left\{ \frac{m(KU)}{m(U)} \mid U\in\mathscr{K},\, m(U)>0 \right\}.$$

He proved that

$$I(G) = \begin{cases} 1 & \text{if } G \text{ has property } P_1, \\ +\infty & \text{if not.} \end{cases}$$

We were not able to relate directly $I(G)$ with ϱ_1, ϱ_1^*, $d(U^{i\{e\}})$ and $d^*(U^{i\{e\}})$.

4. On the C^*-Algebra of G

Let $B(G)$ be the complex linear span of $P(G)$. The supremum norm closure of the convex hull of the left (or right) translates of an arbitrary $u\in B(G)$ contains a unique constant function, denoted $M(u)$. M defines ([6] p. 59–61) a linear functional on $B(G)$ satisfying the following conditions: (i) $M({}_a u_b) = M(u)$, (ii) $M(\bar{u}) = \overline{M(u)}$ and (iii) $|M(u)| \leqslant \|u\|_\infty$.

PROPOSITION 15. *For every finite subset $\{f_1,\ldots,f_n\}$ of $L^1(G)$ we have*

$$\inf_{A\in\mathscr{A}} \max_{1\leqslant j\leqslant n} \|Af_j\|_\Sigma = \max_{1\leqslant j\leqslant n} \left| \int_G f_j(x)\, dx \right| \qquad (*)$$

Proof. We first remark that for $f, g\in L^1(G)$ and $u\in B(G)$ we have

$$\left| \int_G f * u(x)\, g(x)\, dx \right| = \left| \int_G u(x) f^* * g(x)\, dx \right| \leqslant \|u\|\, \|f\|_\Sigma\, \|g\|_\Sigma$$

where $\|u\|$ denotes the norm of u as element of the dual of the C^*-algebra of G. We therefore have $f * u\in B(G)$. From the uniform continuity of u it follows that for every

$\varepsilon > 0$ we can find disjoint Borel subsets $\{A_j\}_{j=1}^m$ of G and $\{x_j\}_{j=1}^m \subset G$ such that

$$\left\| f * u - \sum_{j=1}^{m} c_{j\,x_j}u \right\|_\infty < \varepsilon \quad \text{where} \quad c_j = \int_{A_j} f(x)\,dx.$$

Using (i) and (iii) we obtain

$$\left| M(f * u) - \int_G f\,dx M(u) \right| < \varepsilon, \quad \text{i.e.} \quad M(f * u) = \int_G f\,dx M(u).$$

For every $A \in \mathscr{A}$ and $f \in L^1(G)$ we have

$$\left| \int_G f\,dx \right| = \left| \int_G Af\,dx \right| \leqslant \|Af\|_\Sigma.$$

This implies that the l.h.s. in (*) is not smaller than the r.h.s. To prove the last part of the theorem we can proceed as in prop. 9. Let d be the l.h.s. We can assume $d > 0$. Then there exists a continuous linear functional ϕ on the product of n copies of $B(G)$ such that $\|\phi\| = 1/d$ and $\mathrm{Re}\,\phi(Af) \geqslant 1$ for every $A \in \mathscr{A}$ (where $Af = (Af_1, \ldots, Af_n)$). Clearly $\|\phi\| = \sum_{j=1}^n \|u_j\|$ where $\phi = (u_1, \ldots, u_n)$ and $u_j \in B(G)$. From

$$\phi(A_{x^{-1}}Af) = \sum_{j=1}^{n} \overline{(Af_j)^*} * u_j(x)$$

it follows that

$$1 \leqslant \sum_{j=1}^{n} |M(\overline{(Af_j)^*} * u_j)| = \sum_{j=1}^{n} \left| \int_G f_j(x)\,dx \right| |M(u_j)|$$

i.e.

$$1 \leqslant \frac{1}{d} \max_{1 \leqslant j \leqslant n} \left| \int_G f_j(x)\,dx \right|.$$

COROLLARY 16. *For every $f \in L^1(G)$ and every $K \in \mathscr{K}$ we have* $\inf_{A \in \mathscr{A}} \sup_{x \in K} \|{}_x(Af) - Af\|_\Sigma = 0$.

BIBLIOGRAPHY

[1] DIXMIER, J., *Les C*-algèbres et leurs représentations* (Gauthier-Villars, Paris 1969).
[2] DUNFORD, N. and SCHWARTZ, J. T., *Linear Operators*, Pt. 1 (Interscience, New York 1958).
[3] EYMARD, P., *Moyennes invariantes sur les espaces homogènes et représentations quasi-régulières des groupes localement compacts* (II), Séminaire de l'Institut Elie Cartan, Nancy (1969).
[4] FELL, J. M. G., *The Dual Space of C*-Algebras*, Trans. Amer. Math. Soc. *94* (1960), 365–403.
[5] FELL, J. M. G., *Weak Containment and Induced Representations of Groups*, Canad. J. of Math. *14* (1962), 237–268.

[6] GODEMENT, R., *Les fonctions de type positif et la théorie des groupes*, Trans. Amer. Math. Soc. *63* (1948), 1–84.
[7] GREENLEAF, F. P., *Amenable Actions of Locally Compact Groups*, J. of Functional Analysis *4* (1969), 295–315.
[8] GULICK, S. L., LIU, T. S., and VAN ROOIJ, A. C. M., *Group Algebra Modules*. II, Canad. J. of Math. *19* (1967), 151–173.
[9] REITER, H., *Classical Harmonic Analysis and Locally Compact Groups* (Oxford, at the Clarendon Press, 1968).
[10] REITER, H., *Sur certains idéaux dans $L^1(G)$*, C.R. Acad. Sc. Paris *267* (1968), 882–885.
[11] TAKENOUCHI, O., *Sur une classe de fonctions continues de type positif sur un groupe localement compact*, Math. J. Okayama Univ. *4* (1955), 143–173.

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Received November 26, 1970