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Bundles with Totally Disconnected Structure Group

By JOHN W. WOOD

1. Introduction

We consider the problem of reducing the structure group of a circle bundle over a 2-manifold to a totally disconnected subgroup and find necessary and sufficient conditions in terms of the Euler class of the bundle; the result is stated in Theorems 1.1 and 1.2. In §6 we discuss the relation of such a reduction to foliations and flat connections. The proof of 1.1 and 1.2 depends on a study of products of commutators in certain universal covering groups and, using the same methods, in §7 we characterize the elements of $\widetilde{\mathrm{SL}}(2)$ which can be written as m-fold products of commutators.

By a fibre bundle we mean an equivalence class of coordinate bundles under the equivalence generated by enlarging the set of (admissible) coordinate transformations, see Steenrod [10, § 2]. Given a bundle ξ' with group G' and a homomorphism $\varphi: G' \to G$ there is a bundle ξ with group G over the same base whose coordinate transformations are those of ξ' composed with φ. We say ξ is the bundle induced from ξ' by φ. If φ is a homeomorphism onto its image we say ξ reduces to the subgroup G'.

Let $\mathrm{Top}\,S^1$ be the group of homeomorphisms of S^1 and $\mathrm{Diff}\,S^1$ the subgroup of diffeomorphisms. Let $G \subset \mathrm{Top}\,S^1$ be a group which retracts by deformation to $\mathrm{O}(2) \subset G$. Let M be a 2-manifold and $\mathcal{O}$ be the orientation bundle of integer coefficients twisted by $w_1(M): \pi_1(M) \to \mathrm{Aut}\,\mathbf{Z}$. Poincaré duality gives an isomorphism of $H^2(M; \mathcal{O})$ with $\mathbf{Z}$ by evaluation on the fundamental cycle $[M]$. A bundle ξ over M with fibre S^1 and group G is $\mathcal{O}$-orientable if the total space is orientable or, equivalently, if $w_1(\xi) = w_1(M)$. Such bundles are classified by their Euler class

$$\chi(\xi) \in H^2(M; \mathcal{O}).$$

Our main result is the following.

THEOREM 1.1. *For an $\mathcal{O}$-orientable fibre bundle ξ with fibre S^1, group $G = \mathrm{Top}\,S^1$ or $\mathrm{Diff}\,S^1$, and base a 2-manifold M the following are equivalent:*

(i) *ξ can be reduced to a totally disconnected subgroup.*

(ii) $|\chi(\xi)[M]| \leq -\chi(M)$ *for* $\chi(M) \leq 0$
$ = 0$ *for* $\chi(M) \geq 0$.

(iii) *ξ is induced by a representation $\varphi: \pi_1(M) \to G$.*

(iv) *There is a foliation (smooth if $G = \mathrm{Diff}\,S^1$) of the total space with leaves transverse to the fibres.*

Notice that if M is orientable then we may replace G by the subgroup G^+ of orientation preserving maps. In fact $w_1(\xi)$ is obtained from φ by composition with the map $G \to G/G^+ = \mathbf{Z}_2$.

SL(2), the group of real 2×2 matrices with determinant $+1$, acts faithfully on $\mathbf{R}^2$ and on the set of oriented lines through 0 which we identify with S^1. This gives a natural action of SL(2) on S^1 which preserves antipodal points. Let $\mathrm{Top}_2 S^1$ be the subgroup of $\mathrm{Top}\, S^1$ of elements which commute with the antipodal map.

Define $T: \mathbf{R} \to \mathbf{R}$ by $T(t) = t+1$. T generates the covering transformations of the covering $\mathbf{R} \to S^1$. Any homeomorphism $g: S^1 \to S^1$ lifts to a homeomorphism $f: \mathbf{R} \to \mathbf{R}$ satisfying $fT = Tf$ if g preserves orientation or $fT = T^{-1}f$ if g reverses orientation and such an f covers a well-defined $pf \in \mathrm{Top}\, S^1$. Suppose G acts effectively on S^1. Set $\tilde{G} = \{f: \mathbf{R} \to \mathbf{R} \mid fT^{\pm 1}f, pf \in G\}$. Then $\tilde{G}$ is a group, $p: \tilde{G} \to G$ is a surjective homomorphism and $\ker p = \mathbf{Z}$, hence p is a covering. Since we assume $\mathrm{SO}(2) \subset G^+$ as a deformation retract, $\tilde{G}^+$ is the universal covering group of G^+. Identify $\mathbf{R}$ as a subgroup of $\tilde{G}^+$ by sending $a \in \mathbf{R}$ to the function $f_a(t) = t + a$; then we have:

$$\begin{array}{ccccc} \mathbf{R} & \subset & \tilde{G}^+ & \subset & \tilde{G} \\ \downarrow & & \downarrow & & \downarrow p \\ \mathrm{SO}(2) & \subset & G^+ & \subset & G \end{array}$$

THEOREM 1.2. *ξ as in Theorem* 1.1 *with group G,* $\mathrm{SL}(2) \subseteq G \subseteq \mathrm{Top}_2 S^1$, *then the following are equivalent:*

(i) *ξ can be reduced to a totally disconnected subgroup.*

(ii) $|\chi(\xi)[M]| \leqslant -\frac{1}{2}\chi(M)$ *for* $\chi(M) \leqslant 0$
$\qquad = 0$ *for* $\chi(M) \geqslant 0$

(iii) *ξ is induced by a representation $\varphi: \pi_1(M) \to G$.*

(iv) *If G is a Lie group then the associated principal bundle has a flat connection.*

The theorem for SL(2) and M orientable is due to Milnor and the proof of Theorems 1 and 2 is modeled on his paper [7]. I thank Professor Milnor for several helpful conversations.

2. Proof of theorems

The universal covering space $\tilde{M}$ of M is a principal $\pi_1(M)$ bundle. The G-bundle induced by a homomorphism $\varphi: \pi_1(M) \to G$ reduces to the subgroup image φ which is totally disconnected since $\pi_1(M)$ is countable. The converse is proved in Steenrod [10, §13]; hence i) and iii) are equivalent.

We now describe an algorithm for computing $\chi(\xi)[M]$ from $\varphi: \pi_1(M) \to G$. Regard M as a $2n$-gon with pairs of edges identified so the fundamental group is presented as $\pi_1(M)=\{\alpha_1,\ldots,\alpha_n \mid W(\alpha_1,\ldots,\alpha_n)=1\}$ where W is the word obtained by listing the oriented edges in order. Choose $f_i\in\tilde{G}$ covering $\varphi(\alpha_i)\in G$; then $W(f_1,\ldots,f_n)$ covers $\varphi(W(\alpha_1,\ldots,\alpha_n))=1$ so $W(f_1,\ldots,f_n)$ is translation by some integer.

LEMMA 2.1. $\chi(\xi)[M]=-W(f_1,\ldots,f_n)$.

This is Lemma 2 in [7] in the case M orientable. In §3 we will derive the non-orientable case from this. The algorithm described above can be thought of as a coboundary. Briefly, there is a canonical element $\varepsilon\in H^1(M;\pi_1(M))$ which assigns to each edge the homotopy class it represents. Let δ be the coboundary operator corresponding to the sequence

$$0\to \mathbf{Z}\to\tilde{G}\to G\to 0.$$

Then $\chi(\xi)=-\delta\varphi_*\varepsilon$, (cf. Hirzebruch [6, § 4.3.1] and his sign convention p. 59). The lemma computes δ explicitly.

In the rest of this section we will reduce the theorems in the case M is orientable to two propositions about products of commutators. We may assume $M\neq S^2$ since that case follows already from condition iii). We will apply Lemma 2.1 to the normal presentation of M which gives $\pi_1(M)=\{\alpha_1,\ldots,\alpha_{2m}\mid[\alpha_1,\alpha_2]\ldots[\alpha_{2m-1},\alpha_{2m}]=1\}$ where m is the genus of M and $-\chi(M)=2m-2$.

PROPOSITION 2.2. *If an m-fold product of commutators in* $\widetilde{\mathrm{Top}^+ S^1}$ *is translation by a then* $|a|<2m-1$.

This gives i)$\Rightarrow$ii) in Theorem 1.1.

Define a homomorphism $j:\widetilde{\mathrm{Top}^+ S^1}\to\widetilde{\mathrm{Top}^+ S^1}$ by $j(f)(t)=\frac{1}{2}f(2t)$. Then image j $=\widetilde{\mathrm{Top}_2^+ S^1}$.

COROLLARY 2.3. *If an m-fold product of commutators in* $\widetilde{\mathrm{Top}_2^+ S^1}$ *is translation by a, then* $|a|<m-\frac{1}{2}$.

Proof. Say $[jf_1,jf_2]\ldots[jf_{2m-1},jf_{2m}]\,t=t+a$. By definition of j, $[f_1,f_2]\ldots[f_{2m-1},f_{2m}]\,(2t)=2t+2a$, hence $|2a|<2m-1$.

This gives i)$\Rightarrow$ii) in Theorem 1.2.

Define $S\in\widetilde{\mathrm{SL}}(2)$ by $S(t)=t+\frac{1}{2}$.

PROPOSITION 2.4. S^n *can be written as an m-fold product of commutators in* $\widetilde{\mathrm{SL}}(2)$ *for* $|n|\leqslant 2m-2$.

Now for $|n|\leqslant m-1$, $T^n=S^{2n}=[f_1,f_2]\ldots[f_{2m-1},f_{2m}]$ where $f_i\in\widetilde{\mathrm{SL}}(2)$. Define $\varphi:\pi_1(M)\to\mathrm{SL}$ by $\varphi(a_i)=pf_i$; then $\chi(\xi)[M]=-n$.

This proves the constructive part of Theorem 1.2 for SL(2) and hence for any larger group.

$\mathrm{Top}_2 S^1$ and in particular SL(2) acts also on the set of unoriented lines in $\mathbf{R}^2$ which we may identify with S^1. This action is not effective; $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ acts as the identity. There is a natural homorphism $h:\mathrm{Top}_2 S^1 \to \mathrm{Top}\, S^1$, $h(\mathrm{SL}(2)) \subset \mathrm{Diff}^+ S^1$, and the image $h(\mathrm{SL}(2)) \approx \mathrm{SL}(2)/\{\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\} = \mathrm{PSL}(2)$. h is covered by $\tilde{h}$: $\widetilde{\mathrm{Top}_2 S^1} \to \widetilde{\mathrm{Top}\, S^1}$. If $\tilde{h}(f)$ covers the identity, then f covers $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ so $f = S^n$ for some n. If $|n| \leqslant 2m-2$ write $S^n = [f_1, f_2] \ldots [f_{2m-1}, f_{2m}]$, $f_i \in \widetilde{\mathrm{SL}}$, and define $\varphi: \pi_1(M) \to \mathrm{Diff}^+ S^1$ by $\varphi(\alpha_i) = p\tilde{h}(f_i)$. Then $[\tilde{h}f_1, \tilde{h}f_2] \ldots [\tilde{h}f_{2m-1}, \tilde{h}f_{2m}] = \tilde{h}(S^n) = T^n$. This proves the constructive part of Theorem 1.1.

3. The non-orientable case

Let N be a non-orientable surface and $p: M \to N$ be the orientable double cover. Let $\mathcal{O}$ be the orientation bundle for N. Since $p^* w_1(N) = 0$, $p^* \mathcal{O} = \mathbf{Z}$ and we have a commutative square

$$\begin{array}{ccc} H^2(N;\mathcal{O}) & \xrightarrow{p^*} & H^2(M;\mathbf{Z}) \\ \downarrow{\scriptstyle \cap [N]} & & \downarrow{\scriptstyle \cap [M]} \\ \mathbf{Z} & \xrightarrow{\times 2} & \mathbf{Z} \end{array}$$

Thus $2\chi(\xi)[N] = \chi(p^*\xi)[M]$ for an $\mathcal{O}$-orientable bundle ξ on N, in particular $\chi(M) = 2\chi(N)$. Given $\varphi: \pi_1(N) \to G$, let ξ be the G-bundle induced by φ from the universal covering bundle. Then $p^*\xi$ is the bundle on M corresponding to $\varphi \circ p_\#$. Since ξ is $\mathcal{O}$-orientable, image $\varphi \circ p_\# \subset G^+$ and $p^*\xi$ is orientable. Now the implication i) $\Rightarrow$ ii) in Theorems 1.1 and 1.2 for non-orientable surfaces follows from the case of orientable surfaces.

We now verify Lemma 2.1 for the normal presentation of N which gives $\pi_1(N) = \{\alpha_1, \ldots, \alpha_k \mid W = 1\}$ where $W = \alpha_1^2 \ldots \alpha_k^2$. Here $k = 2 - \chi(N)$ is the genus of N. Let F_k be the free group on $\alpha_1, \ldots, \alpha_k$. Given $\varphi: \pi_1(N) \to G$, choose a lift $\tilde{\varphi}: F_k \to \tilde{G}$. $w_1(\xi)$ is the composition of φ with the projection $G \to G/G^+$, hence $\varphi(\alpha_i)$ reverses orientation on S^1 and $T\tilde{\varphi}(\alpha_i) = \tilde{\varphi}(\alpha_i) T^{-1}$; hence $\tilde{\varphi}(\alpha_i)^2$ is independent of the choice of lift. We must check that $\chi(\xi)[N] = -n$ where $\tilde{\varphi}(W) = T^n$. Let F_{2k-2} be the free group on the generators $\beta_1, \ldots, \beta_{k-1}, \gamma_1, \ldots, \gamma_{k-1}$ and set $V = \beta_1 \ldots \beta_{k-1} \gamma_{k-1}^{-1} (\gamma_{k-2} \beta_{k-1}^{-1} \gamma_{k-2}^{-1}) \ldots (\gamma_1 \beta_2^{-1} \gamma_1^{-1}) \beta_1^{-1} \gamma_{k-1}$. Then $\pi_1(M)$ can be presented as $\{F_{2k-2} \mid V = 1\}$ and the map $p_\#$ is given by

$p_\#(\beta_i)=\alpha_i^2,\ p_\#(\gamma_i)=\alpha_1\alpha_{1+i}^{-1},\ 1\leqslant i\leqslant k-1.$

$$\begin{array}{ccccc} F_{2k-2} & \xrightarrow{\tilde{p}_\#} & F_k & \xrightarrow{\tilde{\varphi}} & \tilde{G} \\ \downarrow & & \downarrow & & \downarrow \\ \pi_1(M) & \xrightarrow{p_\#} & \pi_1(N) & \xrightarrow{\varphi} & G \end{array}$$

Compute $\tilde{p}_\#(V)=W\alpha_k W^{-1}\alpha_k^{-1}$. (Take two $2k$-gons and mark the edges with arrows labeled $\alpha_1, \alpha_1', \alpha_2, \alpha_2', \ldots$ and $\alpha_1', \alpha_1, \alpha_2', \alpha_2, \ldots$ respectively. Then glue them together along α_k'; the result, figure 1, is a presentation of M with two vertices and $4k-2$ edges.

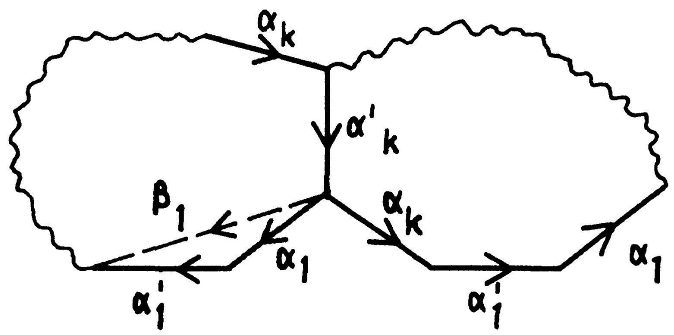


Figure 1

This can be reduced by cutting and pasting, beginning by cutting along β_1 and pasting along α_1', to the presentation of M given above. The image $\tilde{p}_\#(V)$ of the relation for M will be formally equivalent (equal in F_k) to the boundary of the original $(4k-2)$-gon, namely $\alpha_1^2\ldots\alpha_{k-1}^2\alpha_k\alpha_{k-1}^{-2}\ldots\alpha_1^{-2}\alpha_k^{-1}=W\alpha_k W^{-1}\alpha_k^{-1}$.) Now $\tilde{\varphi}(W)$ covers the identity so $\tilde{\varphi}(W)=T^n$ for some n, hence $\tilde{\varphi}\tilde{p}_\#(V)=T^n\tilde{\varphi}(\alpha_k)T^{-n}\tilde{\varphi}(\alpha_k)^{-1}=T^{2n}$. By the lemma applied to M, $\chi(p^*\xi)[M]=-2n$. Hence $\chi(\xi)[N]=-n$. Since we are interested ultimately in the absolute value we have not mentioned the several choices above which affect the sign of this result.

Now exactly as in §2 the constructive parts of Theorems 1.1 and 1.2 follow from

PROPOSITION 3.1. *For* $|n|\leqslant k-2$, $S^n=g_1^2\ldots g_k^2$ *where* $g_i\in\widetilde{\mathrm{Top}_2 S^1}$ *covers an element in the coset* $\begin{pmatrix}+1 & 0\\ 0 & -1\end{pmatrix}\mathrm{SL}(2)$.

4. A retraction $\widetilde{\mathrm{Top}^+ S^1}\to\mathbf{R}$

In this section we prove Proposition 2.2 by studying a retraction $r:\widetilde{\mathrm{Top}^+ S^1}\to\mathbf{R}$. Recall that $f\in\widetilde{\mathrm{Top}^+ S^1}$ iff f is a homeomorphism of $\mathbf{R}$ such that $f(t+1)=f(t)+1$.

An element $a\in\mathbf{R}$ corresponds to the function $f_a(t)=t+a$.

$$\text{Define } r(f)=\int_0^1 f(t)-t\,dt.$$

Clearly $r(f_a)=a$ and r covers a retraction of $\mathrm{Top}^+ S^1$ to $\mathrm{SO}(2)$.

The retraction r has the following properties:

$$r(f)=\int_u^{u+1} f(t)-t\,dt \quad \text{for any } u\in\mathbf{R}. \tag{4.1}$$

There is an x such that $f(x)-x=r(f)$. This is the mean value theorem. (4.2)

Consider the graph (figure 2).

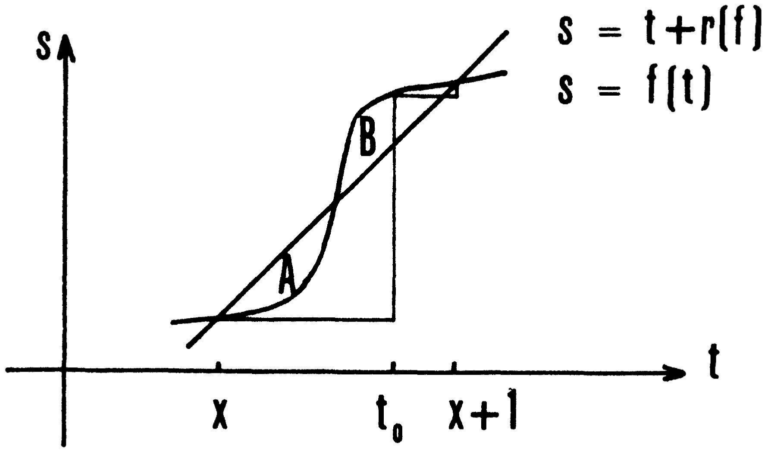


Figure 2

Let $A=\{(t,s):x\leqslant t\leqslant x+1, f(t)\leqslant s\leqslant t+r(f)\}$
$B=\{(t,s):x\leqslant t\leqslant x+1, t+r(f)\leqslant s\leqslant f(t)\}$.

$$r(f^{-1})=-r(f). \tag{4.3}$$

Proof. By (4.1) $\int_x^{x+1} f(t)-(t+r(f))\,dt=0$.

This means area A = area B. Reading from vertical to horizontal the same figure is a graph of $t=f^{-1}(s)$ and $t=s-r(f)$. Hence

$$\int_{x+r(f)}^{x+r(f)+1} f^{-1}(s)-s+r(f)\,ds=0,$$

which gives the result.

$$r(f)-\tfrac{1}{2}<f(t)-t<r(f)+\tfrac{1}{2} \quad \text{for all } t. \tag{4.4}$$

Proof. Say $f(t_0)-t_0-r(f)=u$ and refer to the graph (figure 2). Assume $u>0$. Since f is monotonic, B includes the right isosceles triangle with edges parallel to the axes and vertex at $(t_0, t_0+r(f)+u)$ and with area $\frac{1}{2}u^2$. On the other hand A is included in the two such triangles with vertices at $(t_0, x+r(f))$ and $(x+1, t_0+r(f)+u)$ whose combined area is $\leqslant\frac{1}{2}(1-u)^2$. Hence $\frac{1}{2}u^2<\text{area } A=\text{area } B<\frac{1}{2}(1-u)^2$ which implies $u<\frac{1}{2}$. The case $u<0$ is analogous.

$$r(fg)=r(f)+r(g) \quad \text{if} \quad f,\ g, \text{ or } fg \text{ is a translation}. \tag{4.5}$$

Proof. If $f(t)=t+a$, then $r(fg)=\int_0^1 g(t)+a-t\,dt=a+r(g)$.
If g is a translation $r(fg)=-r(g^{-1}f^{-1})=-r(g^{-1})-r(f^{-1})=r(f)+r(g)$.
If fg is a translation $r(f)=r((fg)\,g^{-1})=r(fg)+r(g^{-1})=r(fg)-r(g)$.

Any matrix in SL(2) can be written uniquely as a product RS of an element $R\in\mathrm{SO}(2)$ and a symmetric, positive definite matrix S. In [7] Milnor used the retraction $r'(RS)=R$. The lift to $\widetilde{\mathrm{SL}}$ of R is a translation and r (lift of S)$=0$ (since the action of S on S^1 is symmetric about the principal axes in the directions of the orthogonal eigenvectors of S), so (4.5) implies that r agrees with the lift of r' on $\widetilde{\mathrm{SL}}$. No retraction is a homomorphism; we now determine how much r differs from a homomorphism.

$$|r(fg)-r(f)-r(g)|<\tfrac{1}{2}. \tag{4.6}$$

Proof. $r(f)-\frac{1}{2}<f(g(t))-g(t)<r(f)+\frac{1}{2}$ by (4.4). Integrate from 0 to 1 and use

$$\int_0^1 f(g(t))-g(t)\,dt=r(fg)-r(g)$$

to get $r(f)-\frac{1}{2}<r(fg)-r(g)<r(f)+\frac{1}{2}$.

$$|r([f_1, f_2]\cdots[f_{2m-1}, f_{2m}])|<2m-\tfrac{1}{2}. \tag{4.7}$$

Proof. Apply (4.6) $4m-1$ times and use (4.3).

This last result is enough to prove i)$\Rightarrow$ii) in Theorem 1.2; for Theorem 1.1 a further fact is necessary.

PROPOSITION 4.8. *There is an x such that $|[f, g](x)-x|<1$.*

Proof.

Case 1: f covers an element of $\mathrm{Top}^+ S^1$ with no fixed points. This means

$$t+m<f(t)<t+m+1 \qquad \text{for some } m\in\mathbf{Z}.$$

f^{-1} is monotonic, so

$$f^{-1}(t)+m=f^{-1}(t+m)<t<f^{-1}(t)+m+1.$$

These inequalities give

$$t - m - 1 < f^{-1}(t) < t - m.$$

Replace t by $g^{-1}(t)$:

$$g^{-1}(t) - m - 1 < f^{-1}g^{-1}(t) < g^{-1}(t) - m$$

and apply g:

$$t - m - 1 < gf^{-1}g^{-1}(t) < t - m.$$

Finally $t-1<f(t)-m-1<[f, g]\,(t)<f(t)-m<t+1$.

Case 2: f covers an element of $\mathrm{Top}^+ S^1$ with a fixed point, so there is a y such that

$$f(y) = y + m \qquad \text{for some} \qquad m \in \mathbf{Z}.$$

Let $x=g(y)$, then $[f,g]\,(x)=fgf^{-1}(y)=fg(y-m)=f(x)-m$. Say $y+l \leqslant x < y+l+1$, $l \in \mathbf{Z}$. Then $y+m+l=f(y+l) \leqslant f(x) < f(y+l+1)=y+m+l+1$. Hence $x-1<y+l \leqslant f(x)-m<y+l+1 \leqslant x+1$, so $x-1<[f, g]\,(x)<x+1$.

COROLLARY 4.9. *There is an x such that*

$$|[f_1, f_2] \dots [f_{2m-1}, f_{2m}]\,(x) - x| < 2m - 1.$$

Proof. Choose x by the proposition so that

$$|[f_{2m-1}, f_{2m}]\,(x) - x| < 1.$$

Write $h=[f_1,f_2]\dots[f_{2m-3},f_{2m-2}]$, then $|r(h)|<2m-\frac{5}{2}$ by (4.7), so $|h(t)-t|<2m-2$ for all t by (4.4). Hence

$$|h\,[f_{2m-1}, f_{2m}]\,(x) - x| < 2m - 1.$$

This gives immediately the proof of

PROPOSITION 2.2. *If an m-fold product of commutators in* $\widetilde{\mathrm{Top}^+ S^1}$ *is translation by a then* $|a|<2m-1$.

5. Proof of Propositions 2.4 and 3.1.

Let K be the conjugacy class in SL(2) of $\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$. A matrix $A \in K$ if and only if trace $A=\frac{5}{2}$, hence also $A^{-1} \in K$. Any product $AB \in K \cdot K$ is equal to a commutator, $AB=ACA^{-1}C^{-1}$, since B is conjugate to A^{-1}.

PROPOSITION 5.1. *If* $A \in \mathrm{SL}(2)$ *and* $A \neq -I$, *then* $A \in K \cdot K$ *and hence is a commutator.*

Proof. $\begin{pmatrix}2 & 0\\ 0 & \frac{1}{2}\end{pmatrix}\begin{pmatrix}\frac{1}{2} & 0\\ 0 & 2\end{pmatrix}=I\in K\cdot K$ so we may assume $A\neq I, -I$. Then A is monogenic, that is, there is a vector $v\in\mathbf{R}^2$ such that v, Av are a basis for $\mathbf{R}^2$. Hence A is conjugate to a matrix of the form $\begin{pmatrix}0 & -1\\ 1 & a\end{pmatrix}$, (using $\det A=1$). Let $x=a/2-\frac{5}{2}$, then $\operatorname{tr}\begin{pmatrix}0 & -1\\ 1 & a\end{pmatrix}\begin{pmatrix}2 & 0\\ x & \frac{1}{2}\end{pmatrix}=-x+a/2=\frac{5}{2}$. Thus $\begin{pmatrix}0 & -1\\ 1 & a\end{pmatrix}\begin{pmatrix}2 & 0\\ x & \frac{1}{2}\end{pmatrix}\in K$ and $\begin{pmatrix}0 & -1\\ 1 & a\end{pmatrix}\in K\cdot K$, hence $A\in K\cdot K$.

Recall the homomorphism $j\colon \widetilde{\text{Top}^+ S^1}\to\widetilde{\text{Top}^+ S^1}$ defined in § 2 by $j(f)(t)=\frac{1}{2}f(2t)$.

LEMMA 5.2. $r(j(f))=\frac{1}{2}r(f)$.

Proof. $r(j(f))=\int_0^1 j(f)\,t-t\,dt=\frac{1}{2}\int_0^1 f(2t)-2t\,dt=\frac{1}{4}\int_0^2 f(u)-u\,du=\frac{1}{2}\int_0^1 f(u)-u\,du=\frac{1}{2}r(f)$.

COROLLARY. *Elements of image j and in particular elements of $\widetilde{\mathrm{SL}}(2)$ satisfy the following stronger versions of the inequalities of* § 4:

$$r(f)-\tfrac{1}{4}<f(t)-t<r(f)+\tfrac{1}{4}. \tag{5.3}$$

$$|r(fg)-r(f)-r(g)|<\tfrac{1}{4}. \tag{5.4}$$

$$|r([f_1,f_2]\ldots[f_{2m-1},f_{2m}])|<m-\tfrac{1}{4}. \tag{5.5}$$

Let f_0 cover $\begin{pmatrix}2 & 0\\ 0 & \frac{1}{2}\end{pmatrix}$ and satisfy $f_0(0)=0$. Let $\tilde{K}$ be the conjugacy class of f_0. Then for $f\in\tilde{K}$, $\operatorname{tr}f=$(by definition) $\operatorname{tr}(pf)=\frac{5}{2}$. f has a fixed point and so $|r(f)|<\frac{1}{4}$. Now $\operatorname{tr}f^{-1}=\frac{5}{2}$ so some translate $T^n f^{-1}\in\tilde{K}$ but then $|r(T^n f^{-1})|=|n-r(f)|<\frac{1}{4}$ so $|n|<\frac{1}{2}$ and $f^{-1}\in\tilde{K}$.

LEMMA 5.6. *$f\in\tilde{K}^{2m}$ implies f is an m-fold product of commutators.*

Proof. Say $f=f_1\ldots f_m$ where $f_i\in\tilde{K}$. Then $f_{2i}=g_{2i}f_{2i-1}^{-1}g_{2i}^{-1}$ so $f_{2i-1}f_{2i}=[f_{2i-1},g_{2i}]$.

LEMMA 5.7. *$|r(f)|<\frac{1}{4}$, $f\neq I$, and f conjugate to f^{-1} implies $S^{-1}f, f, Sf\in\tilde{K}^2$.*

Proof. By Proposition 5.1 $pf\in K^2$ so there exist $g, h\in\tilde{K}$ such that $T^n gh=f$. Then $|n+r(gh)|<\frac{1}{4}$, but $|r(gh)|<\frac{3}{4}$ so $|n|<1$ and $f\in\tilde{K}^2$. Also by 5.1 $pSf\in K^2$ so there exist $g, h\in\tilde{K}$ such that $T^n gh=Sf$. Then $|n-\frac{1}{2}+r(gh)|<\frac{1}{4}$, so $|n-\frac{1}{2}|<1$ and $n=0$ or 1. If $n=0$, $Sf=gh\in\tilde{K}^2$ and $S^{-1}f^{-1}=h^{-1}g^{-1}\in\tilde{K}^2$, but $S^{-1}f^{-1}$ is conjugate to $S^{-1}f$ so $S^{-1}f\in\tilde{K}^2$. If $n=1$, $S^{-1}f=gh\in\tilde{K}^2$ and $Sf^{-1}=h^{-1}g^{-1}\in\tilde{K}^2$, but Sf^{-1} is conjugate to Sf.

COROLLARY 5.8. *$\tilde{K}$, $S\tilde{K}$, $S^{-1}\tilde{K}\subset\tilde{K}^2$ and $f\in\tilde{K}^2$ implies $S^n f\in\tilde{K}^k$ for $|n|\leqslant k-2$.*

Proof. For the first part apply the lemma to $f\in\tilde{K}$. The second part is true by assumption for $k=2$ and the inductive step follows from the first part.

PROPOSITION 2.4. *S^n can be written as an m-fold product of commutators in* $\widetilde{\mathrm{SL}}$ *for* $|n| \leqslant 2m-2$.

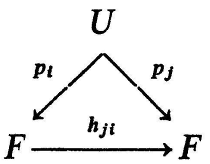
Proof. Take $f=I$ and $k=2m$ in 5.8 and use 5.6.

PROPOSITION 3.1. *For* $|n| \leqslant k-2$, $S^n = g_1^2 \dots g_k^2$ *where* $g_i \in \widetilde{\mathrm{Top}_2 S^1}$ *covers an element in the coset* $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathrm{SL}(2)$.

Proof. It remains to show that for $f \in \tilde{K}$, $f=g^2$ where $g \in \widetilde{\mathrm{Top}_2 S^1}$ covers a matrix of determinate -1. Let g_0 cover $\begin{pmatrix} \sqrt{2} & 0 \\ 0 & -1/\sqrt{2} \end{pmatrix}$, then $g_0^2 = T^n f_0 \cdot g_0$ and hence g_0^2 have fixed points so $|r(g_0^2)| < \frac{1}{4}$ and $n=0$. Finally $f = h^{-1} f_0 h = (h^{-1} g_0 h)^2$.

6. Foliations and flat connections

If the bundle $p: E \to M$ has a totally disconnected structure group and the base M and fibre F are manifolds, there is a naturally associated foliation of E. In the notation of Steenrod [10, §2.3] there are submersions $p_i : p^{-1}(V_i) \to F$; the inverse images of points are leaves of a foliation of $p^{-1}(V_i)$. For a component U of $p^{-1}(V_i \cap V_j)$ there is a commutative triangle



where the diffeomorphism h_{ji} given by $h_{ji}(y) = g_{ji}(x) \cdot y$ for any $x \in U$ is well-defined since the coordinate transformation g_{ji} is constant on components of $V_i \cap V_j$. Hence there is a well-defined foliation on E and p restricted to a leaf is a covering space of B. If the structure group acts smoothly on F, then the foliation is smooth.

A revealing example is given by $M = S^1$, $F = S^1$, and φ takes the generator of $\pi_1(S^1)$ to rotation through the angle α. Although $\pi_1(S^1) = \mathbf{Z}$ has a natural discrete topology, image φ is discrete only when α is rational in which case it is finite cyclic. If α is irrational, image φ is infinite cyclic with a totally disconnected topology which depends on α. The bundle is trivial as an SO(2) bundle and with a suitable choice of coordinates, $E \approx S^1 \times S^1$, the leaves of the associated foliation are the orbits of the constant vector field of slope α.

Ehresmann [3] and [4] defines a *connection* in a differentiable fibre bundle to be an m-dimensional distribution (subbundle $\sigma^m \subset \tau E$, where $m = \dim M$) transverse to the fibres ($\sigma + \ker dp = \tau E$) such that any curve in the base is covered by an integral curve starting from any point in the fibre. He shows that this last condition is automatically satisfied when F is compact. Let ΩM be loops based at x and identify F with the fibre at x. Then this structure gives rise to a homomorphism $\varphi : \Omega M \to \mathrm{Diff} F$.

$\operatorname{im}\varphi$ is called the holomony group. For a bundle with structure group G we also require $\operatorname{im}\varphi \subset G$. This generalizes the notion of a connection in a principal bundle (see Nomizu [8] especially p. 43).

The connection is *flat* if the distribution is integrable. For a principal bundle this is equivalent to the vanishing of the curvature form. In the flat case φ induces a homomorphism $\varphi: \pi_1(M) \to \operatorname{Diff} F$. Using φ, $\pi_1(M)$ acts on the universal covering space $\tilde{M}$ and on F and $E = \tilde{M} \times_{\pi_1(M)} F$. This construction is described by Haefliger in [5, pp. 373–5] including an example with $F = S^1$ and M the Klein bottle. See also Milnor [7, p. 221].

The map sending $A \in \mathrm{GL}$ to $|\det A|^{-1/2} A \in \mathrm{Top}_2 S^1$ is a homomorphism and a deformation retract of GL to the subgroup of matrices with determinate ± 1, hence a plane bundle ξ over M with group GL has a flat connection if and only if the associated $\mathrm{Top}_2 S^1$ bundle does. An affine connection on M is a connection on the GL bundle associated to τM.

COROLLARY 6.1. (Benzecri-Milnor) *A surface M with $\chi(M) \neq 0$ has no flat affine connection.*

Notice that the tangent circle bundle of a surface (the bundle of oriented tangent directions) does satisfy the conditions of Theorem 1.1. It is possible to give a (somewhat bizarre) geometric interpretation to the result in that case. Let ξ be the tangent circle bundle. A point of ξ corresponds to a direction in the tangent space at a point. A section X of ξ over a curve $\lambda: I \to M$ will be called a *direction field* along λ. Suppose we are given a transverse foliation. The direction field X is *parallel* along λ if it lies in one leaf of the foliation. The geometric meaning of *flat* is that if λ is a closed, null-homotopic curve and if X is parallel along λ then $X_{\lambda(0)} = X_{\lambda(1)}$. For any curve λ there is a tangent direction field and λ is a *geodesic* if this field is parallel along λ. If X and Y are each parallel along λ and $X_{\lambda(0)} = -Y_{\lambda(0)}$, in general $X_{\lambda(1)} \neq -Y_{\lambda(1)}$, since generally the action of $\pi_1(M)$ on the circle of tangent directions at each point does not preserve antipodal directions. Thus a curve which is a geodesic (in this sense) when traversed in one direction will not generally be one when traversed in the opposite direction.

It is possible however that, although the action of $g \in \operatorname{im}\varphi$ on S^1 does not preserve antipodal points, the identification of $\tilde{M} \times S^1$ with the tangent circle bundle to $\tilde{M}$ $(M \neq S^2)$ is such that geodesics are independent of orientation; $\lambda(t)$ is a geodesic if and only if $\lambda(-t)$ is. I thank William Casselman and Gheorghe Lusztig for conversations concerning the following example.

PSL(2) can be identified with the group of Möbius transformations with real coefficients, that is complex analytic functions of the form $f(z) = (az+b)/(cz+d)$ where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a non-singular real matrix. These act as isometries on the Poincaré upper half-plane or, after conjugating by a conformal map, on the Poincaré disk

$\Delta=\{z:|z|\leqslant 1\}$. Let M be a compact Riemann surface of genus $\geqslant 2$. Then there is a subgroup $\Gamma\approx\pi_1(M)$ of PSL(2) which acts as a group of covering transformations on Δ such that the quotient Δ/Γ is conformally equivalent to M, see [9, chapter 9]. Further PSL(2) acts on the unit circle $\partial\Delta$ and thus we get a natural representation $\varphi:\pi_1(M)\to \operatorname{Diff} S^1$.

PROPOSITION 6.2. *The bundle induced by the natural representation $\varphi:\pi_1(M)\to$ $\to \operatorname{Diff} S^1$ for a compact Riemann surface of genus $\geqslant 2$ is the tangent circle bundle. Further the geodesics in the (constant negative curvature) metric induced from Δ are also geodesics in the sense described above.*

Proof. The geodesics in Δ are circles which meet $\partial\Delta$ orthogonally. Define $e:\tau\Delta-\Delta$ $\to\Delta\times S^1$ by $e(v_p)=(p,\theta)$ where θ is the first intersection of the geodesic circle through p in the direction v_p with $\partial\Delta$, (see Figure 3).

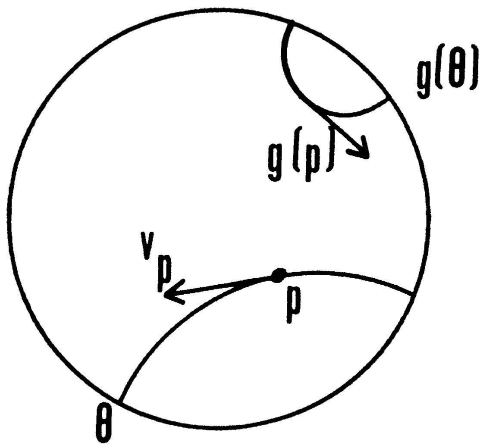


Figure 3

Let $\tau_0\Delta$ be the tangent circle bundles; e induces a bundle equivalence $\tau_0\Delta\to\Delta\times S^1$. Let $g\in\Gamma$, $\varphi(g)=g\mid\partial\Delta$, and $\tau_0 g=dg\mid\tau_0\Delta$. The commutativity of the diagram

$$\begin{array}{ccc} \tau_0\Delta & \xrightarrow{\tau_0 g} & \tau_0\Delta \\ \downarrow & & \downarrow \\ \Delta\times S^1 & \xrightarrow{g\times\varphi g} & \Delta\times S^1 \\ \downarrow & & \downarrow \\ \Delta & \xrightarrow{g} & \Delta \end{array}$$

follows from the fact that g takes circles orthogonal to $\partial\Delta$ into circles orthogonal to $\partial\Delta$. The bundle induced by φ is the quotient of $\Delta\times S^1$ by the action of Γ which is equivalent to the quotient of $\tau_0\Delta$, that is $\tau_0 M$. If v is a vector field tangent to a geodesic $\lambda:I\to\Delta$ then $e(v_{\lambda(t)})=(\lambda(t),\theta)$ where θ does not depend on t. Thus the tangent direction field to λ lies in a leaf of the product foliation of $\Delta \times S^1$ which induces the foliation of $\tau_0 M$, see [5, p. 374].

Ehresmann has shown that if L is a leaf of a differentiable foliation of M, then the normal bundle can be given a discrete structure group, i.e. reduces from GL to a totally disconnected subgroup ([4, p. 38] or [5, p. 384]).

COROLLARY 6.3. *If $L^2 \subset M^4$ with normal bundle υ, M orientable, then L can be a leaf of a differentiable foliation of a neighborhood in M if and only if $|\chi(\upsilon)[L]| \leqslant -\frac{1}{2}\chi(L)$.*

The constructive part of Theorem 1 can be used to give a counter-example to this result when the foliation is not differentiable. Let υ be an $\mathcal{O}$-orientable 2-plane bundle over L with $|\chi(\upsilon)[L]| \leqslant -\chi(L)$. The associated circle bundle reduces to a totally disconnected subgroup of DiffS^1; let $g_{ij}: S^1 \to S^1$ be a transition function. Define $f_{ij}: \mathbf{R}^2 \to \mathbf{R}^2$ by $f_{ij}(r, \theta) = (r, g_{ij}(\theta))$. Then f_{ij} is a homeomorphism but not generally smooth. The transition functions f_{ij} give the total space of υ a topological foliation (smooth in the compliment of L) in which L is a leaf. L is a smoothly embedded submanifold with normal bundle υ but for $|\chi(\upsilon)[L]| > -\frac{1}{2}\chi(L)$ the structure group does not reduce from GL(2) to a totally disconnected subgroup.

As a final application we given the following consequence of Theorem 1.1.

COROLLARY 6.4. *There is a $1-1$ correspondence between pairs of commuting homeomorphisms (diffeomorphisms) of S^1 and (smooth) foliations of the trivial S^1 bundle over $S^1 \times S^1$ transverse to the fibres (equivalent up to choice of coordinates).*

7. Products of commutators in $\widetilde{SL}(2)$

In this section we characterize the elements of $\widetilde{\mathrm{SL}}(2)$ which are m-fold products of commutators. For translations Proposition 2.4 is a special case of

PROPOSITION 7.1. *Translation by a is an m-fold product of commutators in $\widetilde{\mathrm{SL}}(2)$ if and only if $|a| < m - \frac{1}{2}$.*

Recall from §4 that the retraction defined there, when restricted to $\widetilde{\mathrm{SL}}(2)$, covers the retraction $\mathrm{SL}(2) \to \mathrm{SO}(2)$ sending RS to R where we have written an arbitrary matrix in SL(2) as a product, RS, of an orthogonal matrix and a positive definite, symmetric matrix. This property defines r on $\widetilde{\mathrm{SL}}(2)$.

THEOREM 7.2. *$f \in \widetilde{\mathrm{SL}}$ is an m-fold product of commutators in $\widetilde{\mathrm{SL}}$ if and only if there is a g conjugate to f with $|r(g)| < m - \frac{1}{2}$.*

COROLLARY 7.3. *The set of m-fold products of commutators is open.*

Proof. This set equals

$$\bigcup_h hr^{-1}((-m+\tfrac{1}{2}, m-\tfrac{1}{2}))\,h^{-1}$$

which is a union of open sets.

Notice that if f is an m-fold product of commutators then so is any g conjugate to f. Unfortunately r is not constant on conjugacy classes. Denjoy [2] has defined a retraction $i:\widetilde{\mathrm{Top}^+ S^1}\to \mathbf{R}$ which is constant on conjugacy classes and Milnor has used i to give a picture of the conjugacy classes in $\widetilde{\mathrm{SL}}(2)$. For $f\in\widetilde{\mathrm{Top}^+ S^1}$ define

$$i(f)=\lim_{n\to\infty}\frac{1}{n}f^n(x).$$

This limit exists, is independent of x (see [1, p. 406]), and depends only on the conjugacy class of f. Denjoy showed that if $f\in\widetilde{\mathrm{Diff}^+ S^1}$ and $i(f)$ is irrational then f is conjugate in $\widetilde{\mathrm{Top}^+ S^1}$ to translation by $i(f)$ and gave a counter-example for $f\in\widetilde{\mathrm{Top}^+ S^1}$

The trace is also an invariant of conjugacy classes in SL(2) and if $|\mathrm{tr}A|>2$ it is a complete invariant, that is A is conjugate to $\begin{pmatrix} a & 0\\ 0 & a\end{pmatrix}$ for a unique a with $|a|>1$. If $|\mathrm{tr}A|<2$, then A is conjugate to a rotation through θ with $\mathrm{tr}A=2\cos\theta$. There are three classes with $\mathrm{tr}A=2$ represented by the matrices $\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}$, $\begin{pmatrix} 1 & 0\\ 1 & 1\end{pmatrix}$, and $\begin{pmatrix} 1 & 0\\ -1 & 1\end{pmatrix}$. We define $\mathrm{tr}\,f=\mathrm{tr}\,pf$ for $f\in\widetilde{\mathrm{SL}}$.

Milnor's picture is the set of points in $\mathbf{R}^2$ with coordinates $(i(f), \mathrm{tr}(f))$ for $f\in\widetilde{\mathrm{SL}}(2)$, Figure 4.

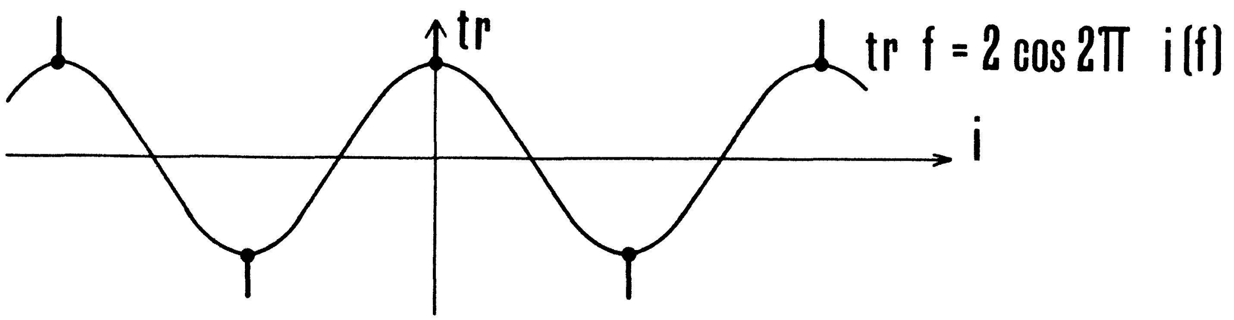


Figure 4. Conjugacy classes in $\widetilde{SL}(2)$.

The points with $|\mathrm{tr}|=2$ represent three classes. Let $U\in\widetilde{\mathrm{Top}^+ S^1}$ be the function covering $\begin{pmatrix} 1 & 0\\ 1 & 1\end{pmatrix}$ with a fixed point. Then U^{-1} covers $\begin{pmatrix} 1 & 0\\ -1 & 1\end{pmatrix}$ and the points with

$|\mathrm{tr}|=2$ and $i=n/2$ are represented by S^n, S^nU, and S^nU^{-1}. Let q be the projection of $\widetilde{\mathrm{SL}}(2)$ onto the set of conjugacy classes. This set inherits a quotient topology and if the central elements, S^n, are deleted it has the structure of a non-hausdorff manifold. A neighborhood of qU meets classes with $\mathrm{tr}>2$ and classes with $i>0$ and a neighborhood of qU^{-1} meets $\mathrm{tr}>2$ and $i<0$.

With respect to the quotient topology on $q\widetilde{\mathrm{SL}}$ the set of m-fold products of commutators can be characterized as follows:

$$\{m\text{-fold products of commutators}\} = q^{-1}(\text{interior } \{qf : |i(f)| \leqslant m - \tfrac{1}{2}\}).$$

This set includes all f with $|i(f)|\leqslant m-\frac{1}{2}$ except $S^{\pm(2m-1)}$ and those conjugate to $S^{2m-1}U$ and $S^{-2m+1}U^{-1}$.

We now prove Theorem 7.2. The characterization in terms of i is a fairly easy consequence.

LEMMA 7.4. *If* $|\mathrm{tr} f|>2$, *then* $|r(f)-n/2<\frac{1}{4}$ *for some* $n\in\mathbf{Z}$ *and* f *is conjugate to* $S^{2n}f^{-1}$.

If $|\mathrm{tr} f|\leqslant 2$, *then* $n/2\leqslant f(t)-t\leqslant (n+1)/2$ *for some* $n\in\mathbf{Z}$ *and all* t.

Proof. To prove the second part it suffices to show that for $A=pf$, either $\theta\leqslant A\theta\leqslant\leqslant\theta+\pi$ or $\theta-\pi\leqslant A\theta\leqslant\theta$ for all $\theta\in S^1$. Recall that A acts on S^1 by acting on oriented rays in $\mathbf{R}^2$, see Figure 5.

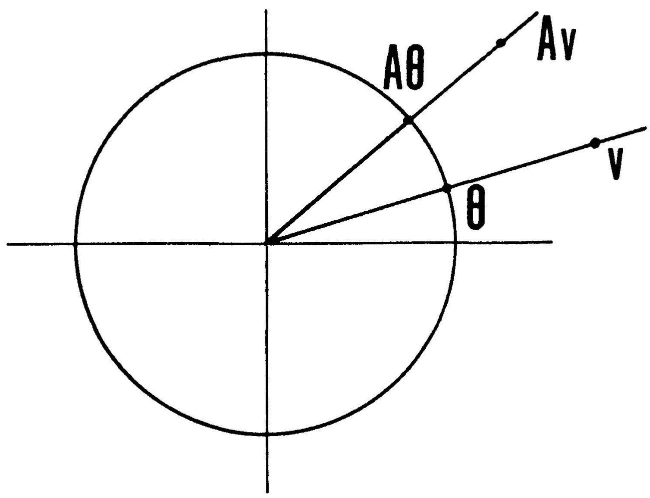


Figure 5

If $v=\begin{pmatrix}x\\y\end{pmatrix}$, then $Av=\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}ax+by\\cx+dy\end{pmatrix}$. Thus it suffices to show that

$$\begin{vmatrix}x & ax+by\\ y & cx+dy\end{vmatrix} = cx^2 + (-a+d)\,xy - by^2$$

is either $\geqslant 0$ or $\leqslant 0$ for all x, y. The discriminant of this quadratic form is $\Delta=(-a+d)^2+4bc=(a+d)^2-4$, since $\det A=1$. If $|\mathrm{tr} A|\leqslant 2$, then $\Delta\leqslant 0$ and hence the quadratic form does not change sign.

If $|\mathrm{tr} f|\geqslant 2$, f covers a matrix A with real eigenvalues λ, $1/\lambda$. Thus there is a $\theta\in S^1$ such that $A\theta=\theta$ if $\lambda>0$ or $A\theta=\theta+\pi$ if $\lambda<0$. Hence there is an x such that $f(x)=S^n x=x+n/2$ for some n. Then 5.3 implies $r(f)-\frac{1}{4}<n/2<r(f)+\frac{1}{4}$, so $|r(f)-n/2|<\frac{1}{4}$. Further $\mathrm{tr}\, S^{2n}f^{-1}=(-1)^{2n}\,\mathrm{tr}\, f^{-1}=\mathrm{tr}\, f$. If $|\mathrm{tr} f|>2$, then $S^{2n}f^{-1}$ is conjugate to some translate $T^m f$. By 5.4 and 4.3 $|r(T^m f)-r(S^{2n}f^{-1})|<\frac{1}{2}$ hence $|m-n+2r(f)|<\frac{1}{2}$, but $|r(f)-n/2|<\frac{1}{4}$, so $|m|<1$ and hence $S^{2n}f^{-1}$ is conjugate to f.

LEMMA 7.5. *If* $|\mathrm{tr} f|>2$ *and* $|r(f)-n/2|<\frac{1}{4}$, *then for any* t *with* $|t-n/2|<\frac{1}{4}$ *there is a* g *conjugate to* f *with* $r(g)=t$.

Proof. Let g be a conjugate to f and cover $\begin{pmatrix} a & 0 \\ \varepsilon & 1/a \end{pmatrix}$. This matrix retracts to

$$\frac{1}{\sqrt{(a+1/a)^2+\varepsilon^2}}\begin{pmatrix} a+1/a & -\varepsilon \\ \varepsilon & a+1/a \end{pmatrix}, \quad \text{see [7, p. 217]}.$$

Hence $\sin 2\pi r(g)=\varepsilon/\sqrt{(a+1/a)^2+\varepsilon^2}$.

Applying the homomorphism j of §2 to Corollary 4.9 gives

COROLLARY 7.6. *If* $f_i\in\widetilde{\mathrm{SL}}$ *there is an* x *such that*

$$|[f_1, f_2]\cdots[f_{2m-1}, f_{2m}](x)-x|<m-\tfrac{1}{2}.$$

This proves part of Proposition 7.1 and the rest follows from 7.2.

LEMMA 7.7. *If* $0\leqslant f(t)-t\leqslant\frac{1}{2}$ *and* $f\neq I, S$ *then* $f, S^{-1}f\in\tilde{K}^2$.

Proof. By Proposition 5.1 $pf\in K^2$ so $f=T^n gh$ for some $g, h\in\tilde{K}$. By 7.6 there is an x such that $|gh(x)-x|<\frac{1}{2}$, hence $n-\frac{1}{2}<f(x)-x<n+\frac{1}{2}$. Also $0\leqslant f(x)-x\leqslant\frac{1}{2}$ so $0<n+\frac{1}{2}<\frac{3}{2}$, hence $n=0$. In the second case $S^{-1}f=T^n gh$, $|gh(x)-x|<\frac{1}{2}$, so $n<f(x)-x<n+1$. Thus $0<n+1<\frac{3}{2}$ so $n=0$.

LEMMA 7.8. *If* $|r(g)|>m-\frac{1}{2}$, *then* $g\in\tilde{K}^{2m}$.

Proof. If $|\mathrm{tr} g|>2$, then by 7.4 $|r(g)-n/2|<\frac{1}{4}$ and g is conjugate to $S^{2n}g^{-1}$. By Lemma 5.7 $S^{-n}g$, $S^{-n\pm 1}g\in\tilde{K}^2$. Also $|n/2|<m-\frac{1}{4}$ so $|n|\leqslant 2m-1$. Then by 5.8 $g\in\tilde{K}^{2m}$.

If $|\mathrm{tr} g|\leqslant 2$, then $n/2\leqslant g(t)-t\leqslant(n+1)/2$. Hence $|r(g)-n/2-\frac{1}{4}|\leqslant\frac{1}{4}$, so $|n/2+\frac{1}{4}|<m-\frac{1}{4}$ so $-2m+1\leqslant n\leqslant 2m-2$. By 2.4 we may assume g is not a power of S. Then $S^{-n}g$ and $S^{-n-1}g\in\tilde{K}^2$ by Lemma 7.7, and by 5.8 $g\in\tilde{K}^{2m}$.

The sufficiency of the condition in Theorem 7.2 now follows from Lemma 5.6. The proof is completed by the following

LEMMA 7.9. *Assume f is an m-fold product of commutators in $\widetilde{\mathrm{SL}}$. Then $|\mathrm{tr}\, f|\leqslant 2$ implies $|r(f)|<m-\frac{1}{2}$ and $|\mathrm{tr}\, f|>2$ implies f is conjugate to g with $|r(g)|<m-\frac{1}{2}$.*

Proof. By 7.6 $|f(x)-x|<m-\frac{1}{2}$ for some x. If $|\mathrm{tr}\, f|\leqslant 2$, then $n/2\leqslant f(t)-t\leqslant (n+1)/2$, so $-\frac{1}{4}\leqslant f(t)-t-n/2-\frac{1}{4}\leqslant\frac{1}{4}$. Hence $|n/2+\frac{1}{4}|<m-\frac{1}{4}$ and $-m+\frac{1}{2}\leqslant n/2\leqslant m-1$. We may assume f is not a power of S, hence $n/2<r(f)<(n+1)/2$. Then $-m+\frac{1}{2}<r(f)<m-\frac{1}{2}$.

If $|\mathrm{tr}\, f|>2$, then $|r(f)-n/2|<\frac{1}{4}$. By 5.5 $|r(f)|<m-\frac{1}{4}$ so $|n/2|<m$, hence $|n/2|\leqslant m-\frac{1}{2}$. Now by 7.5 there is a g conjugate to f with $|r(g)|<m-\frac{1}{2}$.

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