Some Congruence Theorems for Closed Hypersurfaces in Riemann Spaces (Part III: Method baser on Voss' Proof)

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(Part III: Method based on Voss' Proof)

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Introduction

An idea that gives congruence of two hypersurfaces concerning a transformation group by a relation between the invariant of the corresponding points of these hyper surfaces was first introduced by H. Hopf and K. Voss [1], that is, in that paper congruence relations of two closed curves on a plane and of two closed surfaces in 3-dimensional euclidean space have been given by the relation of the mean curvatures.

K. Voss has generalized these theorems to hypersurfaces in an $(m+1)$-dimensional euclidean space $(m+1 \geqq 3)$ and also given the congruence relations in case of Gauss curvatures or the r-th mean curvatures H_r $r=1, 2, \ldots, m$ [2]. A. Aeppli has developed analogous statements for a central transformation group (a homothetic transformation group with the center 0) [3].

The present authors wished to generalize these theorems to Riemann spaces. In the previous papers [4], [5], we gave the generalized theorems relating to the first mean curvature.

The purpose of the present paper is to investigate a general theorem relating to the Gauss curvature or the r-th mean curvature, that is, to generalize to an orientable Riemann space R^{m+1} with constant Riemann curvature the following theorems given by K. Voss:

THEOREM (K. Voss). *Let W^m and $\bar{W}^m$ be two orientable closed hypersurfaces in an $(m+1)$-dimensional euclidean space and let p and $\bar{p}$ be the corresponding points of these hypersurfaces, and let $K(p)$ and $\bar{K}(p)$ be that Gauss curvatures at these points respectively. Assume that the second fundamental forms of W^m and $\bar{W}^m$ are positive definite. If all straight lines $(p\bar{p})$ are parallel to one another and if $K(p)=\bar{K}(\bar{p})$ for all $p \in W^m$, then the hypersurface $\bar{W}^m$ is produced from W^m by simple translation in the direction of $(p\bar{p})$. (W^m and $\bar{W}^m$ are therefore congruent mod the translation group).*

THEOREM (K. Voss). *Let W^m and $\bar{W}^m$ be to orientable closed hypersurfaces in an $(m+1)$-dimensional euclidean space and let p and $\bar{p}$ be the corresponding points of these hypersurfaces, and let $H_r(p)$ and $\bar{H}_r(\bar{p})$ be the r-th mean curvatures at these* *points respectively, for some* $r=1, 2, \ldots, m$. *Assume that the second fundamental forms of* W^m *and* $\bar{W}^m$ *are positive definite. If all straight lines* $(p\bar{p})$ *are parallel to one another and if* $H_r(p)=\bar{H}_r(\bar{p})$ *for all* $p \in W^m$, *then the hypersurface* $\bar{W}^m$ *is produced from* W^m *by simple translation in the direction of* $(p\bar{p})$. (W^m *and* $\bar{W}^m$ *are congruent mod the translation group.*)

§ 1. Generalized Theorems

We suppose an $(m+1)$-dimensional orientable Riemann space with constant curvature S^{m+1} of class C^v $(v \geqq 3)$ which admits an infinitesimal isometric transformation

$$\bar{x}^i = x^i + \xi^i(x)\, \delta\tau \tag{1.1}$$

(where x^i are local coordinate in S^{m+1} and ξ^i are the components of a contravariant vector ξ). We assume that orbits of the transformations generated by ξ cover S^{m+1} simply and that ξ is everywhere continuous and $\neq 0$. Let us choose a coordinate system such that the orbits of the transformations are new x^1-coordinate curves, that is, a coordinate system in which the vector ξ^i has components $\xi^i = \delta^i_1$, where the symbol δ^i_j denotes Kronecker's delta; then (1.1) becomes as follows

$$\bar{x}^i = x^i + \delta^i_1\, \delta\tau \tag{1.2}$$

and S^{m+1} admits a one-parameter continuous group G of transformations which are $1-1$-mappings of S^{m+1} onto itself and are given by the expression $\bar{x}^i = x^i + \delta^i_1 \tau$ in the new special coordinate system ([6]).

Now we consider two orientable closed hypersurfaces W^m and $\bar{W}^m$ of class C^v imbedded in S^{m+1} which are given as follows

$$\left.\begin{aligned} W^m &: x^i = x^i(u^\alpha) \qquad i=1, \ldots, m+1 \qquad \alpha = 1, \ldots, m \\ \bar{W}^m &: \bar{x}^i = \bar{x}^i(u^\alpha) + \delta^i_1 \tau(u^\alpha) \end{aligned}\right\} \tag{1.3}$$

where u^α are local coordinates of W^m and τ is a continuous function attached to each point of the hypersurface W^m. We shall henceforth confine ourselves to Latin indices running from 1 to $m+1$ and Greek indices from 1 to m, and to two hypersurfaces W^m and $\bar{W}^m$ which do not contain a piece of a hypersurface covered by the orbits of the transformations, which is expressed by $f(x^2, \ldots, x^{m+1}) = 0$.

Then we can take the family of the hypersurfaces

$$W^m(t) = (1-t)\, W^m + t\bar{W}^m \qquad 0 \leqq t \leqq 1,$$

generated by W^m and $\bar{W}^m$ whose points correspond along the orbits of the transformations where W^m and $\bar{W}^m$ mean $W^m(0)$ and $W^m(1)$ respectively. Thus according to (1.3), $W^m(t)$ is given by the expression

$$W^m(t): x^i(u^\alpha, t) = (1-t)\, x^i(u^\alpha) + t\bar{x}^i(u^\alpha) \qquad 0 \leqq t \leqq 1, \tag{1.4}$$

and (1.4) may be rewritten as follows

$$W^m(t): x^i(u^\alpha, t) = x^i(u^\alpha) + \delta^i_1 t\tau(u^\alpha) \qquad 0 \leqq t \leqq 1. \tag{1.5}$$

Let us denote the normal unit vector of $W^m(t)$ by $n^i(t)$ and its derivative with respect to t by $n'^i(t)$. Then g_{ij} being the metric tensor of S^{m+1} and differentiating the following relations with respect to t,

$$g_{ij} n^i(t) \frac{\partial x^j(u, t)}{\partial u^\alpha} = 0, \qquad g_{ij} n^i(t)\, n^j(t) = 1,$$

since the transformation group G is isometric, that is, $\partial g_{ij}/\partial x^1 = 0$, we have

$$g_{ij} n'^i \frac{\partial x^j(u, t)}{\partial u^\alpha} + g_{ij} n^i(t) \frac{d}{dt}\left(\frac{\partial x^j(u, t)}{\partial u^\alpha}\right) = 0, \tag{1.6}$$

$$g_{ij} n^i(t)\, n'^j(t) = 0. \tag{1.7}$$

From (1.6), (1.7) and

$$\frac{d}{dt}\left(\frac{\partial x^i(u, t)}{\partial u^\alpha}\right) = \delta^i_1 \frac{\partial \tau}{\partial u^\alpha},$$

we get

$$n'^i(t) = -g^{\alpha\beta}(t)\, \tau_\alpha \delta^l_1 n_l(t) \frac{\partial x^i(u, t)}{\partial u^\beta}, \tag{1.8}$$

where $g^{\alpha\beta}(t)$ is the contravariant metric tensor of $W^m(t)$ and τ_α means $\partial\tau/\partial u^\alpha$. Throughout this paper repeated lower case Latin indices call for summation 1 to $m+1$ and repeated lower case Greek indices for summation 1 to m. And also for its covariant differential along $W^m(t)$ we have

$$\delta n'^i(t) = dn'^i(t) + \Gamma^i_{jl} n'^j(t)\, x^l_\gamma\, du^\gamma, \tag{1.9}$$

where Γ^i_{jl} is the Christoffel symbol with respect to the metric tensor g_{ij} of S^{m+1} and x^l_γ means $\partial x^l(u, t)/\partial u^\gamma$.

Let us give henceforth the derivative with respect to t by the dash. Calculating $(\delta n^i)'$, we have

$$\delta n^i = dn^i + \Gamma^i_{jl} n^j(t)\, x^l_\gamma d\, u^\gamma,$$
$$(\delta n^i)' = (dn^i)' + (\Gamma^i_{jl})'\, n^j(t)\, x^l_\gamma\, du^\gamma + \Gamma^i_{jl} n'^j(t)\, x^l_\gamma\, du^\gamma + \Gamma^i_{jl} n^j(t)\, (x^l_\gamma)'\, du^\gamma,$$

since G is isometric, that is, $\partial g_{ij}/\partial x^1 = 0$, we have $\partial \Gamma^i_{lj}/\partial x^1 = 0$. Consequently we obtain the following relation between $\delta n'^i$ and $(\delta n^i)'$

$$(\delta n^i)' = \delta n'^i + \Gamma^i_{j1} n^j(t)\, \tau_\gamma\, du^\gamma. \tag{1.10}$$

We claim that the following theorems hold

THEOREM 1.1. *Let K and $\bar{K}$ be the Gauss curvature of W^m and $\bar{W}^m$ respectively. Assume that the second fundamental form of $W^m(t)$, $0 \leqq t \leqq 1$ is positive definite. If the relation $K = \bar{K}$ holds for each point $p \in W^m$, then W^m and $\bar{W}^m$ are congruent* mod G.

Proof. We consider the following differential form of degree $m-1$ attached to each point p on the hypersurface $W^m(t)$

$$\left.\begin{aligned} ((n', \delta_1\tau, \delta n, \ldots, \delta n)) &\overset{\text{def.}}{\equiv} \sqrt{g}\,(n', \delta_1\tau, \delta n, \ldots, \delta n) \\ &= (-1)^{m-1}\sqrt{g}\,(n', \delta_1\,\tau, x_{\alpha_1}, \ldots, x_{\alpha_{m-1}}) \\ &\quad \times b^{\alpha_1}_{\beta_1}(t) \ldots b^{\alpha_{m-1}}_{\beta_{m-1}}(t)\, du^{\beta_1} \wedge \ldots \wedge du^{\beta_{m-1}} \end{aligned}\right\} \tag{1.11}$$

where g is the determinant of the metric tensor g_{ij} of S^{m+1}, the symbol ( ) means a determinant of order $m+1$ whose columns are the components of respective vectors, $b_{\alpha\beta}(t)$ is the second fundamental tensor of $W^m(t)$ and $b^\beta_\alpha(t)$ denotes $b_{\alpha\gamma}(t)\,g^{\beta\gamma}(t)$.

Then the exterior differential of the differential form (1.11) becomes as follows

$$\left.\begin{aligned} &d((n', \delta_1\tau, \delta n, \ldots, \delta n)) = ((\delta n', \delta_1\tau, \delta n, \ldots, \delta n)) \\ &\quad + ((n', \delta(\delta_1)\,\tau, \delta n, \ldots, \delta n)) + ((n', \delta_1\, d\tau, \delta n, \ldots, \delta n)), \end{aligned}\right\} \tag{1.12}$$

because since S^{m+1} is a space of constant curvature, we have

$$((n', \delta_1\tau, \delta\delta n, \delta n, \ldots, \delta n)) = 0\,.$$

Because G is isometric, the quantity $n_i(t)\,\delta^i_1\sqrt{g^*(t)}$ is independent of t, where $g^*(t)$ means the determinant of $g_{\alpha\beta}(t)$, we have

$$(((\delta n)', \delta_1\tau, \delta n, \ldots, \delta n)) = (-1)^m(m-1)!\; K' n_i(t)\,\delta^i_1\tau\; dA(t) \tag{1.13}$$

where $dA(t)$ is the area element of $W^m(t)$, and using (1.8), we obtain

$$\left.\begin{aligned} &((n', \delta_1\, d\tau, \delta n, \ldots, \delta n)) = (-1)^m(m-1)! \\ &\quad \times \frac{1}{\sqrt{g^*(t)}} B^{\alpha\beta}(t)\,\tau_\alpha\tau_\beta(n_i(t)\,\delta^i_1)^2\sqrt{g^*(t)}\, dA(t) \end{aligned}\right\} \tag{1.14}$$

where $B^{\alpha\beta}(t)$ means the cofactor of an element $b_{\beta\alpha}(t)$ in the determinant $|b_{\alpha\beta}(t)|$ divided by $g^*(t)$.

By making use of (1.10), (1.12), (1.13), (1.14) and the relation

$$\delta(\delta^i_1) = \Gamma^i_{j1} x^j_\gamma\, du^\gamma,$$

we have

$$\begin{aligned} d((n', \delta_1\tau, \delta n, \ldots, \delta n)) &= (-1)^m(m-1)!\left\{K' n_i(t)\,\delta^i_1\tau\, dA(t)\right. \\ &\quad \left. + \frac{1}{\sqrt{g^*(t)}} B^{\alpha\beta}(t)\,\tau_\alpha\tau_\beta(n_i(t)\,\delta^i_1)^2\sqrt{g^*(t)}\, dA(t)\right\} \\ &\quad + ((n', \tau\Gamma_{j1}x^j_\gamma\, du^\gamma, \delta n, \ldots, \delta n)) \\ &\quad - ((\Gamma_{j1}n^j(t)\,\tau_\gamma\, du^\gamma, \delta_1\tau, \delta n, \ldots, \delta n)). \end{aligned}$$

Next we shall prove that

$$((n', \tau\Gamma_{j1}x^j_\gamma\, du^\gamma, \delta n, \ldots, \delta n)) - ((\Gamma_{j1}n^j(t)\,\tau_\gamma\, du^\gamma, \delta_1\tau, \delta n, \ldots, \delta n)) = 0. \tag{1.15}$$

For the first term of the left-hand member of (1.15), making use of (1.8), we can see the following

$$\left.\begin{aligned} &((n', \tau\Gamma_{j1}x^j_\gamma\, du^\gamma, \delta n, \ldots, \delta n)) = (-1)^{m-1}\,\tau n_l(t)\,\delta^l_1 \\ &\quad\times ((\Gamma_{j1}x^j_\gamma, g^{\alpha\beta}(t)\,\tau_\beta x_\alpha, x_{\alpha_1}, \ldots, x_{\alpha_{m-1}})) \\ &\quad\times b^{\alpha_1}_{\beta_1}(t)\ldots b^{\alpha_{m-1}}_{\beta_{m-1}}(t)\, du^\gamma \wedge du^{\beta_1} \wedge \ldots \wedge du^{\beta_{m-1}}. \end{aligned}\right\} \tag{1.16}$$

Let $\varepsilon_{i_1\cdots i_{m+1}}$ and $\varepsilon_{\alpha_1\cdots\alpha_m}$ be the ε-symbol of S^{m+1} and of $W^m(t)$ respectively,

$$\varepsilon_{i_1\ldots i_{m+1}} \overset{\text{def.}}{=} \sqrt{g}\, e_{i_1\ldots i_{m+1}}, \qquad \varepsilon_{\alpha_1\ldots\alpha_m} \overset{\text{def.}}{\equiv} \sqrt{g^*(t)}\, e_{\alpha_1\ldots\alpha_m},$$

the symbol $e_{i_1\cdots i_{m+1}}$ meaning plus one or minus one, depending on whether the indices $i_1, \ldots i_{m+1}$ denote an even permutation of $1, 2, \ldots, m+1$ or odd permutation, and zero when at least any two indices have the same value, and also the symbol $e_{\alpha_1\cdots\alpha_m}$ meaning similarly for the indices $\alpha_1, \ldots, \alpha_m$ running from 1 to m.

Making use of the relation

$$n_i(t)\,\varepsilon_{\alpha\alpha_1\ldots\alpha_{m-1}} = \varepsilon_{ii_2\ldots i_{m+1}}x^{i_2}_\alpha x^{i_3}_{\alpha_1}\cdots x^{i_{m+1}}_{\alpha_{m-1}}$$

we have

$$\begin{aligned} &((n', \tau\Gamma_{j1}x^j_\gamma\, du^\gamma, \delta n, \ldots, \delta n)) = (-1)^{m-1}\,\tau n_l(t)\,\delta^l_1\Gamma^i_{j1}n_i(t)\, x^j_\gamma\tau_\beta g^{\beta d}(t) \\ &\quad\times \varepsilon_{\alpha\alpha_1\ldots\alpha_{m-1}}b^{\alpha_1}_{\beta_1}(t)\ldots b^{\alpha_{m-1}}_{\beta_{m-1}}(t)\, du^\gamma \wedge du^{\beta_1} \wedge \ldots \wedge du^{\beta_{m-1}} \\ &\quad= (-1)^{m-1}\,\tau n_l(t)\,\delta^l_1\Gamma^i_{j1}n_i(t)\, x^j_\gamma\tau_\beta\varepsilon^\beta_{\alpha_1\ldots\alpha_{m-1}}\,\varepsilon^{\gamma\beta_1\ldots\beta_{m-1}}b^{\alpha_1}_{\beta_1}(t)\ldots b^{\alpha_{m-1}}_{\beta_{m-1}}(t)\, dA(t) \end{aligned}$$

and we can see easily the following relation

$$\begin{aligned} \varepsilon^\beta_{\alpha_1\ldots\alpha_{m-1}}\,\varepsilon^{\gamma\beta_1\ldots\beta_{m-1}}b^{\alpha_1}_{\beta_1}(t)\ldots b^{\alpha_{m-1}}_{\beta_{m-1}}(t) &= \varepsilon^{\beta\gamma_1\ldots\gamma_{m-1}}\,\varepsilon^{\gamma\beta_1\ldots\beta_{m-1}}b_{\gamma_1\beta_1}(t)\ldots b_{\gamma_{m-1}\beta_{m-1}}(t) \\ &= (m-1)!\,B^{\beta\gamma}(t). \end{aligned}$$

Since $B^{\beta\gamma}(t)$ is the symmetric tensor, we have

$$\left.\begin{aligned} &((n', \tau\Gamma_{j1}x^j_\gamma du^\gamma, \delta n, \ldots \delta n)) = (-1)^{m-1}(m-1)!\,\tau n_l(t) \\ &\quad\times \delta^l_1\Gamma_{ji1}n^i(t)\, x^j_{(\gamma}\tau_{\beta)}B^{\beta\gamma}(t)\, dA(t) \end{aligned}\right\} \tag{1.17}$$

where Γ_{ji1} means $g_{il}\Gamma^l_{j1}$ and the symbol $(\gamma\beta)$ denotes the symmetric part for the indices γ and β.

On the other hand, we calculate the second term of the left-hand member of (1.15). Since G is isometric, that is, $\partial g_{ij}/\partial x^1 = 0$, we have

$$\left.\begin{aligned} \Gamma^l_{j1}n^j(t)\, n_l(t) &= \frac{1}{2}\, g^{lk}\left(\frac{\partial g_{kj}}{\partial x^1} + \frac{\partial g_{1k}}{\partial x^j} - \frac{\partial g_{j1}}{\partial x^k}\right) n^j(t)\, n_l(t) \\ &= \frac{1}{2}\frac{\partial g_{kj}}{\partial x^1}\, n^j(t)\, n^k(t) = 0, \end{aligned}\right\} \tag{1.18}$$

and we can give the vector δ_1^i by the expression

$$\delta_1^i = n_l(t)\,\delta_1^l n^i(t) + \varphi^\beta x_\beta^i. \tag{1.19}$$

Substituting (1.19) in the second term of the left-hand member of (1.15) and making use of (1.18), we have

$$\left.\begin{aligned} &-((\Gamma_{j1}n^j(t)\,\tau_\gamma\,du^\gamma, \delta_1\tau, \delta n, \dots, \delta n)) \\ &\quad = -(-1)^{m-1}\,\tau n_l(t)\,\delta_1^l\,((\Gamma_{j1}n^j(t)\,\tau_\gamma, n, x_{\alpha_1}, \dots, x_{\alpha_{m-1}})) \\ &\quad \times b_{\beta_1}^{\alpha_1}(t)\dots b_{\beta_{m-1}}^{\alpha_{m-1}}(t)\,du^\gamma\wedge du^{\beta_1}\wedge\dots\wedge du^{\beta_{m-1}}. \end{aligned}\right\} \tag{1.20}$$

Let us take the relation

$$\varepsilon_{\alpha\alpha_1\dots\alpha_{m-1}}g^{\alpha\beta}(t)\,x_\beta^j g_{ij} = (-1)^m\,\varepsilon_{ii_2\dots i_{m+1}}x_{\alpha_1}^{i_2}\dots x_{\alpha_{m-1}}^{i_m}n^{i_{m+1}}.$$

Then we have

$$\left.\begin{aligned} &-((\Gamma_{j1}n^j(t)\,\tau_\gamma\,du^\gamma, \delta_1\tau, \delta n, \dots, \delta n)) \\ &\quad = (-1)^{m-1}(m-1)!\,\tau n_l(t)\,\delta_1^l\Gamma_{ij1}n^i(t)\,x_{(\beta}^j\tau_{\gamma)}B^{\beta\gamma}(t)\,dA(t). \end{aligned}\right\} \tag{1.21}$$

Thus from (1.17), (1.21) and $\Gamma_{ij1}+\Gamma_{ji1}=\partial g_{ij}/\partial x^1=0$, we can arrive at (1.15) as follows

$$\begin{aligned} &((n', \tau\Gamma_{j1}x_\gamma^j\,du^\gamma, \delta n, \cdots, \delta n)) - ((\Gamma_{j1}n^j(t)\,\tau_\gamma\,du^\gamma, \delta_1\tau, \delta n, \dots, \delta n)) \\ &\quad = (-1)^{m-1}(m-1)!\,\tau n_l(t)\,\delta_1^l\,(\Gamma_{ij1}+\Gamma_{ji1})\,n^i(t)\,x_{(\gamma}^j\tau_{\beta)}B^{\beta\gamma}(t)\,dA(t) = 0. \end{aligned}$$

Finally we have

$$\left.\begin{aligned} &\frac{(-1)^m}{(m-1)!}\,d((n', \delta_1\tau, \delta n, \dots, \delta n)) = K'n_i(t)\,\delta_1^i\tau\,dA(t) \\ &\quad + \frac{1}{\sqrt{g^*(t)}}\,B^{\alpha\beta}(t)\,\tau_\alpha\tau_\beta(n_i(t)\,\delta_1^i)^2\sqrt{g^*(t)}\,dA(t). \end{aligned}\right\} \tag{1.22}$$

Integrating both members of (1.22) over the interval $0\leqq t\leqq 1$, we get

$$\left.\begin{aligned} &\frac{(-1)^m}{(m-1)!}\,d\int_0^1((n', \delta_1\tau, \delta n, \dots, \delta n))\,dt = (\bar{K}-K)\,n_i(0)\,\delta_1^i\tau\,dA(0) \\ &\quad + \sqrt{g^*(0)}\int_0^1 g^*(t)^{-1/2}B^{\alpha\beta}(t)\,dt\,\tau_\alpha\tau_\beta(n_i(0)\,\delta_1^i)^2\,dA(0). \end{aligned}\right\} \tag{1.23}$$

Furthermore integrating both members of (1.23) over W^m and applying Stokes' theorem, since W^m is closed, we have

$$\begin{aligned} &\iint_{W^m}(\bar{K}-K)\,n_i(0)\,\delta_1^i\tau\,dA(0) \\ &\quad + \iint_{W^m}\sqrt{g^*(0)}\int_0^1 g^*(t)^{-1/2}\,B^{\alpha\beta}(t)\,dt\,\tau_\alpha\tau_\beta(n_i(0)\,\delta_1^i)^2\,dA(0) = 0, \end{aligned}$$

making use of the hypothesis $\bar{K}=K$, we obtain

$$\iint_{W^m} \sqrt{g^*(0)} \int_0^1 g^*(t)^{-1/2} B^{\alpha\beta}(t)\, dt \tau_\alpha \tau_\beta (n_i(0)\, \delta_1^i)^2\, dA(0) = 0\,.$$

On the other hand, from that the second fundamental form of $W^m(t)$ is positive definite everywhere in $W^m(t)$, $0 \leqq t \leqq 1$, the quantity

$$\sqrt{g^*(0)} \int_0^1 g^*(t)^{-1/2} B^{\alpha\beta}(t)\, dt v_\alpha v_\beta$$

becomes positive definite. From that two hypersurfaces W^m and $\bar{W}^m$ do not contain a piece of a hypersurface covered by the orbits of transformations, a point on W^m such that $n_i(0)\delta_1^i = 0$ must be an isolate point. Moreover since τ is a continuous function of W^m, we have

$$\tau = \text{constant}$$

for all points of W^m. Consequently we can arrive at the following result

$$W^m \equiv \bar{W}^m \bmod G\,.$$

THEOREM 1.2. *Let H_r and $\bar{H}_r$ be the r-th mean curvature of W^m and $\bar{W}^m$ respectively. Assume that the second fundamental form of $W^m(t)$, $0 \leqq t \leqq 1$, is positive definite. If the relation*

$$H_r = \bar{H}_r$$

holds for each point $p \in W^m$, then W^m and $\bar{W}^m$ are congruent mod G.

Proof. We consider the following differential form of degree $m-1$ attached to each point p on the hypersurface $W^m(t)$

$$\left.\begin{aligned} &((n', \delta_1\tau, \underbrace{\delta n, \ldots, \delta n}_{r-1}, dx, \ldots, dx)) \overset{\text{def.}}{\equiv} \sqrt{g}(n', \delta_1\tau, \delta n, \ldots, \delta n, dx, \ldots, dx) \\ &= (-1)^{r-1} \sqrt{g}(n', \delta_1\tau, x_{\alpha_1}, \ldots, x_{\alpha_{r-1}}, x_{\beta_r} \ldots x_{\beta_{m-1}}) \\ &\times b^{\alpha_r}_{\beta_r}(t) \ldots b^{\alpha_{\gamma-1}}_{\beta\partial-1}(t)\, du^{\beta_1} \wedge \ldots \wedge du^{\beta_{r-1}} \wedge du^{\beta_r} \wedge \ldots \wedge du^{\beta_{m-1}} \end{aligned}\right\} \quad (1.24)$$

The exterior differential of the differential form (1.24) becomes as follows

$$\begin{aligned} d((n', \delta_1\tau, \delta n, \ldots, \delta n, dx, \ldots, dx)) &= ((\delta n', \delta_1\tau, \delta n, \ldots, \delta n, dx, \ldots, dx)) \\ &+ ((n', \delta(\delta_1)\,\tau, \delta n, \ldots, \delta n, dx, \ldots, dx)) + ((n', \delta_1\, d\tau, \delta n, \ldots, \delta n, dx, \ldots, dx)) \end{aligned}$$

because since S^{m+1} is a space of constant curvature, it follows that

$$((n', \delta_1\tau, \delta\delta n, \ldots, \delta n, dx, \ldots, dx)) = 0\,,$$

and also we have

$$((n', \delta_1 \tau, \delta n, \ldots, \delta n, \delta dx, \ldots, dx)) = 0.$$

Making use of (1.8), we have

$$\left.\begin{aligned} &((n', \delta_1 d\tau, \delta n, \ldots, \delta n, dx, \ldots, dx)) \\ &\quad = (-1)^{r-1} g^{\alpha\beta}(t) \tau_\alpha n_i(t) \delta_1^i ((\delta_1 \tau_\gamma, x_\beta, x_{\alpha_1}, \ldots, x_{\alpha_{r-1}}, x_{\alpha_r}, \ldots, x_{\alpha_{m-1}})) \\ &\quad \times b_{\beta_1}^{\alpha_1}(t) \ldots b_{\beta_{r-1}}^{\alpha_{r-1}}(t)\, du^\gamma \wedge du^{\beta_1} \wedge \ldots \wedge du^{\beta_{r-1}} \wedge du^{\alpha_r} \\ &\quad \wedge \ldots \wedge du^{\alpha_{m-1}} \\ &\quad = (-1)^{r-1} g^{\alpha\beta}(t)\, \varepsilon_{\beta\alpha_1 \ldots \alpha_{r-1}\alpha_r \ldots \alpha_{m-1}} \varepsilon^{\gamma\beta_1 \ldots \beta_{r-1}\alpha_r \ldots \alpha_{m-1}} \\ &\quad \times b_{\beta_1}^{\alpha_1}(t) \ldots b_{\beta_{r-1}}^{\alpha_{r-1}}(t)\, (n_i(t)\, \delta_1^i)^2\, \tau_\alpha \tau_\gamma\, dA(t). \end{aligned}\right\} \quad (1.25)$$

On the other hand, from (1.10) we get

$$\left.\begin{aligned} &((\delta n', \delta_1 \tau, \delta n, \ldots, \delta n, dx, \ldots, dx)) \\ &\quad = (((\delta n)', \delta_1 \tau, \delta n, \ldots, \delta n, dx, \ldots, dx)) \\ &\quad - ((\Gamma_{j1} n^j(t)\, \tau_\gamma\, du^\gamma, \delta_1 \tau, \delta n, \ldots, \delta n, dx, \ldots, dx)). \end{aligned}\right\} \quad (1.26)$$

And after some calculations, we have

$$(-1)^r\, m!\, H_r' n_i(t)\, \delta_1^i\, dA(t) = r((\delta_1, (\delta n)', \delta n, \ldots, \delta n, dx, \ldots, dx)), \quad (1.27)$$

because $n_i(t)\, \delta_1^i\, dA(t)$ is independent of t and $dx'^i = \delta_1^i\, d\tau$, that is, the same direction to δ_1. Moreover we can prove similarly the following relation as the proof of (1.15)

$$\left.\begin{aligned} &((n', \delta(\delta_1)\, \tau, \delta n, \ldots, \delta n, dx, \ldots, dx)) \\ &\quad - ((\Gamma_{j1} n^j(t)\, \tau_\gamma\, du^\gamma, \delta_1 \tau, \delta n, \ldots, \delta n, dx, \ldots, dx)) = 0. \end{aligned}\right\} \quad (1.28)$$

Then putting

$$(m-1)!\, c_{(r)}^{\alpha\beta} = \varepsilon^{\alpha}_{\alpha_1 \ldots \alpha_{r-1}\alpha_r \ldots \alpha_{m-1}} \varepsilon^{\beta\beta_1 \ldots \beta_{r-1}\alpha_r \ldots \alpha_{m-1}} b_{\beta_1}^{\alpha_1}(t) \ldots b_{\beta_{r-1}}^{\alpha_{r-1}}(t)$$

and using (1.25), (1.26), (1.27) and (1.28), we have

$$\left.\begin{aligned} &d((n', \delta_1 \tau, \delta n, \ldots, \delta n, dx, \ldots, dx)) \\ &\quad = \frac{(-1)^{r-1}}{r}\, m!\, H_r' n_i(t)\, \delta_1^i \tau\, dA(t) \\ &\quad + (-1)^{r-1} (m-1)!\, c_{(r)}^{\alpha\beta} \tau_\alpha \tau_\beta (n_i(t)\, \delta_1^i)^2\, dA(t). \end{aligned}\right\} \quad (1.29)$$

Integrating both members of (1.29) over the interval $0 \leqq t \leqq 1$, and putting

$$C_{(r)}^{\alpha\beta} = g^*(0)^{1/2} \int_0^1 g^*(t)^{-1/2}\, c_{(r)}^{\alpha\beta}\, dt,$$

we have

$$\left.\begin{aligned} &m(\bar{H}_r - H_r)\, n_i(0)\, \delta_1^i \tau dA(0) + r C_{(r)}^{\alpha\beta} \tau_\alpha \tau_\beta (n_i(0)\, \delta_1^i)^2\, dA(0) \\ &\quad = \frac{r(-1)^{r-1}}{(m-1)!}\, d \int_0^1 ((n', \delta_1 \tau, \delta n, \ldots, \delta n, dx, \ldots, dx))\, dt. \end{aligned}\right\} \quad (1.30)$$

Furthermore integrating both members of (1.30) over W^m and applying Stokes' theorem

$$\frac{m}{r}\iint\limits_{W^m}(\bar{H}_r - H_r)\, n_i(0)\, \delta_1^i\tau\, dA(0) + \iint\limits_{W^m}(n_i(0)\,\delta_1^i)^2\, C_{(r)}^{\alpha\beta}\tau_\alpha\tau_\beta\, dA(0)$$

$$= \frac{(-1)^{r-1}}{(m-1)!}\int\limits_{\partial W^m}\int\limits_0^1((n', \delta_1\tau, \delta n, \ldots, \delta n, dx, \ldots, dx))\, dt.$$

Since W^m is closed, we have

$$\frac{m}{r}\iint\limits_{W^m}(\bar{H}_r - H_r)\, n_i(0)\, \delta_1^i\tau\, dA(0) + \iint\limits_{W^m}(n_i(0)\,\delta_1^i)^2\, C_{(r)}^{\alpha\beta}\tau_\alpha\tau_\beta\, dA(0) = 0,$$

using the hypotheses $H_r = \bar{H}_r$ and that the second fundamental form of $W^m(t), 0 \leqq t \leqq 1$, is positive definite, and from that two hypersurfaces W^m and $\bar{W}^m$ do not contain a piece of a hypersurface covered by the orbits of transformations, we can arrive at

$$\tau_\alpha = 0$$

for all points of W^m, consequently we have

$$\tau = \text{constant}$$

for all points of W^m. Accordingly we can see the following result

$$W^m = \bar{W}^m \bmod G.$$

This proof follows to the method of that due to K. Voss [2].

Remark. In an euclidean space, if G is translation group, that is, a special isometric transformation group, Theorem 1.1 and Theorem 1.2 just coincide with theorems of K. Voss given in the introduction.

REFERENCES

[1] HOPF, H., and VOSS, K., *Ein Satz aus der Flächentheorie im Grossen*, Archiv. der Math. *3* (1952), 187–192.

[2] VOSS, K., *Einige differentialgeometrische Kongruenzsätze für geschlossene Flächen und Hyperflächen*, Math. Ann. *131* (1956), 180–218.

[3] AEPPLI, A., *Einige Ähnlichkeits- und Symmetriesätze für differenzierbare Flächen im Raum*, Comment. Math. Helv. *33* (1959), 174–195.

[4] KATSURADA, Y. *Some Congruence Theorems for Closed Hypersurfaces in Riemann Spaces (Part I: Method based on Stokes' Theorem)*, Comment. Math. Helv. *43* (1968), 176–194.

[5] HOPF H., and KATSURADA, Y., *Some Congruence Theorems for Closed Hypersurfaces in Riemann Spaces (Part II: Method based on a Maximum Principle)*, Comment. Math. Helv. *43* (1968), 217–223.

[6] EISENHART, L. P. *Continuous Groups of Transformations*, (Princeton, London, 1934).

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