Zeitschrift:	Commentarii Mathematici Helvetici
Herausgeber:	Schweizerische Mathematische Gesellschaft
Band:	46 (1971)
Artikel:	Homotopy Equivaleces of Almost Smooth Manifolds
Autor:	Brumfiel, G.
DOI:	https://doi.org/10.5169/seals-35528

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. <u>Mehr erfahren</u>

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. <u>En savoir plus</u>

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. <u>Find out more</u>

Download PDF: 13.07.2025

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

Homotopy Equivalences of Almost Smooth Manifolds

G. BRUMFIEL

§ 1. Introduction. Let M^k , $k \ge 6$, be a simply connected, oriented, closed combinatorial manifold with a differentiable structure in the complement of a point. Let $M_0^k = M^k$ -interior (D^k) , where $D^k \subset M^k$ is a combinatorially embedded disc. M_0^k inherits a differentiable structure from $M^k - (p)$, hence ∂M_0^k belongs to Γ_{k-1} , the group of oriented differentiable structures on S^{k-1} . In general, $\partial M_0^k \in \Gamma_{k-1}$ is not a homotopy invariant of M^k . In this paper we study this non-invariance.

Specifically, let $B_h(M_0) \subset \Gamma_{k-1}$ be the set of boundaries of homotopy smoothings of $M_0[18]$. That is, $\Sigma^{k-1} \in B_h(M_0)$ if and only if there is a smooth manifold M'_0 , with $\partial M'_0 = \Sigma^{k-1}$, and a homotopy equivalence of pairs $h:M'_0$, $\partial M'_0 \to M_0$, ∂M_0 . Then $B_h(M'_0) = B_h(M_0)$, and M^k is homotopy equivalent to a smooth manifold if and only if $0 \in B_h(M_0)$. We will give a homotopy theoretic description of the set of differences $\Delta_h(M_0) = \{\Sigma^{k-1} - \partial M_0 \mid \Sigma^{k-1} \in B_h(M_0)\} \subset \Gamma_{k-1}$, for certain classes of manifolds. If $\partial M_0 \in \Gamma_{k-1}$ is known, for example if $\partial M_0 = 0$, this determines $B_h(M_0)$. In any case, $B_h(M_0)$ and $\Delta_h(M_0)$ have the same number of elements.

Following Sullivan, two homotopy smoothings, $h:M'_0$, $\partial M'_0 \to M_0$, ∂M_0 and $g:M''_0$, $\partial M''_0 \to M_0$, ∂M_0 , are called equivalent if there is a diffeomorphism $f:M'_0 \cong M''_0$ such that h is homotopic to gf. The set of equivalence classes is denoted $hS(M_0)$. In [18], Sullivan constructs a bijection $\theta:hS(M_0) \cong [M_0, F/0]$, where F/0 is the fibre of the map $BSO \to BSF$. Thus, if $h:M'_0 \to M_0$ represents an element of $hS(M_0)$, the formula $d\theta(M'_0, h) = \partial M'_0 - \partial M_0 \in \Gamma_{k-1}$ defines a map $d: [M_0, F/0] \to \Gamma_{k-1}$, and $\Delta_h(M_0) = \text{image}$ $(d) \subset \Gamma_{k-1}$.

The group Γ_{k-1} can be described as follows. If $k \neq 2^j - 1$ or $2^j - 2$ then $\Gamma_{k-1} \simeq \simeq bP_k \oplus (\pi_{k-1}^s/\operatorname{im}(J))$, where $bP_k \subset \Gamma_{k-1}$ is the cyclic subgroup of homotopy spheres that bound π -manifolds [9], [11], [15].

 $\Gamma_{2J-2} \simeq \operatorname{kernel}(\pi_{2J-2}^{s} \xrightarrow{\psi} Z_{2})$, where ψ is the Arf invariant. $\psi \neq 0$ if and only if the element $h_{j-1}^{2} \in \operatorname{Ext}_{A}(Z_{2}, Z_{2})$ is an infinite cycle in the Adams spectral sequence [6]. Mahowald has shown that h_{j-1}^{2} is an infinite cycle if $j \leq 6$. Also, if $\psi \neq 1$, $\Gamma_{2}{}^{j}{}_{-3} = \pi_{2J-3}^{s}/\operatorname{im}(J) (=\pi_{2J-3}^{s} \operatorname{if} j \geq 2)$.

If k is odd then $bP_k = 0$. If k is even, the direct sum decomposition of Γ_{k-1} follows from properties of two homomorphisms, namely, the Kervaire-Milnor map $\varrho: \Gamma_{k-1} \rightarrow \pi_{k-1}^s/\operatorname{im}(J)$, with kernel $(\varrho) = bP_k$ [15], and an invariant $f_R: \Gamma_{k-1} \rightarrow Z_2$ if $k = 4n + 2 \neq 2^j - 2$ [11], or $f_R: \Gamma_{k-1} \rightarrow Z_{\theta_n}$ if k = 4n, where $\theta_n = a_n \cdot 2^{2n-2} \cdot (2^{2n-1} - 1) \operatorname{num}(B_n/4n)$, $a_n = 2$ if n is odd, $a_n = 1$ if n is even, and B_n is the Bernoulli number [9]. The restriction of f_R to $bP_k \subset \Gamma_{k-1}$ is an isomorphism. Thus a homotopy sphere $\Sigma^{k-1} \in \Gamma_{k-1}$ is determined by $\varrho(\Sigma^{k-1}) \in \pi_{k-1}^s/\operatorname{im}(J)$ and $f_R(\Sigma^{k-1}) \in bP_k$. The invariants $f_R: \Gamma_{4n-1} \to Z_{\theta_n}$ and $f_R: b \operatorname{spin}_{8n+2} \to Z_2$ are natural, and can be computed where $\operatorname{bspin}_{8n+2} \subset \Gamma_{8n+1}$ is the subgroup (of index 2) of homotopy spheres that bound spin manifolds. However, $f_R: \Gamma_{8n+5} \to Z_2$ and the extension $f_R: \Gamma_{8n+1} \to Z_2$ depend on choices, and can not be effectively computed. Thus our results on $\Delta_h(M_0^k)$ are complete only if $k \not\equiv 6 \pmod{8}$ and if, when $k \equiv 2 \pmod{8}$, M_0^k is a spin manifold.

The paper is arranged as follows. In §§ 2 and 3, we discuss Sullivan's work on homotopy smoothings and describe the composition $\varrho d: [M_0^k, F/0] \rightarrow \Gamma_{k-1} \rightarrow \pi_{k-1}^s / \operatorname{im}(J)$. In § 4, we give some homotopy theoretic results on F/0. Many of the results in these three sections are well-known. In § 5, we compute the composition $f_R d: [M_0^{4n}, F/0] \rightarrow \Gamma_{4n-1} \rightarrow Z_{\theta_n}$. In § 6, we compute the composition $f_R d: [M_0^{8n+2}, F/0] \rightarrow \Gamma_{8n+1} \rightarrow Z_2$ for spin manifolds, M_0^{8n+2} . The main results of the paper are Propositions 4.4, 4.5, 5.1, 5.2 and 6.5.

In two appendixes, we give applications of the results of §2 through §6. In Appendix I, we set $M^{2k} = CP(k)$ and characterize those homotopy (2k-1)-spheres which admit differentiable, fixed point free, S^1 actions. In Appendix II, we set $M^{k+1} =$ $= S^1 \times N^k$ and compute certain canonical subgroups of the inertia group, $I(N^k) \subset \Gamma_k$, of a smooth manifold N^k .

Many of the ideas in this paper are due to D. Sullivan. I am very grateful to him for many conversations.

§2. Homotopy Smoothings. We first sketch a definition of the bijection $\theta:hS(M_0) \cong$ $\cong [M_0, F/0]$. Let $h:M'_0 \to M_0$ be a homotopy smoothing of M'_0 , and let \bar{h} be a homotopy inverse of h. Homotope the map h to a smooth embedding of M'_0 in the total space, $E(\xi_0)$, of the (stable) vector bundle $\xi_0 = \xi_0(h) = \bar{h}^*(\tau_{M_0'}) - \tau_{M_0}$ over M_0 where τ_{M_0} is the tangent bundle. Then the normal bundle of M'_0 in $E(\xi_0)$ is trivial and choosing a framing of M'_0 in $E(\xi_0)$ determines a fibre homotopy trivialization of ξ_0 . (In fact, it follows from the *h*-cobordism theorem that there is a diffeomorphism $H:M'_0 \times \mathbb{R}^q \cong E(\xi_0^q), q$ large, homotopic to h.) This defines an element $\theta(h) \in [M_0, F/0]$, which depends only on the class of (M'_0, h) in $hS(M_0)$. By construction, the composition $M_0 \to F/O \to BSO$ represents $\xi_0(h) \in KO^0(M_0)$.

Now, h induces a bijection $h_*:hS(M'_0) \cong hS(M_0)$, defined by $h_*(M''_0, g) = (M''_0, hg)$ where $g:M''_0 \to M'_0$. Also, there is the bijection $h^*: [M_0, F/0] \cong [M'_0, F/0]$ induced by the homotopy equivalence $h:M'_0 \to M_0$. Since F/0 is an H-space, h^* is an isomorphism of groups. Consider the diagram

$$\begin{split} & hS(M_0) \stackrel{\theta}{\Rightarrow} \begin{bmatrix} M_0, F/0 \end{bmatrix} \\ & h_* \stackrel{\theta}{\xrightarrow{}} & \downarrow_{h^*} \stackrel{d}{\xrightarrow{}} & \Gamma_{k-1} \\ & hS(M_0') \stackrel{\varphi}{\Rightarrow} \begin{bmatrix} M_0', F/0 \end{bmatrix} \end{split}$$
 (2.1)

This diagram is very non-commutative. In fact, if $g:M''_0 \to M'_0$ is a homotopy smoothing

of M'_0 then $d\theta(h_*(g)) = \partial M''_0 - \partial M_0 = (\partial M''_0 - \partial M'_0) + (\partial M'_0 - \partial M_0) = d\theta(g) + d\theta(h)$. We also have

PROPOSITION 2.2. If $g \in hS(M'_0)$ then $h^*\theta h_*(g) - \theta(g) = h^*\theta(h) \in [M'_0, F/0]$.

This can be equivalently stated as follows. Suppose

$$\begin{array}{ccc}
M_0'' & \xrightarrow{f} & M_0 \\
 & & & & \\
 & g \searrow & & & \\
 & & & & M_0' & & \\
\end{array}$$

is a homotopy commutative diagram and f, g, h are all homotopy equivalences. Then $f=h_*(g)$ and applying the isomorphism \bar{h}^* to the equation in 2.2 gives

$$\theta(f) = \theta(h) + h^*(\theta(g)) \in [M_0, F/0]$$
(2.3)

We will prove 2.3. In §§ 5 and § 6 we give formulas for the difference $d-dh^*$ and for the deviation of d from linearity (that is, in general d is not a homomorphism of groups).

Proof of 2.3. Choose a diffeomorphism $H:M'_0 \times \mathbf{R}^q \cong E(\xi^q(\theta(h)))$ homotopic to h, and, in the diagram below, let $E(\bar{H})$ be the obvious bundle map covering $\bar{H} = H^{-1}$.

Since $\pi_1 \overline{H} \simeq h\pi$, it follows from the bundle covering homotopy theorem that there is a bundle isomorphism, *B*, covering the identity on $E(\xi^q(\theta(h)))$, and a bundle homotopy commutative diagram

$$E(h^*(\xi^q(\theta(g))) + \xi^q(\theta(h))) = E(\pi^*h^*(\xi^q(\theta(g)))) \xrightarrow{E(h\pi)} E(\xi^q(\theta(g)))$$

$$\downarrow^B \qquad \uparrow^{E(\pi_1)}$$

$$E(\bar{H}^*\pi_1^*(\xi^q(\theta(g)))) \xrightarrow{E(H)} E(\pi_1^*(\xi^q(\theta(g))))$$

$$= E(\xi^q(\theta(g))) \times \mathbf{R}^q.$$

Let $G: M_0'' \times \mathbf{R}^q \cong E(\xi^q(\theta(g)))$ be a diffeomorphism homotopic to g. Then $\overline{F} = (\overline{G} \times 1) E(\overline{H}) B: E(\overline{h}^*(\xi^q(\theta(g))) + \xi^q(\theta(h))) \cong M_0'' \times \mathbf{R}^q \times \mathbf{R}^q$ is a diffeomorphism homotopic to $\overline{f} = \overline{gh}$ where $\overline{G} = G^{-1}$. Thus the fibre homotopy trivialization

$$(\pi_2 \times \pi_3) \mathbf{F} : E(h^*(\xi^q(\theta(g))) + \xi^q(\theta(h))) \to \mathbf{R}^q \times \mathbf{R}^q$$

represents $\theta(f)$. On the other hand, bundle homotopy commutativity of the diagram above implies that $(\pi_2 \times \pi_3) F$ is properly homotopic to $(\pi_2 \bar{G}E(\bar{h}) \times \pi_2 \bar{H}) \Delta$ where

$$\Delta : E(h^*(\xi^q(\theta(g))) + \xi^q(\theta(h))) \to E(h^*(\xi^q(\theta(g)))) \times E(\xi^q(\theta(h)))$$

is the diagonal. Since $(\pi_2 \bar{G}E(h) \times \pi_2 \bar{H}) \Delta$ represents $h^*(\theta(g)) + \theta(h)$, we have shown that $\theta(f) = h^*(\theta(g)) + \theta(h)$, as desired.

The tangential homotopy equivalence, that is, $h:M'_0 \to M_0$ with $h^*(\tau_{M_0}) = \tau_{M_0'}$ are particularly important. Let $B_{th}(M_0) \subset \Gamma_{k-1}$ be the set of boundaries of manifolds M'_0 tangentially homotopy equivalent to M_0 , and let $\Delta_{th}(M_0) = \{\Sigma^{k-1} - \partial M_0 \mid \Sigma^{k-1} \in B_{th}(M_0)\} \subset \Gamma_{k-1}$.

There is a fibration $SF \xrightarrow{j} F/0 \xrightarrow{i} BS0$, where $SF = \lim_{\to} SF_q$ and SF_q is the space of base point preserving maps of degree one of S^{q-1} to itself. Thus, given $h:M'_0 \to M_0$, we have $h^*(\tau_{M_0}) = \tau_{M_0'}$ if and only if $\xi_0(h) = h^*(\tau_{M_0'}) - \tau_{M_0} = 0 \in K0^\circ(M_0)$ or, equivalently, if and only if $\theta(h) \in \text{image}([M_0, SF] \xrightarrow{j^*} [M_0, F/0])$. Thus $\Delta_{th}(M_0) = d(\text{image}([M_0, SF] \to [M_0, F/0]))$.

Two other subsets of $B_h(M_0)$ are of geometric interest. Let $B_c(M_0) \subset \Gamma_{k-1}$ be the set of boundaries of smooth manifolds M''_0 combinatorially equivalent to M_0 , and let $B_{tc}(M_0) \subset B_c(M_0)$ be the subset of boundaries of those M'_0 such that some combinatorial equivalence $h:M'_0 \to M_0$ preserves the (smooth) tangent bundles, that is, $h^*(\tau_{M_0}) = \tau_{M_0'}$ as vector bundles. Let $\Delta_c(M_0) = \{\Sigma^{k-1} - \partial M_0 \mid \Sigma^{k-1} \in B_c(M_0)\}$ and let $\Delta_{tc}(M_0) = \{\Sigma^{k-1} - \partial M_0 \mid \Sigma^{k-1} \in B_{tc}(M_0)\}$.

There are spaces SPL and PL/0, and a braid of fibrations



From smoothing theory [14], it follows that $\Delta_c(M_0) = d(\text{image}([M_0, PL/0] \rightarrow M_0, F/0]))$ and that $\Delta_{tc}(M_0) = d(\text{image}([M_0, SPL] \rightarrow M_0, F/0]))$. Also, if $v \in [M_0^k, PL/0]$ then $dv = \partial^*(v) \in \pi_{k-1}(PL/0) = \Gamma_{k-1}$, where $\partial: S^{k-1} \rightarrow M_0^k$ represents the homotopy class of the inclusion of the boundary, $\partial M_0 \rightarrow M_0$.

In particular, $d: [M_0^k, PL/0] \rightarrow \Gamma_{k-1}$ and $d: [M_0^k, SPL] \rightarrow \Gamma_{k-1}$ are group homomorphisms. Also, $\Delta_c(M_0^k)$ and $\Delta_{tc}(M_0^k)$ are homotopy invariants of M_0^k .

Recall that for a simply connected, closed manifold, M^k , there is the surgery obstruction $s: [M^k, F/0] \rightarrow P_k$, where $P_k = \mathbb{Z}$, 0, \mathbb{Z}_2 , 0 if $k \equiv 0, 1, 2, 3 \pmod{4}$, respectively, defined as follows [18]. If $u \in [M^k, F/0]$, represent u by a framing $f: M' \times \mathbb{R}^q \rightarrow F(\xi^q(u))$ of some manifold M' in the total space of the bundle $\xi^q(u) = i_*(u)$ over M.

Then $s(u) \in P_k$ is the obstruction to constructing a homotopy equivalence $M'' \times \mathbb{R}^q \to E(\xi^q(u))$, framed cobordant to $M' \times \mathbb{R}^q$ in $E(\xi^q(u)) = E(\xi^q)$.

PROPOSITION 2.4 (Sullivan). Suppose $u: M_0^k \to F/0$ extends to a map $\bar{u}: M^k \to F/0$. Then $du \in bP_k$. In fact, $du = bs(\bar{u})$ where $b: P_k \to bP_k$ is the natural projection.

PROOF. Represent \bar{u} by a framing of a connected sum M' # W in the vector bundle $E(\xi(\bar{u}))$ over M where the projection $M'_0 \to M_0$ is a homotopy equivalence and where W is an almost parallelizable manifold. Then $s(\bar{u}) = -[W] \in P_n$ where P_n is regarded as the group of cobordism classes of almost parallelizable PL manifolds. By smoothing theory, in the complement of a point, M' # W inherits a smooth structure from $E(\xi(\bar{u}))$ and $\partial (M' \# W)_0 = \partial M_0$. Then $du = \partial M'_0 - \partial M_0 = -\partial W_0 = bs(\bar{u}) \in bP_k$.

REMARK 2.5. If k = 4n and $u \in [M^{4n}, F/0]$ is represented by $f: M' \times \mathbb{R}^q \to E(\xi^q)$, then

$$s(u) = \left(\frac{1}{8}\right) \left(\operatorname{index}\left(M\right) - \operatorname{index}\left(M'\right)\right) = \left(\frac{1}{8}\right) \left\langle L(M)\left(1 - L(\xi)\right), \left[M^{4n}\right] \right\rangle \in \mathbb{Z}$$

since $\tau_{M'} = f^*(\tau_M + \xi)$.

If k=4n+2 and $u \in [M^{4n+2}, F/0]$, there is also a cohomology formula for s(u); namely,

$$s(u) = \langle v^2(M) \cdot u^*(K), [M]_2 \rangle \in \mathbb{Z}_2$$

where $v(M) = 1 + v_1(M) + v_2(M) + ... \in H^*(M, \mathbb{Z}_2)$ is the total Wu class, and $K = k_2 + k_6 + k_{10} + ... \in H^{4*+2}(F/0, \mathbb{Z}_2)$ is a suitable class [18].

§ 3. The composition $\varrho d: [M_0^k, F/0] \rightarrow \Gamma_{k-1} \rightarrow \pi_{k-1}^s / \operatorname{im} (J)$

Let $\partial: S^{k-1} \to M_0^k$ represent the homotopy class of the inclusion of the boundary, $\partial M_0^k \to M_0^k$. Then ∂ induces $\partial^*: [M_0^k, F/0] \to [S^{k-1}, F/0] = \pi_{k-1}(F/0)$. Further, image (∂^*) is contained in the torsion subgroup of $\pi_{k-1}(F/0)$, which is isomorphic to $\pi_{k-1}^s/(I)$.

PROPOSITION 3.1. Let $u \in [M_0^k, F/0]$. Then $\varrho(du) = \partial^*(u) \in \pi_{k-1}^s / \operatorname{im}(J) \subset \pi_{k-1}(F/0)$.

Proof. Let $u = \theta(h)$, where $h: M'_0 \to M_0$. Then *u* is represented by a fibre homotopy trivialization of $\xi_0(h) = \xi_0$, defined by a framing $H: M'_0 \times \mathbb{R}^q \cong E(\xi_0^q)$. The restriction of ξ_0 to ∂M_0^k is trivial. For, if $k-1\equiv 0$ or 4 (mod 8), the Pontrjagin class of $\xi_0|_{\partial M_0^k}$ is zero, and if $k-1\equiv 1$ or 2 (mod. 8) $\xi_0|_{\partial M_0^k}$ is fibre homotopically trivial. Thus, *H* induces a framing $\partial H: \partial M'_0 \times \mathbb{R}^q \cong \partial M_0 \times \mathbb{R}^q$, which represents $\partial^*(u) \in \pi_{k-1}(F/0)$. It now

G. BRUMFIEL

follows from the definition of the Kervaire-Milnor map, ρ , and a little smoothing theory, that $\partial^*(u) = \rho (\partial M'_0 - \partial M_0) = \rho (du)$.

COROLLARY 3.2. The composition $\varrho d: [M_0^k, F/0] \rightarrow \pi_{k-1}^s / \operatorname{im}(J)$ is a homomorphism of groups. Thus, if $u, v \in [M_0^k, F/0]$ then $du + dv - d(u+v) \in bP_k \subset \Gamma_{k-1}$.

COROLLARY 3.3. Let $h:M'_0 \to M_0$ be any degree one map (not necessarily a homotopy equivalence). Then $\varrho(dh^*(u)) = \varrho(du)$, where $u \in [M_0, F/0]$ and $h^*: [M_0, F/0] \to [M'_0, F/0]$. Thus $dh^*(u) - du \in bP_k \subset \Gamma_{k-1}$.

§ 4. Discussion of F/0. If we are to apply the results of § 2 and § 3 (and those in § 5 and § 6 below), we must be able to compute $[M_0^k, F/0]$. In general, this is difficult. The following discussion relates the group $[M_0^k, F/0]$ to more familiar homotopy invariants of M_0^k .

There are fibrations $S0 \xrightarrow{\Omega J} SF \xrightarrow{j} F/0 \xrightarrow{i} BS0 \xrightarrow{J} BSF$. These induce an exact sequence of groups

$$K0^{-1}(X) \rightarrow [X, SF] \xrightarrow{j_*} [X, F/0] \xrightarrow{i_*} K0^0(X) \rightarrow J(X) \rightarrow 0$$

for any finite complex X. Further, since SF_{q+1} is a component of $\Omega^q S^q$, $[X, SF] = \lim_{\to} [S^q \wedge X, S^q] = \pi_0^s(X)$, as sets, where $\pi_0^s(X)$ is the 0th stable cohomotopy group of X. Actually, $\pi_0^s(X)$ is a ring, and, as groups, $[X, SF] \simeq 1 + \pi_0^s(X)$ where the addition on the right is given by $(1+\alpha)(1+\beta) = 1 + \alpha + \beta + \alpha\beta$ [13].

The Adams conjecture on $J: K0^0(X) \rightarrow J(X)$ can be stated as follows ([1]):

4.1 Let $\xi \in K0^0(X)$. Then there is an integer, $e(k, \xi)$, such that $J(k^{e(k,\xi)}(\psi^k - 1)(\xi)) = 0$ where ψ^k is the Adams operation.

Since $K0^{0}(X)$ is finitely generated, we may choose $e(k, \xi) = e(k)$ independent of ξ . For any function e(k), Adams has proved that kernel $(J) = i_*([X, F/0])$ is contained in the subgroup of $K0^{0}(X)$ generated by the elements $k^{e(k)}(\psi^k - 1)(\xi), \xi \in K0^{0}(X)$. The Adams conjecture 4.1 has recently been proved by Sullivan and Quillen.

PROPOSITION 4.2. If $K0^0(M^k) \rightarrow K0^0(M_0^k)$ is surjective (e.g., if $k-1 \neq 1$ or 2 (mod 8) or if M^k is a spin manifold), then each element $w \in [M_0^k, F/0]$ can be written as a sum, w = u + v, where $u \in \text{image}([M^k, F/0])$ and $v \in \text{image}([M_0, SF)]$.

Proof. $J(\xi_0(w)) = J(i_*(w)) = 0$. It follows that there is an element $\xi \in K0^0(M^k)$ such that $J(\xi) = 0$ and $\xi \mid_{M_0} = \xi_0(w) = \xi_0$. Then $\xi = i_*(\bar{u})$ for some $\bar{u} \in [M^k, F/0]$. Let $u = \bar{u} \mid_{M_0}$. Then $w - u \in \text{kernel}$ $(i_*) = \text{image}(j_*)$, and 4.2 is proved.

Remark 4.3. It is a consequence of the Adams conjecture that for each prime p, there is a homotopy equivalence $(F/0)_{(p)} \sim BSO_{(p)} \times Cok(J)_{(p)}$ where $X_{(p)}$ denotes the

localization of X at p. Morevoer, $SJ_{(p)} \sim \operatorname{im}(J)_{(p)} \times Cok(J)_{(p)}$, and the map $j_{(p)} : SF_{(p)} \rightarrow (F/0)_{(p)}$ is a product map $j_{(p)} \times Id : \operatorname{im}(J)_{(p)} \times Cok(J)_{(p)} \rightarrow BSO_{(p)} \times Cok(J)_{(p)}$. This factoring of $(F/0)_{(p)}$ enables one to also establish the conclusion of 4.2 in the case $(k-1) \equiv 2 \pmod{8}$.

PROPOSITION 4.4. If $u, v \in [M_0^k, F/0]$, with $u \in \text{image}([M_0^k, F/0])$ and $v \in \text{image}([M_0, SF])$, then $d(u+v) = du + dv \in \Gamma_{k-1}$.

Proof. Let $v = \theta(h)$, and let $h^*(u) = \theta(g)$ where $h: M'_0 \to M_0$ and $g: M''_0 \to M'_0$ are homotopy equivalences. By 2.3, $\theta(f) = u + v$ where $f = hg: M''_0 \to M_0$. Thus, $d(u+v) = \partial M''_0 - \partial M_0 = (\partial M''_0 - \partial M'_0) + (\partial M'_0 - \partial M_0) = dh^*(u) + dv$.

By the hypothesis, $h: M'_0 \to \mathbf{M}_0$ is a tangential homotopy equivalence. Also, the maps $M'_0 \xrightarrow{h} M_0 \xrightarrow{u} F/0$ extend to maps $M' \xrightarrow{h} M \xrightarrow{\bar{u}} F/0$. By Proposition 2.4, du and $dh^*(u)$ belong to $bP_k \in \Gamma_{k-1}$. Since $h^*(L(M)) = L(M')$ and $h^*(v^2(M)) = v^2(M')$, it follows from the formulas in Remark 2.5 that $du = dh^*(u)$. Thus $d(u+v) = dh^*(u) + dv = = du + dv$.

The following is an immediate consequence of Propositions 2.4, 4.2, 4.4, and Remark 4.3, and is one of our main results.

PROPOSITION 4.5. Assume that $k \not\equiv 2 \pmod{8}$ or that M_0^k is a spin manifold. Then

$$\Delta_h(M_0^k) = (\Delta_h(M_0^k) \cap bP_k) + \Delta_{th}(M_0^k) \subset \Gamma_{k-1}$$

Here, by the sum of the two subsets, we mean all elements $\Sigma + \Sigma'$ where $\Sigma \in \Delta_h(M_0^k) \cap bP_k$ and $\Sigma' \in \Delta_{th}(M_0^k)$.

Remark 4.6. Note that the map $\partial^*: [M_0^k, SF] \to \pi_{k-1}(SF) = \pi_{k-1}^s$ is an invariant of the stable homotopy of M_0^k and can be computed as

 $\partial^*: [S^q \wedge M_0^k, S^q] \to \pi_{q+k-1}(S^q) = \pi_{k-1}^s, q \text{ large}.$

We will need the following familiar invariant. Consider the subgroup of elements $(\xi, \alpha) \in K0^0(X) \otimes \pi_{4k-1}(X)$ such that $ph_k(\xi) = 0 \in H^{4k}(X, Q)$ and $\alpha^* = 0: H^{4k-1}(X) \rightarrow H^{4k-1}(S^{4k-1})$. Let $\bar{X} = X \bigcup_{\alpha} e^{4k}$, and let $\bar{\xi} \in K0^0(\bar{X})$ restrict to $\xi \in K0^0(X)$. Then $ph_k(\bar{\xi}) \in p^*(H^{4k}(S^{4k}, Q)) = Q$, where $p: \bar{X} \rightarrow S^{4k}$ is the projection. Further, since $\bar{\xi}$ is well-defined modulo $p^*(K0^0(S^{4k}))$, $ph_k(\bar{\xi})$ is well-defined modulo $p^*(H^{4k}(S^{4k}, a_kZ))$. It follows that $e_R(\xi, \alpha) = (1/a_k) ph_k(\bar{\xi}) \in Q/Z$ is a well-defined homomorphism. Moreover, the diagram

$$\begin{array}{c}
K0^{0}(X) \otimes \pi_{4k-1}(X) & & \\
\downarrow \mathscr{P} \otimes s & & & Q/Z \\
K0^{0}(S^{8} \wedge X) \otimes \pi_{4k+7}(S^{8} \wedge X) & & e_{R}
\end{array}$$
(4.7)

commutes (when e_R is defined), where \mathscr{P} is the periodicity isomorphism and s is suspension. e_R can be interpreted as a functional operation from K0-theory to cohomology. If $X = S^{8n}$ and $\xi \in K0^0(S^{8n})$ is a generator, we recover the Adams homomorphism $e_R: \pi_{8n+4k-1}(S^{8n}) \rightarrow Q/Z$ [2]. If $X = M_0^{4n}$ and $\alpha \in \pi_{4n-1}(M_0^{4n})$ represents the inclusion of the boundary, we get a homomorphism $e_R: K0^0(M_0^{4n}) \rightarrow Q/Z$.

The following K0-theory invariant of F/0 bundles will also be essential.

PROPOSITION 4.8. There is an element $\gamma \in 1 + K0^{\circ}(F/0)$ such that $ph(\gamma) = \hat{A} \in H^{**}(F/0, Q) \simeq H^{**}(BS0, Q)$. Further, if $u, v \in [X, F/0]$ then $\gamma(u+v) = \gamma(u) \cdot \gamma(v) \in 1 + K0^{\circ}(X)$, where by $\gamma(u)$ we mean $u^{*}(\gamma) \in 1 + K0^{\circ}(X)$.

Proof. The universal bundle over F/0 admits a unique spin structure. Thus, the Thom space M(F/0) has two canoncial K0-theory orientations, namely, an orientation $U_1 \in K0^0(M(F/0))$ induced from M Spin, with $ph(U_1) = \Phi(\hat{A}^{-1}) \in H^{**}(M(F/0), Q)$, and an orientation, U_2 , with $ph(U_2) = \Phi(1)$, induced from the sphere spectrum via a fibre homotopy trivialization. Define $\gamma \in 1 + K0^0(F/0)$ by the equation $\gamma \cdot U_1 =$ $= U_2 \in K0^0(M(F/0))$. Then $\Phi(1) = ph(U_2) = ph(\gamma)ph(U_1) = \Phi(ph(\gamma) \cdot A^{-1})$, hence $ph(\gamma) = \hat{A}$.

The second statement follows from universal multiplicative properties of the orientations U_1 and U_2 .

The final three results in this section are technical results about the invariants e_R and γ which we will need in §5.

Let $u \in [M_0^k, F/0]$ correspond to a homotopy equivalence $h: M'_0 \to M_0$. Homotope h to an embedding $h: M'_0 \to M_0 \times \mathbb{R}^{8q}$. The normal bundle of M'_0 in $M_0 \times \mathbb{R}^{8q}$ is $h^*(-\xi_0(u))$, and we have the "collapsing map" $c: T(e_{M_0}^{8q}) \to T(h^*(-\xi_0)_{M_0'}^{8q})$. Since ξ_0 is a spin vector bundle there are Thom isomorphisms $\Phi_{K0}: KO(M'_0) \cong KO^0(T(h^*(-\xi_0)_{M_0'}^{8q})))$ and $\Phi_{K0} = \mathscr{P}: KO(M_0) \cong KO^0(T(e_{M_0}^{8q}))$, and a Gysin homomorphism $h_*: KO(M'_0) \to KO(M_0)$ defined by $h_*(x) = \mathscr{P}^{-1}c^*\Phi_{K0}(x)$.

PROPOSITION 4.9. If $u \in [M_0, F/0]$ corresponds to $h: M'_0 \to M_0$ then $h_*(1) = = \gamma(u) \in KO(M_0)$.

Proof. This follows from the definition of $\gamma(u)$ and the observation that the fibre homotopy trivialization

$$T\left(\xi_0^{8q} + e_{M_0}^{8q}\right) \xrightarrow{\overline{c}} T\left(h^*\left(\xi_0^{8q}\right) + h^*\left(-\xi_0^{8q}\right)\right) = T\left(e_{M_0}^{16q}\right) \xrightarrow{\pi} S^{16q}$$

represents $u \in [M_0, F/0]$, where \bar{c} is defined by embedding $M_0 \times \mathbb{R}^{8q} \subset E(\xi_0^{8q}) \times \mathbb{R}^{8q}$ and extending c, and π is the projection.

PROPOSITION 4.10(i) Let $u, v \in [M_0^{4n}, F/0]$. If $v \in [M_0, PL/0]$ or $v \in [M_0, SF]$, then $e_R(\gamma(u+v)) = e_R(\gamma(u)) + e_R(\gamma(v)) \in Q/\mathbb{Z}$. (ii) Suppose M_0^{4n} is a spin manifold. If $u \in [M_0^{4n}, SF]$ or $u \in [M_0^{4n}, PL/0]$, then $e_R(\gamma(u)) = e_R(\xi_0(u)) = 0$.

Proof. Let $\overline{\gamma(u)}$, $\overline{\gamma(v)} \in K0(M^{4n})$ extend $\gamma(u)$, $\gamma(v) \in K0(M_0^{4n})$. By 4.8, $\gamma(u+v) = \gamma(u) \cdot \gamma(v)$, so $\overline{\gamma(v)} \cdot \overline{\gamma(v)} \in K0(M^{4n})$ is an extension of $\gamma(u+v)$. Then

$$e_{R}(\gamma(u+v)) = (1/a_{n}) \langle ph(\overline{\gamma(u)}, \overline{\gamma(v)}), [M^{4n}] \rangle$$

= $(1/a_{n}) \langle ph(\overline{\gamma(u)}) ph(\overline{\gamma(v)}), [M^{4n}] \rangle \in Q/\mathbb{Z}.$

From the assumption, it follows that $ph(\overline{\gamma(v)}) = 1 + ph_n(\overline{\gamma(v)})$; hence

$$(1/a_n) \langle ph(\overline{\gamma(u)}) ph(\overline{\gamma(v)}), [M^{4n}] \rangle = (1/a_n) \langle ph_n(\overline{\gamma(u)}) + ph_n(\overline{\gamma(v)}), [M^{4n}] \rangle \in Q/\mathbb{Z},$$

and 4.10(i) follows immediately.

For 4.10(ii), note that the Thom space of the normal bundle of M_0 , $T(v_{M_0}^{8q})$, has a canonical K0-orientation. This extends to some K0-orientation, U, of $T(v_M^{8q})$. Then, since there is a degree one map $S^{8q+4n} \rightarrow T(v_M^{8q})$, we have

$$(1/a_n) \langle ph(\overline{\gamma(u)} - 1) ph(U), [T(v_M)] \rangle \in \mathbb{Z}.$$

Since $ph(\overline{\gamma(u)}) - 1 = ph_n(\overline{\gamma(u)})$, it follows that

$$e_{R}(\gamma(u)) = (1/a_{n}) \langle ph_{n}(\overline{\gamma(u)}), [M^{4n}] \rangle$$

= $(1/a_{n}) \langle ph_{n}(\overline{\gamma(u)}) ph(U), [T(v_{M})] \rangle = 0 \in Q/\mathbb{Z}.$

Similarly, $e_R(\xi_0(u)) = (1/a_n) \langle ph_n(\overline{\xi_0(u)}) ph(U), [T(v_M)] \rangle = 0 \in Q/\mathbb{Z}$, and 4.10(ii) is proved.

PROPOSITION 4.11. Let $u \in [M_0^{4n}, SF]$. Then $e_R(\gamma(u)) = e_R(\partial^*(u))$ where $\partial^*(u) \in \pi_{4n-1}(SF) = \pi_{4n-1}^s$. Moreover, $e_R(\gamma(u))$ has order a power of 2.

Proof. Let $v: M_0 \times S^{8q} \to S^{8q}$ be the adjoint of $u: M_0 \to SF_{8q+1}$, and let $\alpha \in K0^0$ (S^{8q}) be the generator. Then $\gamma(u) \cdot \pi^*(\alpha) = v^*(\alpha)$, where $\pi: M_0 \times S^{8q} \to S^{8q}$ is the projection. Thus $v^*(\alpha) - \pi^*(\alpha) = \mathscr{P}(\gamma(u) - 1) \in K0^0$ ($S^{8q} \wedge M_0$). It follows that there is a homotopy commutative diagram



From the definitions and diagram 4.7, one sees that $e_R(\partial^*(u)) = e_R(\gamma(u))$.

G. BRUMFIEL

For the second statement, it is only necessary to observe that there are spin manifolds, N_0^{4n} , with $\partial N_0^{4n} = S^{4n-1}$, and maps $g: N_0^{4n}, \partial N_0^{4n} \to M_0^{4n}, \partial M_0^{4n}$ of degree a power of 2, say 2^r. Then $2^r e_R(\gamma(u)) = 2^r e_R(\partial^*(u)) = e_R(2^r \partial^*(u)) = e_R(\partial^*(g^*(u))) = e_R(\gamma(g^*(u))) = 0$, by 4.10(ii).

§5. The composition $f_R d: [M_0^{4n}, F/0] \to \mathbb{Z}_{\theta_n}$. The invariant $f_R: \Gamma_{4n-1} \to \mathbb{Z}_{\theta_n}$ is defined as follows. Given $\Sigma^{4n-1} \in \Gamma_{4n-1}$, let $\Sigma^{4n-1} = \partial W_0^{4n}$, where W_0^{4n} is a smooth spin manifold such that the decomposable Pontryagin numbers of W^{4n} vanish. Then

 $f_R(\Sigma^{4n-1}) = (\frac{1}{8}) \operatorname{index} (W^{4n}) \in \mathbb{Z}/\theta_n \cdot \mathbb{Z}.$

(It is proved in [9] that such manifolds W_0^{4n} exist and that f_R is well-defined.)

It will be convenient to regard f_R as a homomorphism $f_R: \Gamma_{4n-1} \to Q/\mathbb{Z}$. Namely, define $f_R(\Sigma^{4n-1}) = (\frac{1}{8}\theta_n)$ index $(W^{4n}) \in Q/\mathbb{Z}$, where W^{4n} is as above.

Recall that the L-genus is given by

$$L_n(p_1 \dots p_n) = (8\theta_n p_n / a_n (2n-1)! j_n) + L_n(p_1 \dots p_{n-1}, 0).$$

PROPOSITION 5.1. Let $u \in [M_0^{4n}, F/0]$. Then

 $f_{R}(du) = (\frac{1}{8}\theta_{n}) \langle L(M) (1 - L(\xi)), [M^{4n}] \rangle \in Q/\mathbb{Z},$

where $L(\xi) = L(p_1(\xi_0(u))...p_{n-1}(\xi_0(u)), p_n(\xi))$ and $p_n(\xi)/a_n(2n-1)!j_n \in Q/\mathbb{Z}$ is d determined (formally) by the equations

$$(1/a_n)\langle \hat{A}(\xi), [M^{4n}]\rangle = e_R(\gamma(u))\in Q/\mathbb{Z}$$

and

$$(1/a_n)\langle ph(\xi), [M^{4n}]\rangle = e_R(\xi_0(u))\in Q/\mathbb{Z}.$$

The proof of Proposition 5.1 will require some preliminary results.

First, note that since

$$(1/a_n) \hat{A}_n(p_1 \dots p_n) = (- \operatorname{num} (B_n/4n) p_n/a_n(2n-1)! j_n) + \hat{A}_n(p_1 \dots p_{n-1}, 0)$$

and

$$(1/a_n) ph_n(p_1 \dots p_n) = ((-1)^{n-1} j_n p_n / a_n (2n-1)! j_n) + ph_n(p_1 \dots p_{n-1}, 0),$$

and since num $(B_n/4n)$ and $j_n =$ denom $(B_n/4n)$ are relatively prime, it follows that the equations in 5.1 for $p_n(\xi)/a_n(2n-1)!j_n \in Q/\mathbb{Z}$ have at most one solution.

Secondly, the computation of $p_n(\xi)/a_n(2n-1)!j_n$ in Proposition 5.1 is purely formal. That is, we do not assert the existence of a vector bundle ξ with the properties indicated. However, Proposition 5.1 and Remark 2.5 are closely related. If $u \in [M_0^{4n}, F/0]$ extends to $\bar{u} \in [M^{4n}, F/0]$, then $\xi = \xi(\bar{u})$ is an extension of $\xi_0 = \xi_0(u)$. Remark 2.5 asserts that $f_{R}(du) = (\frac{1}{8}\theta_{n}) \langle L(M)(1-L(\xi)), [M^{4n}] \rangle \in Q/\mathbb{Z}. \text{ Moreover, } \gamma(\bar{u}) \in KO(M) \text{ extends}$ $\gamma(u) \in KO(M_{0}), \text{ hence } e_{R}(\gamma(u)) = (1/a_{n}) \langle ph(\gamma(\bar{u})), [M] \rangle = (1/a_{n}) \langle \hat{A}(\xi), [M] \rangle \text{ and}$ also, of course, $e_{R}(\xi_{0}) = (1/a_{n}) \langle ph(\xi), [M] \rangle.$

Recall that the image of the Adams homomorphism $e_R: \pi_{4n-1}^s \to Q/\mathbb{Z}$ consists of integral multiples of $1/j_n = 1/\text{denom} (B_n/4n)$ [2]. Thus, there is a unique homomorphism $\tilde{e}_R: \pi_{4n-1}^s \to Q/\mathbb{Z}$, defined by num $(B_n/4n) \tilde{e}_R(\alpha) = e_R(\alpha)$. If α is the image of the generator of $\pi_{4n-1}(S0) = \mathbb{Z}$, then $e_R(\alpha) = (B_n/4n) = \text{num} (B_n/4n)/\text{denom} (B_n/4n)$. Thus, \tilde{e}_R is a normalization of e_R , with $\tilde{e}_R(\alpha) = 1/j_n$.

PROPOSITION 5.2. If $u \in [M_0^{4n}, SF]$, then $f_R(du) = \tilde{e}_R(\partial^*(u)) \in Q/\mathbb{Z}$. In particular, $f_R(du)$ has order a power of 2.

Proof. Represent u by a tangential homotopy equivalence $h_0: M'_0 \to M_0$. Let h denote the obvious extension $h: M' \to M$. Then $\tau_{M'} = h^*(\tau_M + p^*(\sigma))$ as PL bundles, where $p: M^{4n} \to S^{4n}$ is a map of degree one and $\sigma \in \pi_{4n}(BSPL)$. Since h_0 is a tangential homotopy equivalence, and since index (M') = index (M), it is easy to see that the Pontrjagin class $p_n(\sigma) = 0$. That is, σ is a torsion element of $\pi_{4n}(BSPL)$. Further, $J_{PL}(\sigma) = \partial^*(u)$, where $J_{PL}: \pi_{4n}(BSPL) \to \pi_{4n}(BSF) = \pi^s_{4n-1}$, and $\beta(\sigma) = du$, where $\beta: \pi_{4n}(BSPL) \to \pi_{4n-1}(PL/0) = \Gamma_{4n-1}$. It then follows from [9; Theorems 4.7, 4.8] that num $(B_n/4n) f_R(du) = e_R(\partial^*(u))$. This relation, together with 4.11, proves Proposition 5.2.

Note that if $u \in [M_0^{4n}, SF]$, then Proposition 5.1 asserts that $f_R(du) = -p_n(\xi)/a_n(2n-1)!j_n \in Q/\mathbb{Z}$, where

$$(1/a_n) \langle \hat{A}(\xi), [M^{4n}] \rangle = -\operatorname{num}(B_n/4n) p_n(\xi)/a_n(2n-1)! j_n = e_R(\gamma(u)) \in Q/\mathbb{Z}.$$

Thus 5.2 and 4.11 imply 5.1 in the case $u \in [M_0^{4n}, SF]$.

COROLLARY 5.3(i). The map $d: [M_0^{4n}, SF] \rightarrow \Gamma_{4n-1}$ is a group homomorphism. (ii) If $h: M'_0 \rightarrow M_0$ is any degree one map, then the diagram

$$\begin{bmatrix} M_0, SF \end{bmatrix}_{\substack{h^* \\ M_0', SF \end{bmatrix}}^d \Gamma_{4n-1}$$

commutes.

Proof. This follows from 5.2 and 3.1 since $f_R \oplus \varrho: \Gamma_{4n-1} \to \mathbb{Z}_{\theta_n} \oplus (\pi_{4n-1}^s / \operatorname{im}(J))$ is an isomorphism.

COROLLARY 5.4. If $u \in [M_0^{4n}, F/0]$ and $v \in [M_0^{4n}, SF]$, then d(u+v) = du+dv. Proof. This follows from 4.2, 4.4 and 5.3(i).

We can also prove Proposition 5.1. By 2.5 and 5.2, Proposition 5.1 is true if

 $u \in \text{image}([M^{4n}, F/0])$ or if $u \in \text{image}([M_0^{4n}, SF])$. By 4.4, it suffices to prove that

$$(\frac{1}{8}\theta_n) \langle L(M) (1 - L(\xi(u+v))), [M^{4n}] \rangle$$

$$(\frac{1}{8}\theta_n) \langle L(M) (1 - L(\xi(u))), [M^{4n}] \rangle + (\frac{1}{8}\theta_n) \langle L(M) (1 - L(\xi(v))), [M^{4n}] \rangle$$

if $u \in \text{image}([M^{4n}, F/0])$ and $v \in \text{image}([M_0^{4n}, SF])$. Since $L(\xi(v)) = 8\theta_n p_n(\xi(v))/a_n(2n-1)!j_n$, this is equivalent to proving that $p_n(\xi(u+v))/a_n(2n-1)!j_n = p_n(\xi(u))/a_n(2n-1)!j_n + p_n(\xi(v))/a_n(2n-1)!j_n$. But, by 4.10(i), $e_R(\gamma(u+v)) = e_R(\gamma(u)) + e_R(\gamma(v))$, and, of course, $e_R(\xi_0(u+v)) = e_R(\xi_0(u+v)) = e_R(\xi_0(u)) + e_R(\xi_0(v))$. The equations given in 5.1 which determine $p_n(\xi)/a_n(2n-1)!j_n$ now yield the desired additivity result.

Remark 4.6 and Propositions 3.1 and 5.2 show that $\Delta_{th}(M_0^{4n})$ is computable in terms of the stable homotopy theory invariant $\partial^* : [S^q \wedge M_0^{4n}, S^q] \to \pi_{q+4n-1}(S^q) = \pi_{4n-1}^s$. Proposition 2.4 and Remark 2.5, together with the Adams conjecture, show that $\Delta_h(M_0^{4n}) \cap bP_{4n}$ is computable in terms of L(M) and $ph(KO(M^{4n})) \subset H^{**}(M^{4n}, Q)$. Thus, $\Delta_h(M_0^{4n}) = (\Delta_h(M_0^{4n}) \cap bP_{4n}) + \Delta_{th}(M_0^{4n})$ is computable in terms of familiar invariants.

It is interesting that by using the Riemann-Roch theorem for spin maps, Proposition 5.1 can be proved without using Proposition 4.2 or the Adams conjecture. Then 3.1 and 5.1 provide, in a sense, a homotopy theoretic computation of the geometric map $d: [M_0^{4n}, F/0] \rightarrow \Gamma_{4n-1}$. However, use of the Adams conjecture gives the more practical description of $\Delta_h(M_0^{4n})$ above.

We now give some corollaries of the results above.

COROLLARY 5.5(i). If M_0^{4n} is a spin manifold and $u \in [M_0^{4n}, SF]$ or $u \in [M_0^{4n}, PL/0]$, then $f_R(du) = 0$. Hence $du \in \pi_{4n-1}^s / im(J) \subset \Gamma_{4n-1}$.

(ii) If M_0^{4n} is a weakly complex manifold and $u \in [M_0^{4n}, SF]$, then $a_n f_R(du) = 0$.

Proof. In the notation of Proposition 5.1, it follows from 4.10(ii) that $p_n(\xi)/a_n(2n-1)!j_n=0$. Hence, $L(\xi)=1$ and $f_R(du)=0$.

We will give an alternate proof of 5.5(i). Let $h: M'_0 \to M_0$ represent *u*. Then $h^*(\tau_{M_0}) = \tau_{M_0'}$ as vector bundles if $u \in [M_0, SF]$, and as *PL* bundles if $u \in [M_0, PL/0]$. In either case, $W_0 = M'_0 \# (-M_0)$ is a spin manifold, $\partial W_0 = \partial M'_0 - \partial M_0$, and all the Pontrjagin numbers of *W*, including $p_n(W)$, vanish. Then $f_R(du) = f_R(\partial M'_0 - \partial M_0) = (\frac{1}{8}\theta_n)$ index (W) = 0.

5.5(ii) can be proved by an argument similar to the second proof of 5.5(i). Namely, if M_0 is weakly complex and M'_0 , W_0 are as above, then M'_0 and W_0 are weakly complex, and all the Chern numbers of W vanish. An invariant $f_c: \Gamma_{4n-1} \rightarrow Q/\mathbb{Z}$ is defined in [9], using weakly complex manifolds instead of spin manifolds, and $f_c = a_n f_R$. It follows that $0 = f_c(du) = a_n f_R(du)$.

COROLLARY 5.6. If $u \in [M_0^{4n}, PL/0]$, then num $(B_n/4n) f_R(du) = e_R(\gamma(u))$, and $f_R(du)$ has order a power of 2.

Proof. The first statement follows from Proposition 5.1, since $f_R(du) = -p_n(\xi)/a_n(2n-1)!j_n \in Q/\mathbb{Z}$ and $(1/a_n) \langle \hat{A}(\xi), [M^{4n}] \rangle = -\operatorname{num} (B_n/4n) p_n(\xi)/a_n(2n-1)!j_n = e_R(\gamma(u)) \in Q/\mathbb{Z}$.

For the second statement, let $g: N_0^{4n}$, $\partial N_0^{4n} \to M_0^{4n}$, ∂M_0^{4n} be a map of degree 2^r where N_0^{4n} is a spin manifold. Then $2^r f_R(du) = f_R(dg^*(u)) = 0$ by 5.5(i).

COROLLARY 5.7. If M_0^{4n} is a spin manifold with $f_R(\partial M_0^{4n}) \neq 0$ (or if M_0^{4n} is any manifold and $f_R(\partial M_0^{4n})$ has order not a power of 2), then $0 \notin B_{th}(M_0^{4n})$ and $0 \notin B_c(M_0^{4n})$; that is, M_0^{4n} is not tangentially homotopy equivalent or combinatorially equivalent to a smooth manifold.

Proof. This follows from 5.2 and 5.6.

Here is an example to show that $f_R d: [M_0^{4n}, SF] \to \mathbb{Z}_{\theta_n}$ is not zero in general. Adams has defined elements $\mu_k \in \pi_{8k+2}^s$ such that $2\mu_k = 0$, $\mu_k \eta \neq 0$ and $\mu_k \eta \in im(J) \subset \pi_{8k+3}^s$ [2]. If M^{8k+4} is not a spin manifold (for example, $M^{8k+4} = \mathbb{C}P(4k+2)$), choose $x \in H^{8k+2}$ (M, Z_2) such that $S_q^2(x) \neq 0$ and let $g: M_0 \to S^{8k+2}$ be a map such that $g^*(\sigma) = x$, where $\sigma \in H^{8k+2}(S^{8k+2})$. Then the composition $S^{8k+3} \xrightarrow{\partial} M_0^{8k+4} \xrightarrow{g} S^{8k+2} \xrightarrow{\mu_k} SF$ represents $\partial^*(\mu_k g) = \mu_k \eta$, since $g\partial = \eta$. Since $\tilde{e}_R(\mu_k \eta) = \frac{1}{2} \in Q/\mathbb{Z}$, 5.2 implies $f_R(d(\mu_k g)) = \frac{1}{2} \in Q/\mathbb{Z}$.

In [10] we showed that the element μ_k could, in fact, be defined in $\pi_{8k+2}(SPL)$. Thus, in the example above, we actually have $u = \mu_k g \in [M_0^{8k+4}, SPL]$ and $du \in \Delta_{tc}(M_0^{8k+6})$ is the element of order 2 in bP_{8k+4} . I do not know of an example of $u \in [M_0^{4n}, SF]$ or $u \in [M_0^{4n}, PL/0]$ such that $a_n \cdot f_R(du) \neq 0$.

We next give a somewhat simpler formula for $f_R d: [M_0^{4n}, F/0] \to \mathbb{Z}_{\theta_n}$, when M_0^{4n} is a spin manifold, generalizing 5.5(i).

COROLLARY 5.8. Let $u \in [M_0^{4n}, F/0]$, where M_0^{4n} is a spin manifold. Then $f_R(du) = (\frac{1}{8}\theta_n) < L(M)(1-L(\xi)), [M] > \in Q/\mathbb{Z}$, where $L(\xi)$ is as in 5.1 and $(p_n(\xi)/a_n(2n-1)!j_n) \in Q/\mathbb{Z}$ is determined by the equations

 $(1/a_n)\langle (\hat{A}(\xi)-1)\,\hat{A}(M),[M]\rangle = 0 \in Q/\mathbb{Z}$

and

 $(1/a_n)\langle ph(\xi) \hat{A}(M), [M] \rangle = 0 \in Q/\mathbb{Z}.$

Proof. This follows from 4.4, 5.5(i), and 2.4, and the Riemann-Roch Theorem for manifolds with framed boundary.

The point of 5.8 is that for spin manifolds, $f_R(du)$ depends only on the Pontrjagin classes of M_0^{4n} and $\xi_0(u)$, and not on the K0-theory invariants $\gamma(u)$ and $\xi_0(u)$. This is because if $W_0 = M'_0 \# (-M_0)$ then W_0 is a spin manifold, $\partial W_0 = \partial M'_0 - \partial M_0$, and the

G. BRUMFIEL

Pontrjagin numbers of W, including $p_n(W)$, are functions of the Pontrjagin classes of M_0 and $\xi_0(u)$. Thus $f_R(du) = f_R(\partial W_0)$ can be computed in terms of Pontrjagin classes alone. 5.8 gives a specific formula.

In the next result, we study the deviation of $d: [M_0^{4n}, F/0] \rightarrow \Gamma_{4n-1}$ from linearity.

COROLLARY 5.9. Let
$$u, v \in [M_0^{4n}, F/0]$$
. Then
 $du + dv - d(u + v) = (\frac{1}{8}) \langle L(M) (L(\xi_0(u)) - 1) (L(\xi_0(v)) - 1), [M] \rangle$
 $\in \mathbb{Z}/\theta_n \mathbb{Z} = bP_{4n}$.

Proof. By 3.2, it suffices to prove that

$$f_{R}(du) + f_{R}(dv) - f_{R}(d(u+v)) = \left(\frac{1}{8}\theta_{n}\right) \langle L(M)(L(\xi_{0}(u)) - 1) \rangle \\ \times \left(L(\xi_{0}(v)) - 1\right), [M] \rangle \in Q/\mathbb{Z}.$$

By 4.4 and 5.3(i), we may assume that $u, v \in \text{image}([M^{4n}, F/0])$. The formula now follows from 2.4 since $L(\xi(u+v)) = L(\xi(u)) L(\xi(v))$, hence

$$L(\xi(u+v)) - 1 = (L(\xi(u)) - 1) (L(\xi(v)) - 1) + (L(\xi(u) - 1) + (L(\xi(v)) - 1))$$

= $(L(\xi_0(u)) - 1) (L(\xi_0(v)) - 1) + (L(\xi(u)) - 1))$
+ $(L(\xi(v)) - 1).$

Finally, we investigate the non-commutativity of d with maps.

COROLLARY 5.10. Let $u \in [M_0^{4n}, F/0]$ and let $h: M'_0 \to M_0$ be a map of degree one. Then

$$dh^*(u) - du = (\frac{1}{8}) \langle \left(h^*(L(M)) - L(M')\right) \left(h^*L(\xi_0(u)) - 1\right), [M'] \rangle$$

$$\in \mathbb{Z}/\theta_n \mathbb{Z} = bP_{4n}.$$

Proof. By 3.3 it suffices to compute $f_R(dh^*(u)) - f_R(du)$. By 4.4 and 5.3(ii) we may assume that u extends to $\bar{u} \in [M^{4n}, F/0]$. Then, by 2.4

$$f_{R}(dh^{*}(u)) - f_{R}(du) = (\frac{1}{8}\theta_{n}) \langle (h^{*}L(M) - L(M')) \cdot (L(\xi(h^{*}(u)) - 1), [M'] \rangle \\ = (\frac{1}{8}\theta_{n}) \langle (h^{*}L(M) - L(M')) \cdot (L(\xi_{0}(h^{*}(u))) - 1), [M'] \rangle \in Q/\mathbb{Z}.$$

COROLLARY 5.11. If $h:M'_0 \rightarrow M_0$ is a degree one map of 4n-manifolds which corresponds rational Pontrajagin classes, then the diagram

$$\begin{bmatrix} M_0, F/0 \\ {}^{h^*\downarrow} \end{bmatrix} \overset{d}{}^{\Gamma_{4n-1}} \\ \begin{bmatrix} M'_0, F/0 \end{bmatrix} \overset{d}{}^{J_d}$$

commutes. Thus, if h is a homotopy equivalence which corresponds rational Pontrjagin classes then $\Delta_h(M_0) = \Delta_n(M'_0)$.

§ 6. The composition $f_R d: [M_0^{8n+2}, F/0] \rightarrow \mathbb{Z}_2$. In this section we consider spin manifolds of dimension 8n+2. The main result is Proposition 6.5.

In [4], K0-characteristic numbers $\pi^J(M^{8n+2}) \in \mathbb{Z}_2$, where $J = (j_1...j_r)$ and $\pi^J = \pi^{j_1}...\pi^{j_r} \in K0^0$ (BS0) are defined for smooth spin manifolds. In [10], the definition is extend to almost smooth manifolds, provided that $J \neq (0)$. Roughly, this is done as follows.

Let M_0^{8n+2} be a spin manifold with $\partial M_0^{8n+2} \in \Gamma_{8n+1}$. Since $v_{M_0}^{8q}$ is a spin vector bundle, the Thom space $T(v_{M_0}^{8q})$ has a canonical K0-orientation. This extends to a K0-orientation $U_M \in K0^0(T(v_M^{8q}))$. Also, v_{M_0} extends to a vector bundle v_M^* over Mand we have $v_M = v_M^* + p^*(\sigma)$ as PL bundles, where $p: M^{8n+2} \to S^{8n+2}$ is a map of degree one and $\sigma \in \pi_{8n+2}(BSPL)$. Moreover, v_M^* is well-defined by the additional assumption that $e_R J_{PL}(\sigma) = 0$, where $J_{PL}: \pi_{8n+2}(BSPL) \to \pi_{8n+2}(BSF) = \pi_{8n+1}^s$ is the PL J-homomorphism and $e_R: \pi_{8n+1}^s \to Z_2$ is the homomorphism defined by Adams, which splits off image (J) as a direct summand [2]. Set

$$\pi^{J}(M^{8n+2}) = c^{*} \Phi_{K0}(\pi^{J}(v_{M}^{*})) \in K0^{0}(S^{8q+8n+2}) = \mathbb{Z}_{2},$$

where $\Phi_{K0}: K0(M) \cong K0^0(T(v_M^{8q}))$ is the Thom isomorphism defined by multiplication by U_M , and $c: S^{8q+8n+2} \to T(v_M^{8q})$ is the map of degree one defined by an embedding $M^{8n+2} \to S^{8q+8n+2}$. If $J \neq (0)$, the K0-operation π^J has filtration greater than zero, hence the product $\pi^J(v_M^*) \cdot U_M \in K0^0(T(v_M^{8q}))$ is independent of the choice of the extension U_M .

We will also use the notation

$$\pi^{J}(M^{8n+2}) = \langle \pi^{J}(v_{M}^{*}), [M]_{K0} \rangle \in \mathbb{Z}_{2}$$

where $[M]_{K0}$ is the fundamental K0-homology class dual to U_M .

E. Brown has defined a homomorphism $\psi: \Omega_{\text{spin}}^{8n+2} - \mathbb{Z}_2$, extending the Kervaire-Arf invariant $\Omega_{\text{framed}}^{8n+2} \rightarrow \mathbb{Z}_2$ [7]. In fact, Brown's definition of ψ applies to *PL* manifolds M^{8n+2} , with $w_1(M) = w_2(M) = 0$. From the main results of [4], it follows that for smooth M^{8n+2} ,

$$\psi(M^{8n+2}) = \sum \alpha_J \cdot \pi^J(M^{8n+2}) + \sum \beta_I \cdot w^I(M^{8n+2}) \in \mathbb{Z}_2$$

where α_J , $\beta_I \in \mathbb{Z}_2$, $J = (j_1 \dots j_r)$, $1 < j_1 \leq \dots \leq j_r$, and the w^I are Stiefel-Whitney numbers.

LEMMA 6.1. The coefficients β_I , α_J can be chosen such that $\alpha_J = 0$ if $n(J) = j_1 + ...$ $\dots + j_r \neq 2n$ and $\sum_{n(J)=2n} \alpha_J \pi^J \equiv (L^{-1})_{2n} (0, \pi^2 \dots \pi^{2n}) \pmod{2}$ where $L = 1 + L_1 + L_2 + ...$ is the Hirzebruch L-polynomial.

Proof. We only outline the proof of this lemma, and refer to [4] and [8] for details. The homotopy elements in π_{8n+2} (*M* spin) which have Adams spectral sequence

filtration greater than 2 are precisely the classes $\{M^{8n+2}\}$ with $w^{I}(M^{8n+2}) = \pi^{J}(M^{8n+2}) = 0$ for $n(J) \ge 2n$. It can be shown that $\psi(\{M^{8n+2}\}) = 0$ if $\{M^{8n+2}\} \in \Omega_{\text{spin}}^{8n+2} = \pi_{8n+2}(M \text{ spin})$ represents such a homotopy element. Thus $\alpha_{J} = 0$ if n(J) < 2n. If n(J) = = 2n+1, then the K0-characteristic number π^{J} coincides with a Stiefel-Whitney number for all (8n+2)-spin manifolds. Thus we may choose the coefficients β^{I} such that $\alpha_{J} = 0$. Finally, if T^{2} is the torus with the exotic spin structure and N^{8n} is a spin manifold, then $\psi(N^{8n} \times T^{2}) = \text{index}(N^{8n}) \pmod{2}$. Since the Stiefel-Whitney numbers of $N^{8n} \times T^{2}$ vanish, it follows that $\Sigma_{n(J)=2n} \alpha_{J} \pi^{J} = (L^{-1})_{2n} (0, \pi^{2} \dots \pi^{2n})$.

Let $b \operatorname{spin}_{8n+2} \subset \Gamma_{8n+1}$ be the subgroup consisting of homotopy spheres that bound spin manifolds. In [10], we showed that $\Gamma_{8n+1} = b \operatorname{spin}_{8n+2} \oplus \mathbb{Z}_2$. An invariant $f_R: b \operatorname{spin}_{8n+2} \to \mathbb{Z}_2$, splitting off $\mathbb{Z}_2 = bP_{8n+2} \subset b \operatorname{spin}_{8n+2}$ as a direct summand, can be defined as follows. Given $\Sigma^{8n+1} \in b \operatorname{spin}_{8n+2}$, let $\Sigma^{8n+1} = \partial M_0^{8n+2}$, where M_0^{8n+2} is a spin manifold such that all the Stiefel-Whitney numbers of M^{8n+2} vanish. Then

$$f_{R}(\Sigma^{8n+1}) = \psi(M^{8n+2}) - (L^{-1})_{2n}(0, \pi^{2} \cdots \pi^{2n})(M^{8n+2}) \in \mathbb{Z}_{2}.$$

Let $h: M'_0 \to M_0$ be a homotopy equivalence with $\theta(h) = u \in [M_0^{8n+2}, F/0]$. The spin structure on M_0 induces a spin structure on M'_0 and, since $h: M'_0 \to M_0$ is a homotopy equivalence, $\psi(M') = \psi(M)$. Further $h^*(w^I(M)) = w^I(M')$, hence

$$f_R(du) = f_R(\partial M'_0 - \partial M_0) = (L^{-1})_{2n}(M) - (L^{-1})_{2n}(M') \in \mathbb{Z}_2.$$

We now seek a formula expressing the K0-characteristic numbers of M' in terms of invariants of M and of the map $u: M_0^{8n+2} \rightarrow F/0$.

PROPOSITION 6.2. Let $u \in [M_0^{8n+2}, F/0]$ correspond to the homotopy equivalence $h: M'_0 \to M_0$, where M_0 is a spin manifold. Then

$$\pi^{J}(M') = \langle \pi^{J}(\nu_{M}^{*} - \xi_{0}^{*}(u)) \gamma^{*}(u), [M]_{K0} \rangle \in \mathbb{Z}_{2}$$

where $h^*(v_M^* - \xi_0^*(u)) = v_{M'}^* \in K0^0(M')$ and $\gamma^*(u) \in K0(M)$ extends $\gamma(u) \in K0(M_0)$.

Proof. Homotope $h: M' \to M$ to an embedding $h: M' \to M \times \mathbb{R}^{8q}$. The *PL* normal bundle of M' in $M \times \mathbb{R}^{8q}$ is $h^*((-\xi)^{8q})$, where $h^*(v_M - \xi) = v_{M'}$. By the *h*-cobordism theorem, the embedding *h* extends to a *PL* isomorphism $H: E(h^*(-\xi)^{8q}) \cong M \times \mathbb{R}^{8q}$. Let $c_1 = H^{-1}: T(e_M^{8q}) \to T(h^*(-\xi)^{8q})$ be the induced collapsing map.

Now, $\xi|_{M_0} = \xi_0(u) = \xi_0$ and the canonical K0-orientation of the Thom space $T(h^*(-\xi_0)_{M_0'}^{8q})$ extends to a K0-orientation $U \in K0^0(T(h^*(-\xi)_{M'}^{8q})))$. For, $h^*(-\xi) = v_{M'} - h^*(v_M) = (v_{M'}^* - h^*(v_M^*)) + (p')^*(\sigma' - \sigma)$, where $p': M' \to S^{8n+2}$, and the Thom space of the PL bundle $\sigma' - \sigma$ over S^{8n+2} is K0-orientable. Further, by 4.9, $c_1^*(U) \in K0^0(T(e_M^{8q}))$ restricts to $\Phi_{K0}(\gamma(u)) \in K0^0(T(e_{M_0}^{8q}))$.

There is a homotopy commutative diagram



where the diagonal $\Delta: M \to M \times M$ and the composition $(h \times Id) \Delta: M' \to M' \times M' \to M \times M'$ are covered by bundle maps $\Delta: v_M^{16q} \to v_M^{8q} \times e_M^{8q}$ and $(h \times Id) \Delta: v_{M'}^{16q} \to v_M^{8q} \times h^* (-\xi)_{M'}^{8q}$.

The proof of homotopy commutativity is similar to the proof of 2.3 and will be ommitted.

We thus have

$$\pi^{J}(M') = (c')^{*} (\pi^{J}(v_{M'}^{*}) \cdot U_{M'}) = (c')^{*} (h^{*} (\pi^{J}(v_{M}^{*} - \xi_{0}^{*})) \cdot U_{M'})$$

$$= (c')^{*} (\Delta^{*} (h \times Id)^{*} ((\pi^{J} (v_{M}^{*} - \xi_{0}^{*}) \cdot U_{M}) \cdot U))$$

$$= c^{*} (\Delta^{*} (\pi^{J} (v_{M}^{*} - \xi_{0}^{*}) \cdot U_{M} \cdot c_{1}^{*}(U)))$$

$$= c^{*} (\pi^{J} (v_{M}^{*} - \xi_{0}^{*}) \cdot \gamma^{*} (u) \cdot \Delta^{*} (U_{M} \cdot \Phi_{K0}(1))) = c^{*} \Phi_{K0} (\pi^{J} (v_{M}^{*} - \xi_{0}^{*}) \cdot \gamma^{*} (u))$$

and Theorem 6.2 is proved.

LEMMA 6.3. If n(J) = 2n then $\langle \pi^J(v_M^* - \xi_0^*(u)) \cdot \gamma^*(u), [M]_{K0} \rangle = \langle \pi^J(v_M^*) \cdot \gamma^*(u), [M]_{K0} \rangle \in \mathbb{Z}_2.$

Proof. It suffices to prove that $\pi^J(v_M^* - \xi_0^*) \equiv \pi^J(v_M^*) \pmod{2}$ in $K0^0(M)$.

First, $\pi^J(v_M^* - \xi_0^*)$ is independent of the choice of ξ_0^* , extending $\xi_0 \in K0^0(M_0)$. For, if $\alpha = p^*(\sigma)$, where $\sigma \in K0^0(S^{8n+2})$, and $\eta \in K0^0(M)$ then $\pi^J(\eta + \alpha) = \Sigma \pi^{J'}(\eta) \pi(\alpha)$. But if $J'' \neq (0), \pi^{J''}(\eta) \pi^{J''}(\alpha) = 0$ unless J'' = J, and $\pi^J(\alpha) = 0$ unless J = (2n), since products of elements of high filtration vanish. But also $\pi^{(2n)}(\sigma) = 0$ because $\sigma = \mu \eta^2$, where $\mu \in K0^0(S^{8n})$ and $\eta^2 : S^{8n+2} \to S^{8n}$, and $\pi^{(2n)}(\mu) = (4n-1)!\mu$. Thus $\pi^J(\eta + \alpha) = \pi^J(\eta)$.

Secondly, since $J(\xi_0) = 0$, $\xi_0 = \Sigma_k k^e (\psi^k - 1) (\xi_k)$ for some (arbitrarily) large integer *e* and $\xi_k \in K0^0(M_0)$. Since $2\xi_2$ and $(\psi^k - 1)\xi_k$, *k* odd, extend to $K0^0(M)$ and since $\psi^{2k} - 1 = (\psi^2 \psi^k - \psi^k) + (\psi^k - 1)$, it suffices to prove $\pi^J(\eta_1 + 2^e(\psi^2 - 1)\eta_2) \equiv \pi^J(\eta_1)$ (mod 2) and $\pi^J(\eta_1 + (\psi^k - 1)\eta_k) \equiv \pi^J(\eta_1) \pmod{2}$, *k* odd, where $\eta_1, \eta_2 \in K0^0(M)$ and $\eta_k \in K0^0(M_0)$.

If we set $\pi_t = \sum_{j \ge 0} \pi^j t^j$ then

$$\pi_t(\eta_1 + 2^e(\psi^2 - 1)\eta_2) = \pi_t(\eta_1) \cdot \pi_t((\psi^2 - 1)\eta_2)^{2^e} \equiv \pi_t(\eta_1) \pmod{2},$$

because e is large, hence 2^e-fold powers vanish in $K0^0(M)$. It follows that $\pi^J(\eta_1 + 2^e(\psi^2 - 1)\eta_2) \equiv \pi^J(\eta_1) \pmod{2}$.

If k is odd it suffices to prove that all products $x \cdot \pi^j (((\psi^k - 1)\eta_k) \equiv 0 \pmod{2}))$, where $j \ge 1$, filtration (x) = 8n - 4j if j is even, and filtration (x) = 8n - 4j - 2 if j is odd. Now,

$$\pi_t ((\psi^k - 1) \eta) = 1 + [\pi^1 (\psi^k (\eta)) - \pi^1 (\eta)] t + [(\pi^2 (\psi^k (\eta)) - \pi^2 (\eta)) - \pi^1 (\eta) (\pi^1 (\psi^k (\eta)) - \pi^1 (\eta))] t^2 + \cdots.$$

An easy induction shows that it suffices to prove $x \cdot (\pi^j (\psi^k(\eta)) - \pi^j(\eta)) \equiv 0 \pmod{2}$. But a computation in $K0^0 (BS0)$ shows that

$$\pi^{j}\psi^{k} - k^{2j}\pi^{j} - \left(2k^{2j}(k^{2}-1)/4!\right)\left(\pi^{(j,1)} - j\pi^{j+1}\right)$$

has filtration greater than 4j+4. Since k is odd, $2k^{2j}(k^2-1)/4!$ and $k^{2j}-1$ are even integers, hence

$$x \cdot (\pi^{j}(\psi^{k}(\eta)) - \pi^{j}(\eta)) = x \cdot ((k^{2j} - 1) \pi^{j}(\eta) - (2k^{2j}(k^{2} - 1)/4!)$$

$$\times (\pi^{(j, 1)} - j\pi^{j+1})(\eta)) \equiv 0 \pmod{2.}.$$

LEMMA 6.4.
$$\langle (L^{-1})_{2n}(v_M^*)(\gamma^*(u)-1). [M]_{K0} \rangle = \langle v_{4n}^2(M) w_2(\gamma(u)), [M] \rangle \in \mathbb{Z}_2.$$

Proof. Let $\gamma^*(u) = 1 + \tilde{\gamma}$. Then $L_{2n}^{-1}(v_M^*)\tilde{\gamma}$ has filtration 8n + 2, and we have a homo topy commutative diagram

The product $L_{2n}^{-1}(v_M^*) \cdot \tilde{\gamma}$ can thus be computed by evaluating the cohomology map $\mathbb{Z}_2 = H^{8n+2}(BS0\langle 8n+2\rangle, \mathbb{Z}_2) \rightarrow H^{8n+2}(M, \mathbb{Z}_2)$ in the diagram. The results of [4] on the operations $\pi^J \colon BS0 \rightarrow BS0\langle 8n \rangle$, n(J) = 2n, can be used to show that this coincides with $\langle v_{4n}^2(M) \cdot w_2(\gamma(u)), [M] \rangle \in \mathbb{Z}_2$.

Note that since $(\gamma - 1): F/0 \rightarrow BS0$ is a homotopy equivalence on the 5-skeltons, $w_2(\gamma(u)) = u^*(k_2)$, where $u \in [M_0^{8n+2}, F/0]$ and $k_2 \in H^2(F/0, \mathbb{Z}_2) = \mathbb{Z}_2$ is the generator.

PROPOSITION 6.5. Let $u \in [M_0^{8n+2}, F/0]$, where M_0^{8n+2} is a spin manifold. Then $f_R(du) = \langle v_{4n}^2(M) \cdot u^*(k_2), [M] \rangle \in \mathbb{Z}_2$.

Proof. This follows immediately from 6.2, 6.3, 6.4 and the formula

$$f_{R}(du) = (L^{-1})_{2n}(M) - (L^{-1})_{2n}(M').$$

COROLLARY 6.6. $d: [M_0^{8n+2}, F/0] \rightarrow \Gamma_{8n+1}$ is a group homomorphism. Proof. This follows from 3.2 and 6.5 and the fact that $k_2 \in H^2(F/0, \mathbb{Z}_2)$ is primitive.

COROLLARY 6.7. Let $h: M'_0 \to M_0$ be a map of degree one. Then $dh^*(u) - du = = \langle (v_{4n}^2(M') - h^*(v_{4n}^2(M))) \cdot h^*u^*(k_2), [M'] \rangle \in bP_{8n+2} = \mathbb{Z}_2$, where $u \in [M_0^{8n+2}, F/0]$. In particular, if h is a tangential map or a homotopy equivalence, then $dh^*(u) = du$. Thus $\Delta_h(M_0)$ is a homotopy invariant of 8n+2 spin manifolds.

Proof. This follows from 3.3 and 6.5.

COROLLARY 6.8 Let $u \in [M_0^{8n+2}, PL/0]$. Then $f_R(du) = 0$. Proof. PL/0 is 6-connected, hence $u^*(k_2) = 0$ and 6.8 follows from 6.5.

Remark 6.9. In § 5, we showed that for 4*n*-spin manifolds, $f_R(\Delta_c(M_0^{4n})) = f_R(\Delta_{th}(M_0^{4n})) = 0$. For (8n+2)-spin manifolds, $f_R(\Delta_{th}(M_0^{8n+2}))$ need not be zero. For example, if $M_0^{8n+2} = (N^{8n} \times S^2)_0$ and index (N^{8n}) is odd, and $u: (N^{8n} \times S^2)_0 \xrightarrow{\pi_2} S^2 \xrightarrow{h^2} SF$, then $f_R(du) = 1$.

Remark 6.10. Let M^{8n+2} be a closed, smooth spin manifold. The above results, along with Proposition 2.4, determine the exact sequence of Sullivan [18],

$$0 \to hS(M^{8n+2}) \xrightarrow{\theta} [M^{8n+2}, F/0] \xrightarrow{s} \mathbb{Z}_2.$$

Namely, if $u \in [M^{8n+2}, F/0]$, then

$$s(u) = \langle v_{4n}^2(M) \cdot u^*(k_2), [M] \rangle \in \mathbb{Z}_2.$$

Thus, the cohomology formula of 2.5 simplifies for 8n+2 spin manifolds.

The Adams conjecture, and the resulting factoring $(F/0)_{(2)} = BS0_{(2)} \times (CokJ)_{(2)}$, implies that s=0 if and only if $v_{4n}^2(M) w_2(\gamma) = 0$ for all $\gamma \in K0^0(M)$.

Appendix I. S^1 actions on homotopy spheres

It is known that equivariant diffeomorphism classes of differentiable, fixed point free S^1 actions on homotopy (2n-1)-spheres, $n \ge 4$, correspond bijectively with equivalence classes of homotopy smoothings of CP(n-1) [12]. The correspondence is defined as follows. If S^1 acts on Σ^{2n-1} , there is a diagram

G. BRUMFIEL

where h classifies the principal S^1 bundle over P^{2n-2} given by the action of S^1 on Σ^{2n-1} . An easy spectral sequence argument shows that h is a homotopy equivalence.

There are homotopy equivalences $CP(n-1) \xrightarrow{i} CP(n)_0 \xrightarrow{\pi} CP(n-1)$, since $CP(n)_0$ is the total space of a D^2 bundle, H, over CP(n-1). (If CP(n-1) is regarded as the space of lines in C^n then H is the dual of the "canonical" line bundle.) Consider the diagram

$$hS(\mathbf{CP}(n-1)) \xrightarrow{\theta} [\mathbf{CP}(n-1), F/0] \xrightarrow{s} P_{2n-2}$$

$$\downarrow^{i_{*}} \qquad \uparrow \wr^{i_{*}} \qquad \qquad (I.2)$$

$$hS_{\psi}\mathbf{CP}(n)_{0}) \xrightarrow{\theta} [\mathbf{CP}(n)_{0}, F/0] \xrightarrow{d} \Gamma_{2n-1}$$

where, if $h: P^{2n-2} \to \mathbb{C}P(n-1)$ then $i_*(P^{2n-2}, h)$ is the homotopy equivalence $\tilde{h}: P_0^{2n} = E(h^*H) \to E(H) = \mathbb{C}P(n)_0$.

LEMMA I.3(i). Diagram I.2 commutes.

(ii) $d\theta i_*(P^{2n-2}, h) = \Sigma^{2n-1} \in \Gamma_{2n-1}$, where $\Sigma^{2n-1} \to P^{2n-2}$ is as in diagram I.1.

(iii) $si^*\theta:hS(CP(n)_0) \rightarrow P_{2n-2}$ is the geometric obstruction to finding a codimension 2, homotopy CP(n-1) in a homotopy $CP(n)_0$.

The proof of I.3 is relatively straightforward and will be omitted. It follows from I.3 that the set of homotopy (2n-1)-spheres which admit free S^1 actions coincides with $d(\theta i_*(hS(\mathbb{CP}(n-1)))) = d((si^*)^{-1}(0)) \subset \Delta_h(\mathbb{CP}(n)_0) = B_h(\mathbb{CP}(n)_0) \subset \Gamma_{2n-1}$. Denote this set by $\widetilde{B}_h(\mathbb{CP}(n)_0)$.

We now want to apply the results of §2 through §6 to compute $\tilde{B}_h(CP(n)_0)$. First, it follows from the exact sequence

$$K0^{-1}(\mathbb{C}P(n)_0) \to [\mathbb{C}P(n)_0, SF] \to [\mathbb{C}P(n)_0, F/0] \to K0^0(\mathbb{C}P(n)_0)$$
$$\to J(\mathbb{C}P(n)_0) \to 0$$

and results of [3] that $[\mathbb{C}P(n)_0, F/0] = \mathbb{Z}^{[(n-1)/2]} \oplus [\mathbb{C}P(n)_0, SF]$, where $\mathbb{Z}^{[(n-1)/2]} \subset \Box$ image $([\mathbb{C}P(n), F/0] \rightarrow [\mathbb{C}P(n)_0, F/0])$ and image $(\mathbb{Z}^{[(n-1)/2]} \rightarrow K0^0 (\mathbb{C}P(n)_0))$ is generated by elements $k^e(\psi^k - 1)(\xi)$, $\xi \in K0^0 (\mathbb{C}P(n)_0)$. In theory it is thus possible to compute the fibre homotopically trivial bundles over $\mathbb{C}P(n)_0$. We have done this for $n \leq 8$ [12]. Let $\omega = r(H-1) \in K0^0 (\mathbb{C}P(n))$, where r forgets the complex structure.

LEMMA I.4. Kernel $(K0^{\circ}(\mathbb{C}P(8)_{0}) \rightarrow J(\mathbb{C}P(8)_{0}) = \mathbb{Z}^{3}$ has generators $\xi_{1} = 24\omega + 98\omega^{2} + 111\omega^{3}$, $\xi_{2} = 240\omega^{2} + 380\omega^{3}$, and $\xi_{3} = 504\omega^{3}$. If n < 8, kernel $(K0^{\circ}(\mathbb{C}P(n)^{\circ}) \rightarrow J(\mathbb{C}P(n)_{0}))$ is generated by $\xi_{1}, \xi_{2}, \xi_{3}$ restricted to $K0^{\circ}(\mathbb{C}P(n)_{0})$. Next, we need to compute $si^{*}: [\mathbb{C}P(n)_{0}, F/0] \rightarrow P_{2n-2}$.

LEMMA I.5. If $n \equiv 1$ or 3 (mod 4) and $u \in [CP(n)_0, F/0]$ then $si^*(u) = (\frac{1}{8}) \langle L(CP(n-1))(1-L(\xi_0(i^*(u)))), [CP(n-1)] \rangle \in \mathbb{Z}.$

In particular,

(i) $si^*([CP(n)_0, SF]) = 0$ (ii) If n = 5 and $\xi_0(i^*(u)) = m\xi_1 + n\xi_2$ then $si^*(u) = -4m^2 + 10m + 28n \in \mathbb{Z}$.

In particular, if $si^*(u) = 0$ then $10m \equiv 0 \pmod{4}$, or, $m \equiv 0 \pmod{2}$. (iii) If n = 7 and $\xi_0(i^*(u)) = m\xi_1 + n\xi_2 + q\xi_3$ then

 $si^*(u) = (-m(32m^2 + 301)/3) + 84m^2 + 224mn - 384n - 496q \in \mathbb{Z}.$

Proof. The formula for s was given in Remark 2.5. Statements (ii) and (iii) follow from I.4 and explicit computation of theL-polynomials in the formula.

LEMMA I.6. If $n \equiv 2 \pmod{4}$ and $u \in [CP(n)_0, F/0]$ then $si^*(u) = \langle v_{n-2}^2(CP(n-1)) i^*u^*(k_2), [CP(n-1)] \rangle \in \mathbb{Z}_2$. Thus $si^*(u) = 0$ if and only if $w_2(\gamma(i^*(u))) = i^*u^*(k_2) = 0$, or equivalently, if and only if $p_1(\xi_0(i^*(u))) \equiv 0 \pmod{48}$. In particular,

(i) $si^*([CP(n)_0, SF])=0$,

(ii) If n = 6 and $\xi_0(i^*(u)) = m\xi_1 + n\xi_2$ then $si^*(u) = m \pmod{2}$.

Proof. The formula follows from 6.5 and 6.10. If $n \equiv 2 \pmod{4}$ then

 $v_{n-2}^2(\mathbb{C}P(n-1))\neq 0$ and the second statement follows. Statements (i) and (ii) also follow easily.

We do not have general results with which to compute si^* if $n \equiv 0 \pmod{4}$. The following conjecture is probably true.

Conjecture I.7(i). If $n \equiv 0 \pmod{4}$, $n \neq 2^{j}$, then $si^*([CP(n)_0, F/0]) = 0$.

(ii) There are elements $h_j^2 \in \pi_{2^{j+1}-1}(SF)$ such that if $u: CP(2^j)_0 \xrightarrow{p\pi} S^{2^{j+1}-2} \xrightarrow{h^{2_j}} SF$ then $si^*(u) = 1 \in \mathbb{Z}_2$. The summand $\mathbb{Z}^{(2^{j-1}-1)} \subset [CP(2^j)_0, F/0]$ can be chosen so that $si^*(\mathbb{Z}^{(2^{j-1}-1)}) = 0$.

I.7(ii) is true if $j \leq 6$. For example $h_1^2 = \eta^2 \in \pi_2^s$, $h_2^2 = v^2 \in \pi_6^s$, and $h_3^2 = \sigma^2 \in \pi_{14}^s$.

We can use the results 2.5, 3.1, 4.4, 5.2, and 6.10 to compute $d: [CP(n)_0, F/0] = \mathbb{Z}^{[(n-1)/2]} \oplus [CP(n)_0, SF] \rightarrow \Gamma_{2n-1} = bP_{2n} \oplus (\pi_{2n-1}^s/\operatorname{im}(J)).$

LEMMA I.8. We have $d(\mathbf{Z}^{[(n-1)/2]}) \subset bP_{2n}$. Specifically, (i) If $u \in \mathbf{Z} \subset [\mathbf{C}P(4)_0, F/0]$ and $\xi_0(u) = m\xi_1$ then $du = 10m - 4m^2 \in \mathbf{Z}/28Z = bP_8$. (ii) If $u \in \mathbf{Z}^2 \subset [\mathbf{C}P(5)_0, F/0]$ and $\xi_0(u) = m\xi_1 + n\xi_2$, then $du = m \in \mathbf{Z}/2\mathbf{Z} = bP_{10}$. (iii) If $u \in \mathbf{Z}^2 \subset [\mathbf{C}P(6)_0, F/0]$ and $\xi_0(u) = m\xi_1 + n\xi_2$, then $du = (-m(32m^2 + 301)/3) + 84m^2 + 224mn - 384n \in \mathbf{Z}/992Z = bP_{12}$. (iv) If $u \in \mathbf{Z}^3 \subset [\mathbf{C}P(7)_0, F/0]$ then du = 0, since $bP_{14} = 0$. *Proof.* $\mathbb{Z}^{[(n-1)/2]} \subset \operatorname{image}([\mathbb{C}P(n), F/0] \rightarrow [\mathbb{C}P(n)_0, F/0])$, hence the first statement follows from 2.4 and 6.10. Statements (i) and (iii) follow from I.5 and 2.4 and (ii) follows from I.6 and 6.10.

Specific formulas for $d(\mathbb{Z}^{[(n-1)/2]})$, $n \ge 8$, would only require extending the computations of I.4 and I.5.

Recall that as a set $[\mathbb{C}P(n)_0, SF] = \pi_s^0 (\mathbb{C}P(n)_0)$. In [12] we computed the *p*-primary summand $_p\pi_s^0 (\mathbb{C}P(n)_0)$ and the map $_p\pi_s^0 (\mathbb{C}P(n)_0) \xrightarrow{\partial^*}{\to} p\pi_s^0 (S^{2n-1}) = _p\pi_{2n-1}^s$ for $n \leq (p^2 + 2p)(p-1) - 2$, *p* odd, and we computed $_2\pi_s^0 (\mathbb{C}P(n)_0) \xrightarrow{\partial^*}{\to} 2\pi_{2n-1}^s$ for $n \leq 11$. Thus, using 5.2 and 6.9, we also computed $d: [\mathbb{C}P(n)_0, SF] \to \Gamma_{2n-1}$ if $n \equiv 0, 1$, or 2 (mod 4) or if $n = 2^j - 1$. (note that by 5.5(ii), $a_n f_R(d[\mathbb{C}P(2n)_0, SF]) = 0$ and by 6.9, $f_R(d[\mathbb{C}P(4n+1)_0, SF]) = 0$.) These results involve computations in stable homotopy theory and are too complicated to reproduce here. We will state the conclusions for $n \leq 7$.

LEMMA I.9(i).
$$[CP(4)_0, SF] = Z_2$$
 and $d([CP(4)_0, SF]) = 0$.
(ii) $[CP(5)_0, SF] = \mathbb{Z}_2^2$ and $d([CP(5)_0, SF]) = \mathbb{Z}_2 = \{v^3\} \subset (\pi_9^s/\operatorname{im}(J)) \subset \Gamma_9$.
(iii) $[CP(6)_0, SF] = \mathbb{Z}_2^2 + \mathbb{Z}_3$ and $d([CP(6)_0, SF]) = \mathbb{Z}_2 \subset bP_{12} = \Gamma_{11}$.

(iv) $[CP(7)_0, SF] = \mathbb{Z}_2 + \mathbb{Z}_3$ and $d([CP(7)_0, SF]) = \mathbb{Z}_3 = \{\alpha_1 \beta_1\} = \pi_{13}^s = \Gamma_{13}$.

The construction of the non-zero element of $d([CP(6)_0, SF])$ is described in § 5, following the proof of 5.7.

Finally, we combine the results I.5 through I.9 to describe the set of homotopy spheres of dimensions 7, 9, 11, and 13 which admit free S^1 actions. That is, we compute $\tilde{B}_h(\mathbb{C}P(n)_0) = d((si^*)^{-1}(0)) \subset d([\mathbb{C}P(n)_0, F/0]) = B_h(\mathbb{C}P(n)_0) \subset \Gamma_{2n-1}$, for n = 4, 5, 6, and 7.

THEOREM I.10(i). $\Gamma_7 = bP_8 = Z/28Z$ and $\tilde{B}_h(CP(4)_0) = \{10m - 4m^2/m \in \mathbb{Z}\} = \{0, 4, \pm 6, \pm 8, -10, 14\} \subset \mathbb{Z}/28\mathbb{Z}.$ (ii) $\Gamma_9 = bP_{10} \oplus (\pi_9^s/\text{im}(J)) = \mathbb{Z}_2 \oplus \mathbb{Z}_2^2$ and $\tilde{B}_h(CP(5)_0) = \mathbb{Z}_2 = \{v^3\} \subset (\pi_9^s/\text{im}(J)) \subset \mathbb{Z}_9.$ (iii) $\Gamma_{11} = bP_{12} = \mathbb{Z}/992\mathbb{Z}$ and $\tilde{B}_h(CP(6)_0) = \{(-m(32m^2 + 301/3) + 84m^2 + 224mn - 384n \mid m, n \in \mathbb{Z}, m \text{ even}\} \subset \mathbb{Z}/992\mathbb{Z}.$ (iv) $\Gamma_{13} = \pi_{13}^s = \mathbb{Z}_3$ and $\tilde{B}_h(CP(7)_0) = \mathbb{Z}_3 = \{\alpha_1\beta_1\} = \Gamma_{13}.$

Appendix II. Applications to inertia groups

Given a smooth manifold N^k , the inertia group of N^k , $I(N^k) \subset \Gamma_k$, is defined to be the group of homotopy spheres $\Sigma^k \in \Gamma_k$ such that there is a diffeomorphism $N^k \cong N^k \# \Sigma^k$. Define $I_h(N^k) \subset I(N^k)$ to be the subgroup of homotopy spheres $\Sigma^k \in I(N^k)$ such that some diffeomorphism $N^k \cong N^k \# \Sigma^k$ is homotopic to the identity. (By the "identity" $N^k = N^k \# \Sigma^k$ we mean the obvious *PL* identification.) Similarly, define $I_c(N^k) \subset I_h(N^k)$ to be the subgroup of homotopy spheres Σ^k such that some diffeomorphism $N^k \cong N^k \# \Sigma^k$ is *PL* isotopic to the identity. Equivalently, $\Sigma^k \in I_c(N^k)$ if the smoothings N^k and $N^k \# \Sigma^k$ are concordant.

The group Γ_k is naturally isomorphic to the group of isotopy classes of orientation preserving diffeomorphisms of S^{k-1} . If $\Sigma^k \in \Gamma_k$ corresponds to the diffeomorphism $\sigma: S^{k-1} \cong S^{k-1}$ then $\Sigma^k \in I(N^k)$ if and only if there is a diffeomorphism $h: N_0^k \cong N_0^k$ such that $h \mid_{\partial N_0 = S^{k-1}} = \sigma$. Let $h: N^k \to N^k$ also denote the *PL* extension of *h* defined by coning $h \mid_{\partial N_0}$ over $D^k \subset N^k$. It is easy to see that the mapping torus of *h*, $T_h = N^k \times I/(x, 0) \equiv$ $\equiv (h(x), 1)$, is an almost smooth manifold, with $\partial (T_h)_0 = \Sigma^k$. Further, $\Sigma^k \in I_h(N^k)$ (resp. $\Sigma^k \in I_c(N^k)$) if and only if *h* can be chosen such that there is a homotopy equivalence (resp. a *PL* isomorphism) $H: T_h \to N^k \times S^1$, with $H \mid_{N^k \times 0} = Id$. Then $H: (T_h)_0 \to$ $\to (N^k \in S^1)_0$ is a homotopy smoothing of $(N^k \times S^1)_0$.

Now $N^k \times S^1$ is not simply connected. However, if N^k is simply connected, the map $\theta: hS((N^k \times S^1)_0) \rightarrow [(N^k \times S^1)_0, F/0]$ is still useful. There is a natural decomposition $[(N^k \times S^1)_0, F/0] \simeq [N^k, F/0] \oplus [N_0^k \wedge S^1, F/0]$. The first summand contains the image under θ of the homotopy smoothings $g \times Id: (N' \times S^1)_0 \rightarrow (N \times S^1)_0$, where $g: N' \rightarrow N$ is a homotopy equivalence. The second summand corresponds bijectively with the homotopy smoothings described above, $H: (T_h)_0 \rightarrow (N^k \times S^1)_0, H|_{N^k \times 0} = Id$, where $h: N_0^k \cong N_0^k$ is a diffeomorphism homotopic to the identity. Denote this second set of homotopy smoothings of $(N^k \times S^1)_0$ by $hS((N^k \times S^1)_0)$.

PROPOSITION II.1. $I_h(N^k) = d(\theta(hS(N^k \times S^1)_0)) = d([N_0^k \wedge S^1, F/0]) \subset \Gamma_k$. Also, $I_c(N^k) = d([N_0^k \wedge S^1, PL/0]).$

Proof. This follows from the discussion in the three paragraphs above.

We can thus use the results of § 2 through § 6 to compute $I_h(N^k)$. If $u \in [N_0^k \wedge S^1, F/0]$, k odd, the formulas in 5.1 and 6.5 for $f_R(du)$ simplify.

PROPOSITION II.2. If N^{8n+1} is a simply connected spin manifold and $u \in [N_0^{8n+1} \land S^1, F/0]$ then $f_R(du) = 0$. Thus $I_h(N^{8n+1})$ is contained in the summand $(\pi_{8n+1}^s/im(J)) \subset \Gamma_{8n+1}$ and $I_h(N^{8n+1}) \cong \varrho(I_h(N^{8n+1}))$ is a homotopy invariant of N^{8n+1} . *Proof.* Since $u^*(k_2) = 0$, the result follows from 6.5.

PROPOSITION II.3. If $u \in [N_0^{4n-1} \land S^1, F/0]$ then

$$f_R(du) = \left(-\frac{1}{8}\right) \left\langle L\left(N^{4n-1} \times S^1\right) \left(\sum_{k=1}^n \left(\frac{8\theta_k}{a_k}(2k-1)! j_k\right) p_k(\xi)\right), \left[N^{4n-1} \times S^1\right] \right\rangle$$

 $\in \mathbb{Z}/\theta_n \mathbb{Z},$

where $p_n(\xi)$ is as in 5.1 and $p_k(\xi) = p_k(\xi_0(u))$ if k < n.

Proof. Since cohomology products vanish in $N^{4n-1} \wedge S^1$, we have $(1-L(\xi)) = -(\sum_{k=1}^n (8\theta_k/a_k(2k-1)!j_k)p_k(\xi))$ and the result follows from 5.1. We point out that $p_n(\xi)$ is determined by the equations $(-\operatorname{num}(B_n/4n)/a_n(2n-1)!j_n)p_n(\xi) = e_R(\gamma(u)) \in \mathbb{Q}/\mathbb{Z}$ and $((-1)^{n-1}j_n/a_n(2n-1)!j_n)p_n(\xi) = e_R(\xi_0(u)) \in \mathbb{Q}/\mathbb{Z}$.

Note that by 5.9, $d: [N_0^k \wedge S^1, F/0] \to \Gamma_k$ is a group homomorphism if k = 4n - 1. Actually, if $u, v \in [N_0^k \wedge S^1, F/0]$ correspond to $H: (T_h)_0 \to (N^k \times S^1)_0$ and $G: (T_g)_0 \to (N^k \times S^1)_0$, respectively, where $h, g: N_0^k \cong N_0^k$ are diffeomorphisms, then $d(u+v) \in \Gamma_k$ corresponds to the diffeomorphism $(h \mid_{\partial N_0}) \cdot (g \mid_{\partial N_0}): S^{k-1} \cong S^{k-1}$. Since this composite diffeomorphism also corresponds to du + dv, we have that $d: [N_0^k \wedge S^1] \to \Gamma_k$ is a group homomorphism for all N^k .

There is a braid of four interlocking exact sequences



Here, $\alpha: \Gamma_k \to hS(N^k)$ is defined by $\alpha(\Sigma^k) = (N^k \# \Sigma^k, \text{ Id } \#(\text{point})) \in hS(N^k), \Sigma^k \in \Gamma_k$. Since kernel $(\alpha) \cap bP_{k+1} = bs([N^k \wedge S^1, F/0]) = d\theta(hS((N^k \times S^1)_0)) \cap bP_{k+1} = I_h(N^k) \cap OBP_{k+1}$, we see that $I_h(N^k)$ is very useful for computing $hS(N^k)$.

If we replace F/0 by PL/0, the cofibrations $S^{k-1} \rightarrow N_0^k \rightarrow N^k \rightarrow S^k \rightarrow N_0 \wedge S^1$ yield an exact sequence $[N_0^k \wedge S^1, PL/0] \xrightarrow{d} \Gamma_k \rightarrow [N^k, PL/0] \rightarrow [N_0^k, PL/0] \xrightarrow{d} \Gamma_{k-1}$. Since $[N^k, PL/0]$ and $[N_0^k, PL/0]$ correspond to concordance classes of smoothings of N^k and N_0^k , respectively, it is clear that $I_c(N^k) = d([N_0^k \wedge S^1, PL/0]) = \{\Sigma^k \in \Gamma_k \mid \text{the smoothings } N^k \text{ and } N^k \# \Sigma^k \text{ are concordant}\}$. The following is also clear.

PROPOSITION II.4. $I_c(N^k)$ is a homotopy invariant of N^k .

There are natural subgroups $I_{th}(N^k) \subset I_h(N^k)$ and $I_{tc}(N^k) \subset I_c(N^k)$ defined by $I_{th}(N^k) = d([N_0^k \wedge S^1, SF])$ and $I_{tc}(N^k) = d([N_0^k \wedge S^1, SPL])$. Geometrically, $I_{th}(N^k) \subset \Gamma_k$ (resp. $I_{tc}(N^k) \subset \Gamma_k$) corresponds to those diffeomorphisms $\sigma: S^{k-1} \cong S^{k-1}$ such that there is a diffeomorphism $h: N_0^k \cong N_0^k$, with $h|_{\partial N_0} = \sigma$, and a tangential homotopy equivalence (resp. *PL* equivalence preserving the smooth tangent bundles) $H: (T_h)_0 \to (N^k \times S^1)_0$ with $H|_{N^k \times 0} = Id$.

PROPOSITION II.5(i). $f_R(I_c(N^{4n-1}))$ and $f_R(I_{th}(N^{4n-1})) \subset Z_{\theta_n}$ are 2-primary groups.

- (ii) If N^{4n-1} is a spin manifold then $f_R(I_c(N^{4n-1})) = f_R(I_{th}(N^{4n-1})) = 0$
- (iii) $I_{th}(N^{4n-1})$ and $I_{tc}(N^{4n-1})$ are homotopy invariants.

Proof. These results follow from 5.2, 5.5, and 5.6. It follows from the construction given after the proof of 5.7 that if $w_2(N^{8k+3}) \neq 0$ then the element of order 2 in bP_{8k+4} belongs to $I_{tc}(N^{8k+3})$.

PROPOSITION II.6. $I_{th}(N^{8n+1}) \cong \varrho I_{th}(N^{8n+1})$ and $I_{tc}(N^{8n+1}) \cong \varrho I_{tc}(N^{8n+1})$ are homotopy invariants of (8n+1)-spin manifolds.

Proof. This follows from II.2.

Next we consider manifolds with a trivial stable normal bundle (π -manifolds) or a fibre homotopically trivial stable normal bundle (*fht*-manifolds).

LEMMA II.7. M^k is an fht-manifold if and only if there is a π -manifold M' and a degree one map $M' \rightarrow M$.

Proof. By transverse regularity, such a manifold M', with $M' \times R^q \subset E(v_M^q)$, exists if and only if there is a fibre homotopy trivialization $T(v_M^q) \rightarrow S^q$.

Boardman and Vogt have shown that PL/0 and F/0 are infinite loop spaces [5]. It follows easily that the suspension maps $\pi_*(F/0) \rightarrow \pi_*^s(F/0) = \Omega_*^{\text{framed}}(F/0)$ and $\pi_*(PL/0) \rightarrow \pi_*^s(PL/0) = \Omega_*^{\text{framed}}(PL/0)$ are monomorphisms onto direct summands.

LEMMA II.8. If M^k is an almost smooth, fht-manifold then $\Delta_c(M^k)=0$ and $\Delta_h(M^k) \subset bP_k$. If k=8n+2 then $\Delta_h(M^k)=0$.

Proof. Let $u \in [M_0^k, PL/0]$ and let $h: M'_0 \to M_0$ be a degree one map where M' is a π -manifold. Then by the above remark $du = \partial^*(u) = \partial^*h^*(u) = 0 \in \pi_{k-1}(PL/0) = \Gamma_{k-1}$. Similarly, if $u \in [M_0^k, F/0]$ then by 3.1 $\varrho(du) = \partial^*(u) = \partial^*h^*(u) = 0 \in \pi_{k-1}(F/0)$. The second statement follows from the first and the fact that the surgery obstruction $s: [M^{8n+2}, F/0] \to Z_2$ is given by $s(u) = \langle v_{4n}^2(M)u^*(k_2), [M] \rangle = 0$, since the Wu class $v_{4n}(M) = 0$.

PROPOSITION II.9. If N^k is a smooth, fht-manifold then $I_c(N^k)=0$ and $I_h(N^k) \subset bP_{k+1}$. If k = 8n+1 then $I_h(N^k)=0$. If N^k is a π -manifold and $k \neq 5 \pmod{8}$ then $I_h(N^k)=0$.

Proof. The first two statements follow from II.8 since $N^k \times S^1$ is an *fht*-manifold. If N^{4n-1} is a π -manifold and $u \in [N_0^{4n-1} \wedge S^1, F/0]$ then $f_R(du) = 0$ by 5.8. Thus $I_h(N^k) = I_h(N^k) \cap bP_{k+1} = 0$ if $k \equiv 1, 3$, or 7 (mod 8) and the third statement follows. (I am grateful to D. Sullivan for pointing out the first statement of II.9.)

Finally, as an example, we compute, $I_h(\mathbb{C}P(3) \times S^1) \subset \Gamma_7 = bP_8 = \mathbb{Z}_{28}$. $(\mathbb{C}P(3) \times S^1)$ is not simply connected, but our methods remain valid for special cases with simple fundamental groups.) Now $(\mathbb{C}P(3) \times S^1) \wedge S^1$ is homotopy equivalent to $(\mathbb{C}P(3) \wedge S^2) \vee (\mathbb{C}P(3) \wedge S^1) \vee S^2$. Thus, since $K0^0(\mathbb{C}P(3) \wedge S^1) = 0$, image $([(\mathbb{C}P(3) \times S^1) \wedge S^1, F/0] \rightarrow K0^0((\mathbb{C}P(3) \times S^1) \wedge S^1)) = \text{image}([\mathbb{C}P(3) \wedge S^2, F/0] \rightarrow K0^0(\mathbb{C}P(3) \wedge S^2)) = \mathbb{Z}^2$, with generators ξ_1 and ξ_2 which satisfy $P(\xi_1) = 1 + 1$ $+p_1(\xi_1)+p_2(\xi_1)=1+48(z\cdot\sigma)+32\cdot15(z^3\cdot\sigma)$ and $P(\xi_2)=1+32\cdot45(z^3\cdot\sigma)$, where $z\in H^2(\mathbb{C}P(3),\mathbb{Z})$ and $\sigma\in H^2(S^2,\mathbb{Z})$ are generators. Thus if $u\in [(\mathbb{C}P(3)\times S^1)_0\wedge S^1, F/0]$ extends to $\bar{u}\in [(\mathbb{C}P(3)\times S^1)\wedge S^1, F/0]$ and $\xi=\xi(\bar{u})=m\xi_1+n\xi_2$ then

$$du = s(\tilde{u}) = (\frac{1}{8}) \langle L(\mathbb{C}P(3) \times S^{1} \times S^{1}) (1 - L(\xi)), [\mathbb{C}P(3) \times S^{1} \times S^{1}] \rangle$$

= $(-\frac{1}{8}) \langle (1 + (\frac{4}{3}) z^{2}) ((48m/3) (z\sigma) + (7(32 \cdot 15m + 32 \cdot 45n)/45) (z^{3}\sigma), [\mathbb{C}P(3) \times S^{1} \times S^{1}] \rangle = -12m - 28n \in \mathbb{Z}/28\mathbb{Z}.$

It follows that $I_h(\mathbb{C}P(3) \times S^1) = \mathbb{Z}_7 \subset \mathbb{Z}_{28}$.

Remark II.10. R. Lee [16] has shown that every self-homotopy equivalence of $CP(n) \times S^1$ is homotopic to a diffeomorphism. If a manifold M^k has this property it is easy to see that $I_h(M^k) = I(M^k)$. Thus $I(CP(3) \times S^1) = \mathbb{Z}_7 \subset \mathbb{Z}_{28}$.

Remark II.11. Let π_0^+ (Diff (CP(n))) denote the group of pseudo-isotopy classes of diffeomorphisms of CP(n) which leave fixed a generator of $H^2(CP(n), \mathbb{Z})$. Lee has shown that π_0^+ (Diff CP(n)) is isomorphic to the equivariant diffeomorphism classes of differentiable, semi-free S^1 actions on homotopy (2n+2)-spheres, with fixed point set S^0 . (A group action is semi-free if it is free outside the fixed point set.) It follows from results of Sullivan that the natural map $\Gamma_7 = \pi_0(\text{Diff}(S^6)) \xrightarrow{\gamma} \pi_0^+$ (Diff(CP(3))) is a surjection, where, if $\Sigma^7 \in \Gamma_7$ corresponds to a diffeomorphism $\sigma: D^6 \cong D^6$, with $\sigma |_{S^5} = \text{Id}$, then $\gamma(\Sigma^7) |_{D^6} = \sigma$ and $\gamma(\Sigma^7) |_{CP(3)-D^6} = \text{Id}$, where $D^6 \subset CP(3)$. It is not difficult to see that the mapping torus of $\gamma(\Sigma^7)$ is $(CP(3) \times S^1) \# \Sigma^7$. Hence, $\gamma(\Sigma^7) = 0 \in \pi_0^+$ (Diff(CP(3))) if and only if $\gamma(\Sigma^7)$ is pseudo-isotopic to the identity, or equivalently, if and only if there is a diffeomorphism $(CP(3) \times S^1) \# \Sigma^7 = T_{\gamma(\Sigma^7)} \cong$ $\cong CP(3) \times S^1$ which is the identity on $CP(3) \times 0$. Since any diffeomorphism $(CP(3) \times S^1) \# \Sigma^7 \cong CP(3) \times S^1$ is pseudo-isotopic to one which fixes $CP(3) \times 0$ [19; Lemma 4], this proves that kernel(γ) = $I(CP(3) \times S^1) = \mathbb{Z}_7 \subset \mathbb{Z}_{28}$ and that π_0^+ (Diff(CP(3))) = \mathbb{Z}_4.

Remark II.12. For each integer j there is a manifold P_j^6 homotopy equivalent to CP(3) with $p_1(P_j^6) = (4+24j)z^2$. Thus if $u \in [(P_j^6 \times S^1)_0 \wedge S^1, F/0]$ with $\xi(\bar{u}) = m\xi_1 + n\xi_2$ then $du = s(\bar{u}) = -(12+16j)m - 28n \in \mathbb{Z}/28\mathbb{Z}$. It follows that $I_h(P_j^6 \times S^1) = 0$ if $j \equiv 1 \pmod{7}$ and $I_h(P_j^6 \times S^1) = \mathbb{Z}_7$ if $j \not\equiv 1 \pmod{7}$. In particular, $I_h(N^k)$ is not a homotopy invariant of N^k .

REFERENCES

- [1] J. F. ADAMS, On the groups J(X)-I, Topology 2 (1963) 181–195.
- [2] J. F. ADAMS, On the groups J(X)-IV, Topology 5 (1966), 21–71.

- [3] J. F. ADAMS and G. WALKER, On complex Stiefel manifolds, Proc. Cambridge Phil. Soc. 6 (1965), 81–103.
- [4] D. W. ANDERSON, E. H. BROWN, JR., and F. P. PETERSON, The structure of the spin cobodism ring, Ann. of Math. 86 (1967), 271-298.
- [5] J. M. BOARDMAN and R. M. VOGT, Homotopy everything H-spaces, Bull. Amer. Math. Soc. 74 (1968), 1117–1122.
- [6] W. BROWDER, The Arf invariant and its generalization, Ann. of Math. 90 (1969), 157-186.
- [7] E. H. BROWN, Jr., Note on an invariant of Kervaire, Michigan Math. J. 12 (1965), 23-24.
- [8] E. H. BROWN, JR. and F. P. PETERSON, The Kervaire invariant of (8k + 2)-manifolds, Amer. J. Math. 88 (1966), 815–826.
- [9] G. BRUMFIEL, On the homotopy groups of BPL and PL/O, Ann. of Math. 88 (1968), 291-311.
- [10] G. BRUMFIEL, On the homotopy groups of BPL and PL/O-II, Topology 8 (1969), 305-311.
- [11] G. BRUMFIEL, On the homotopy groups of BPL and PL/O-III, Michigan Math. J. 17 (1970), 217-224.
- [12] G. BRUMFIEL, Differentiable S^1 actions on homotopy spheres, mimeographed (Univ. of Calif., Berkeley, 1968).
- [13] B. HARRIS, J-homomorphisms and cobordism groups, Inventiones Math. 7 (1969), 313-320.
- [14] M. HIRSCH and B. MAZUR, Smoothings of piecewise linear manifolds, mimeographed (Cambridge University, 1964).
- [15] M. A. KERVAIRE and J. W. MILNOR, Groups of homotopy spheres, Ann. of Math. 77 (1963), 504-537.
- [16] R. LEE, Thesis, Univ. of Michigan 1968.
- [17] M. MAHOWALD, On the order of the image of J. Topology, 6 (1967), 371-378.
- [18] D. SULLIVAN, Smoothings homotopy equivalences, Univ. of Warwick, 1966.
- [19] C. T. C. WALL, Unknotting tori, Proc. Cambridge Phil. Soc., 61 (1965), 659-664.

Stanford University, Stanford, California.

Received January 11, 1971.