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by R. A. SHAPIRO

§1. Introduction

In contrast to Riemannian symmetric spaces very little is known about the detailed geometric structure of non-Riemannian or so called affine symmetric spaces. In [2] Berger has classified all the affine symmetric spaces on the Lie algebra level, but globally little is known. Much of the difficulty arises from the fact that when the symmetric space is represented as a homogeneous space G/R the isotropy group R is not compact if G/R is not Riemannian. By restricting ourselves to the generalizations of Hermitian symmetric spaces the problems become more tractable. In the semi-simple case the Hermitian symmetric spaces are singled out among all the others by the presence of central elements in the isotropy subgroup. This is the key fact that carries through to the case of pseudo-Hermitian symmetric spaces.

In section 2 we discuss the relationship between Lie algebra and global descriptions of pseudo-Hermitian symmetric spaces and show that all these spaces are simply connected and hence there is a 1-1 correspondence between them and the algebras they define.

Section 3 deals with an extension of the Borel embedding theorem, which says that a non-compact Hermitian symmetric space may be holomorphically embedded in its compact dual. First we define the associated Riemannian symmetric spaces of non-compact and compact type, A^* and A. Here A and A^* are dual Hermitian symmetric spaces and their definition depends only upon the complexification (g_C, r_C).

THEOREM. *Let G/R be an irreducible pseudo-Hermitian symmetric space. Then there exist injections $\psi: A^* \to G/R$ and $\psi_1: G/R \to A$.*

Here ψ_1 is the generalized Borel embedding and $\psi_1 \cdot \psi$ is the ordinary Borel embedding of A^* into A. Hence any pseudo-Hermitian symmetric space is "sandwiched" between its associated Riemannian spaces. Alternately, this theorem may be regarded as a factoring of the standard Borel embedding of A^* into A through the space G/R. We should note that A has a representation as a complex flag manifold G_C/B and that the images of ψ and $\psi_1\psi$ are just the orbits through the origin of the appropriate groups.

In Section 4 we examine Berger's fibering theorem in the special case of a pseudo-Hermitian symmetric space. The fiber is a Hermitian symmetric space of non-compact type and the base is a Hermitian symmetric space of compact type. These are not to be confused with the associated spaces of Section 3. We show how a pseudo-Hermitian symmetric space induces a non-trivial fibering in the associated space of non-compact type and discuss the relation of these fiberings with the generalized Borel embedding. Next we give a generalization of the Cartan-Harish Chandra realization of a Hermitian symmetric space as a bounded symmetric domain. An irreducible pseudo-Hermitian symmetric space is embedded in a holomorphic vector bundle over a compact Hermitian symmetric space and in each fiber the image is a bounded symmetric domain.

Now we consider a reducible pseudo-Hermitian symmetric space $G_{\mathbf{C}}/R_{\mathbf{C}}$. We show that the fibering $G_{\mathbf{C}}/R_{\mathbf{C}} \to G_{\mathbf{C}}/B$ is equivalent to the holomorphic cotangent bundle of A. The inclusions of A^* and G/R into $G_{\mathbf{C}}/R_{\mathbf{C}}$ thus define sections over the orbits $\psi_1 \psi(A)$ and $\psi_1(G/R)$. These sections are then related to the Harish-Chandra realizations of A^* and G/R.

In section 5 we employ a technique of Griffiths and Schmid, [5], [10], to show that G/R is $k+1$ complete in the sense of [1], where $k = \dim_{\mathbf{C}} K/L$. Then this implies $H^n(G/R, \mathscr{g}) = 0$ $n > k$ where $\mathscr{g}$ is any coherent analytic sheaf on G/R.

I would like to thank Tom Sherman for many helpful hints and suggestions and Jim Lepowski for a great simplification of Proposition (2.4). The proof given is his.

§2. A pseudo-Hermitian symmetric space M is an affine globally symmetric space supplied with an almost complex structure J and an indefinite Hermitian structure h such that the symmetries are isometries of h. The group of isometries $I(M)$ is a transitive real Lie group acting on M and we can write $M = G/R$ where G is the identity component of $I(M)$ and R is the isotropy group at some point of M. The real part of h is a G-invariant pseudo-Riemannian metric on G/R and so its metric connection defines the symmetric space structure on M, [9] Theorem 15.6.

(2.1) PROPOSITION. *The metric h is Kähler, J is integrable and the symmetries are holomorphic.*

Proof. Let (X, Y) denote the real part of $h(X, Y)$. Let s be any isometry. We have $(X, Y) + i(X, JY) = h(X, Y) = h(ds\,X, ds\,Y) = (ds\,X, ds\,Y) + i(ds\,X, J\,ds\,Y)$. Since h is non-degenerate it follows that ds commutes with J. To prove h Kähler it suffices to show that J is invariant under parallel translation. Let p_1 and p_2 be any two points on a geodesic and let s be the symmetry at the point midway between p_1 and p_2. Then $-ds$ is parallel translation from p_1 to p_2, [6] p. 164. But since J commutes with ds it must be invariant under parallel translation from p_1 to p_2. It follows immediately that J is integrable, as is shown in [6] p. 302 for example. Q.E.D.

Recall that for any affine symmetric space the Lie algebra g of G can be decomposed as a direct sum $g = q + r$ where r is the Lie algebra of R and q is an r-module, identified with the tangent space to M at the origin, and $[q, q] \subset r$. That is, G/R is a reductive homogeneous space with the additional symmetry condition that $[q, q] \subset r$. The involutive automorphism θ defined by this decomposition is the differential of the symmetry at the origin. The Lie algebra g together with such a decomposition is called either an involutive Lie algebra or a symmetric Lie algebra. The subalgebra r is called a symmetric subalgebra. R may be identified with the holonomy group of the canonical symmetric connection and r with the holonomy algebra. In order to suitably decompose an affine symmetric space into irreducible objects we make the standing assumption that g be semi-simple.

By choosing a basis compatible with the symmetric decomposition $g=q+r$ and looking at the matrix representations of $\operatorname{ad}x$ and $\operatorname{ad}y$ for $x\in q$ and $y\in r$ it is easy to see that q and r are orthogonal under the Killing form. If g is semi-simple the Killing form is thus non-degenerate on both r and q. Since q is completely determined as the orthogonal complement to r the involutive Lie algebra (g, θ) may also be denoted by (g, r). It will frequently be convenient to do so.

If (g, θ) is a semi-simple involutive Lie algebra it is well known that there exists a Cartan decomposition τ of g such that τ and θ commute. From this fact it follows that r acts completely reducibly on q and that r must be a reductive Lie algebra. An alternate proof of these facts may be obtained as follows. Since the Killing form is non-degerate on r, r must be reductive by [4] p. 79 Prop. #5 That the semisimple part of r acts completely reducibly is well known. Now let x be central in r. Then the semisimple and nilpotent parts of $\operatorname{ad}x$ are polynomials in $\operatorname{ad}x$ and so must be central in r. But any nilpotent central element of r must be orthogonal to all of g and so the center of r must contain only semisimple elements.

The involutive Lie algebra (g, θ) is called simple if there do not exist any θ-stable non-trivial ideals. If $g=\sum g_i$ is a decomposition of g into simple ideals then either g_i is θ-stable or θ interchanges two ideals g_i and g_j. Clearly any central element of the θ-fixed point set of $g_i+\theta(g_i)$ must be central in all of $g_i+\theta(g_i)$ and so if (g, θ) is the involutive Lie algebra of a pseudo-Hermitian symmetric space each g_i must be θ-stable. Indeed, if $g=\sum (g_i, r_i)$ is a decomposition into simple involutive Lie algebras, the almost complex structure J must have a non-zero projection on the center of each r_i. Hence we may limit ourselves to the case where g is simple.

(2.2) PROPOSITION. *If $g_{\mathbf{C}}$ is simple then $r_{\mathbf{C}}$ has at most a one-dimensional center (over $\mathbf{C}$) which may be spanned by an element z with eigenvalues $\pm i$ on $q_{\mathbf{C}}$. The eigenspaces q^+ and q^- are abelian, isotropic and dually paired under the Killing form, and irreducible as $r_{\mathbf{C}}$-modules.*

Proof. Let $q_{\mathbf{C}}=\sum V_\alpha$ be a simultaneous eigenspace decomposition of $q_{\mathbf{C}}$ under the action of the center of $r_{\mathbf{C}}$. Since $[V_\alpha, V_\beta]\subset r_{\mathbf{C}}$, on applying $\operatorname{ad}z$ with z central in $r_{\mathbf{C}}$ we conclude $0=(\alpha(z)+\beta(z))[V_\alpha, V_\beta]$ and so either $\alpha=-\beta$ or $[V_\alpha, V_\beta]=0$. In particular each V_α with $\alpha\neq 0$ is abelian. Pick some V_α with $\alpha\neq 0$. Let $V=V_\alpha+V_{-\alpha}$ and $W=\sum V_\beta$ where $\beta\neq\pm\alpha$. Now $V+W=q_{\mathbf{C}}$ and $[V, W]=0$ and so $V+[V, V]$ is a non-trivial ideal of g_C and so must equal g_C. Let z and z' be any two central elements of r_C. Then for some complex number c $\alpha(z)=c\alpha(z')$ so $\alpha(z-cz')=0$ so $z-cz'$ is central in g_C and so $z=cz'$. Since $V_\alpha=[z, V_\alpha]$ we get $(V_\alpha, V_\alpha)=([z, V_\alpha], V_\alpha)=(z, [V_\alpha, V_\alpha])=0$, and similarly for $V_{-\alpha}$. Now $V_{-\alpha}$ must be non-zero and indeed dually paired with V_α since the Killing form is non-degenerate on q_C. Letting $V_{\pm\alpha}=q^{\pm}$, everything is proved except the irredicubility of $q^{\pm}$. Let $q^+=V_1+V_2$ be a decomposition of q^+ into two r_C submodules. Then we can decompose q^- into two submodules $q^-=W_1+W_2$ with $(V_i, W_i)=0$, $i=1, 2$. We then have $[V_i, W_i]=0$ since $(r_C, [V_i, W_i])=([r_C, V_i], W_i)\subset \subset(V_i, W_i)=0$ and the Killing form is non-degenerate on r_C. Hence $V_1+W_2+[V_1, W_2]$ is an ideal in g_C so either $V_1=0$ or $V_1=q^+$ and $W_2=q^-$.

(2.3) COROLLARY. *If (g, r) is a simple involutive Lie algebra and r is not semi-simple then one of the following must be true*

1) *g_C is simple and r has a one dimensional center.*
2) *g already is a complex simple Lie algebra.*

Proof. We have already noted that g must be simple. If g_C is not simple it is well known that case 2) holds. So assume that g_C is simple and that r has central elements z and z' which are linearly independent over the reals. By the eigenvalues of $\operatorname{ad} z$ and $\operatorname{ad} z'$ must appear in positive-negative pairs. But they must also appear in conjugate pairs so the only possibilities are pure imaginary or real. We may assume that $\operatorname{ad} z$ has eigenvalues ± 1 and $\operatorname{ad} z'$ has eigenvalues $\pm i$. Let x be an element of q. Then there are elements $x'\in q^+$ and $x''\in q^-$ such that $x=x'+x''$. Then $ix=\operatorname{ad} z'(x'-x'')=\operatorname{ad} z' \operatorname{ad} z(x)\in q$ so q is closed under multiplication by i, and since $r=[q, q]$ all of g is.

Q.E.D.

Remark. In case 1) r acts on q irreducibly. For any decomposition of q into r submodules would lead to a decomposition of q^+ which we know is irreducible if g_C is simple. The preceding discussion justifies the following definitions.

DEFINITION. A simple involutive Lie algebra is called a simple irreducible pseudo-Hermitian Lie algebra if r has a one dimensional center whose adjoint action on q has pure imaginary eigenvalues. It is called a simple reducible pseudo-Hermitian Lie algebra in the case where g is complex.

DEFINITION. A pseudo-Hermitian Lie algebra (p.h.l.a.) is a finite direct sum of irreducible or reducible simple pseudo-Hermitian Lie algebras. If all the simple summands are either irreducible or reducible we can begin the definition with the appropriate adjective.

If $(g, r)=(g_1, r_1)+\cdots+(g_n, r_n)=(g_1+\cdots+g_n, r_1+\cdots+r_n)$ then the isotropy Lie algebra r has at least an n dimensional center. Let z_j be the element in the center of r_j with eigenvalues $\pm i$ on q_j. Then $z=z_1+\cdots+z_n$ is a central element of r with eigenvalues $\pm i$ on q, called the canonical central element of r.

We know that every pseudo-Hermitian symmetric space determines a p.h.l.a. We now investigate the converse question and see that there is exactly one pseudo-Hermitian symmetric space associated to every p.h.l.a. and that it is simply connected.

Given a p.h.l.a. (g, θ) let G be any connected group with Lie algebra g. Let T be the 1-parameter subgroup $\{\exp tz\}$. If G has finite center then T is a torus. Let $G_T=$ $=$centralizer of T.

(2.4) PROPOSITION. *G_T is connected.*

Proof. We will first prove this under the assumption that G has a finite center. Let $g=p+k$ be a Cartan decomposition of g which commutes with the canonical decomposition. Let K be the connected subgroup of G corresponding to k. Then $T\subset K$ and K is compact because G has finite center. Let $P=\exp p$. It is well known that G has a unique "polar decomposition" $G=P\cdot K$ and that exp is 1-1 on p. Since $z\in k$ it follows that $G_T=P_TK_T$ where $K_T=$centralizer of T in K, etc. It is also well known that in a compact, connected group the centralizer of a torus is connected so all we need to worry about is P_T. Let $a\in T$ and $\exp X\in P_T$. Then $\exp X=a\ \exp Xa^{-1}=\exp$ Ad aX and so $(\mathrm{Ad}\,a)\ X=X$ since exp is 1-1 on p. Thus for all real t we have $a(\exp tX)\times$ $\times a^{-1}=\exp t\ \mathrm{Ad}\,aX=\exp tX$ so the whole 1-parameter subgroup $\exp tX$ is in P_T hence P_T is path connected.

Now let G have an infinite center Z. Let $G^*=G/Z$ and $T^*=\{\exp tz\}$ in G^*. Then $Z\subset G_T$ and $(G_T)^*=G_T/Z$ is the centralizer of T^* in G^*. Since G^* has no center $(G_T)^*$ is connected and so $G/G_T\cong G^*/G_T^*$ is simply connected. It follows that G_T must be connected. Q.E.D.

(2.5) PROPOSITION. *G/G_T is a pseudo-Hermitian symmetric space.*

Proof. Let Θ be conjugation in G by the element $\exp\pi z$. It is easily seen that Θ is an involutive automorphism of G whose differential at the origin is θ. G_T is the identity component of the Θ-fixed point set and thus is a closed subgroup such that G/G_T is an affine globally symmetric space where the symmetry at the origin s_0 is induced by Θ. Let $J_0=\mathrm{ad}\,z\mid q$. Since J_0 centralizes G_T, this defines a G-invariant almost complex structure J on G/G_T by homogeneity. The Killing form is non-degenerate on q and admits J_0 and Θ as isometries. Since J is G-invariant this may be G-translated to give the real part of a G-invariant indefinite Hermitian metric on G/G_T. To prove that we have a pseudo-Hermitian symmetric space it will suffice by homogeneity to show that at every point p, J commutes with ds_0. Let g be an element of G such that $p=gG_T$. We have $L_{\Theta(g)}\cdot\Theta=\Theta\cdot L_g$ where L denotes left multiplication in G. Taking differentials and passing to G/R this becomes $dL_{\Theta(g)}\cdot ds_0=ds_0\ dL_g$. Now using this formula, the invariance of J and the fact that J_0 commutes with s_0 we easily compute that $J_{s0(p)}ds_0 = ds_0 J_p$. Q.E.D.

(2.6) PROPOSITION. *Let (g, r) be a pseudo-Hermitian Lie algebra. If G is a group with Lie algebra g and R is a closed subgroup with Lie algebra r, then G/R has a G-invariant almost complex structure if and only if $R = G_T$.*

Proof. Assume G/R has a G-invariant almost complex structure J. Then J restricted to the tangent space to the origin of G/R gives an endomorphism J_0 of q such that

a) $J_0^2 = -1$

b) J_0 commutes with Ad R restricted to q.

Let $q_C = \sum q_j$ be a decomposition of q_C into irreducible AdR modules. Let a_j be the algebra of endomorphisms of q_j generated over the complex numbers by the identity component of AdR and consider the commuting algebras $\mathrm{Hom}_{a_j}(q_j, q_j)$.

By Schur's lemma these are all isomorphic to the complex numbers. Now both ad$z$ and J_0 are in $\mathrm{Hom}_{a_j}(q_j, q_j)$ so on q_j we must have ad$z = c_j J_0$ where c_j is some complex number (which indeed must be $+1$ since $J_0^2 = -1$). Since J_0 commutes with AdR this means ad$z$ must and thus so must $\exp t\, \mathrm{ad} z = \mathrm{Ad} \exp tz$. If $a \in R$ then $\mathrm{Ad}((\exp tz)\, a (\exp -tz)\, a^{-1})$ is the identity on q and since the centralizer of q in AdR is discrete the curve $(\exp tz)\, a(\exp -tz)\, a^{-1}$ must be a single point, which indeed must be the identity. Hence $a \in G_T$. Q.E.D.

COROLLARY (to proof). *On a simple pseudo-Hermitian symmetric space the almost complex structure is unique up to sign.*

Remark. The result of the last three propositions is that every pseudo-Hermitian symmetric space is simply connected and that there is a 1-1 correspondence between pseudo-Hermitian Lie algebras and pseudo-Hermitian symmetric spaces. From now on we shall always denote G_T by R.

§3. Let (g, r) be a pseudo-Hermitian Lie algebra and $g = p + k$ a Cartan decomposition whose involution τ commutes with the canonical involution θ. Then we have

$$r = (r \cap p) \oplus (r \cap k), \quad q = (q \cap p) \oplus (q \cap k),$$
$$k = (q \cap k) \oplus (r \cap k), \quad p = (q \cap p) \oplus (r \cap p).$$

Let u = the compact algebra which is dual to the Cartan decomposition of g. That is, $u = ip \oplus k = i(q \cap p) \oplus i(r \cap p) \oplus (q \cap k) \oplus (r \cap k)$. The algebra u contains the algebra $r^* = i(r \cap p) \oplus (r \cap k)$ as a compact subalgebra. Let u^* be the non-compact algebra which is dual to (u, r^*). $u^* = (q \cap p) \oplus i(q \cap k) \oplus i(r \cap p) \oplus (r \cap k)$. Since the canonical central element of r, z, is contained in $r \cap k$ it is clear that z is also the canonical central element of r^* in the algebras u and u^*. Hence the involutive Lie algebras (u, r^*) and (u^*, r^*) define Riemannian Hermitian symmetric spaces of the compact and non-compact type, respectively. By the fact that Hermitian symmetric spaces are simply connected there is indeed a unique Hermitian symmetric space associated with each of these algebras. Denote these by A and A^* respectively.

DEFINITION. Given (g, r), the space A defined above is called the associated Riemannian space of compact type. The space A^* is called the associated Riemannian space of non-compact type.

It is clear that the definitions of A and A^* depend only on the pair of complex Lie algebras (g_C, r_C). That is, if (g_1, r_1) and (g_2, r_2) are two pseudo-Hermitian Lie algebras such that $(g_1)_C=(g_2)_C$ and $(r_1)_C=(r_2)_C$, then the spaces A and A^* will be the same for (g_1, r_1) and (g_2, r_2). Indeed, we may get them from (g_C, r_C) as follows: Let r^* be a compact real form of r_C and extend this to a compact real form u of g_C. Then since r^* is a symmetric subalgebra we may form the dual of u with respect to r^*. Call it u^*. Then the spaces associated to (u, r^*) and (u^*, r^*) will give A and A^*.

Let (g, r) be a pseudo-Hermitian Lie algebra and (g_C, r_C) its complexification. Let G_C be the connected, simply connected real Lie group with Lie algebra g_C. Let the groups G, U, U^*, R, R^*, R_C and Q^- be the connected subgroups of G_C corresponding to the subalgebras g, u, u^*, r, r^*, r_C and q^- respectively. It should be noted here that U and R^* are compact and that exp is a diffeomorphism of q^- with Q^-, as follows from the Iwasawa decomposition for example. G/R is the pseudo-Hermitian symmetric space associated with (g, r) and the spaces A and A^* are given by U/R^* and U^*/R^* respectively.

The Borel embedding of A^* into A tells us: 1) $B=R_CQ^-$ is a closed parabolic subgroup such that U/R^* is holomorphically diffeomorphic to the complex flag manifold G_C/B. 2) U^*/R^* is holomorphically embedded in G_C/B as the open U^* orbit of the identity coset. In particular $U^*\cap B=R^*$ [6] Theorem 7.13.

(3.1) THEOREM (Generalized Borel embedding). *Let G/R be an irreducible pseudo-Hermitian symmetric space. Then G/R is holomorphically diffeomorphic to the G-orbit of the identity coset in G_C/B. This orbit is open.*

Proof. Since $R\subset B$ the inclusion of G into G_C defines a map ψ of G/R onto the G orbit of the identity in G_C/B. To show ψ monomorphic we need $G\cap B=R$. This is the content of the next three lemmas.

LEMMA. $G\cap Q^-=\{1\}$.

Proof. First note that g and q^- are disjoint, for if $x\in g\cap q^-$ then $-ix=[zx]$ is in g which contradicts the fact that g is a real form of g_C. Hence the group $G\cap Q^-$ must be discrete. Assume $a\in G\cap Q^-$ with $a\neq 1$. Then there is an $A\in q^-$ such that

$a = \exp A$. Conjugate a by the elements of T: $(\exp tz)\, a(\exp -tz) = \exp(\mathrm{Ad}\, \exp tz)\, A = \exp(\exp t\, \mathrm{ad}\, z)\, A = \exp e^{it} A$. Because z is in g the left-hand side of this equation must be in G for all real t. But $\exp e^{it} A$ is in Q^- for all real t and since exp is 1-1 on q^- this proves that $G \cap Q^-$ contains the curve $\exp e^{it} A$ which contradicts its being discrete. Q.E.D.

LEMMA. *$G \cap B \subset R_C \cap G$.*

Proof. Let $a \in G \cap B$. Then there is an $A \in q^-$ and $x \in R_C$ such that $a = (\exp A)\, x$. Again conjugate by the elements of T, remembering that this group centralizes R_C: $(\exp tz)\, a(\exp -tz) = (\exp e^{it} A)\, x$ for all real t. Therefore $(\exp -A)\, a = x = (\exp -e^{it} A)\, (\exp tz)\, a(\exp -tz)$ and so $(\exp tz)\, a(\exp -tz)\, a^{-1} = (\exp e^{it} A)(\exp -A) \in Q^-$ for all real t. But since a is in G the left-hand side of this equation must be in G, and so the last lemma implies that each side of this equation is the identity. Thus $\exp A = \exp e^{it} A$ for all real t and so A must be zero since exp is 1-1 on q^-. Hence $a = x \in R_C$. Q.E.D.

LEMMA. *$R_C \cap G = R$.*

Proof. $R \subset R_C \cap G$ and both of these groups have the same Lie algebra, r. They both centralize T and so are connected, hence equal.

Now a dimension count shows that $\psi(G/R)$ is open. To show ψ holomorphic it suffices to show that it is almost complex, and by homogeneity it is enough to show this at the origin. The tangent spaces to the origins of G/R and G_C/B may be respectively identified with q and g_C/b where $b = r_C + q^-$. Let x be in q and write $x = x^+ + x^-$ with $x^+ \in q^+$ and $x^- \in q^-$. Then $d\psi: q \to g_C/b$ is given by $d\psi(x) = x \bmod b = x^+ \bmod b$ and so $d\psi(J_0 x) = d\psi(ix^+ - ix^-) = ix^+ \bmod b = i\, d\psi(x)$. Q.E.D.

The remainder of this section will be devoted to proving the following theorem.

(3.2) THEOREM. *Let ψ_1 denote the generalized Borel embedding of G/R into its associated Riemannian space of compact type, U/R^*. Let ψ_2 denote the Borel embedding of the associated Riemannian space of non-compact type U^*/R into U/R^*. Then $\psi_2(U^*/R^*) \subset \psi_1(G/R)$.*

Before giving the proof we will recall some work of G. D. Mostow that will be used heavily.

DEFINITION. Let g be any Lie algebra and e a subspace of g such that $[x[x, y]] \in e$ for all $x, y \in e$. Then e is called a Lie triple system.

(3.3) THEOREM (Mostow). *Let $g = p + k$ be a Cartan decomposition of a semi-simple Lie algebra. Let e be a Lie triple system contained in p and $f = \{x \in p \mid (x, e) = 0\}$ be the orthogonal complement to e in p. Let K be the connected Lie subgroup correspond-*

ing to k. *Then* $G = K \cdot \exp(f) \cdot \exp(e)$ *is a topological decomposition of* G. *The expression of* g *as the product* $k \cdot a \cdot b$ *with* $k \in K$, $a \in \exp(f)$ *and* $b \in \exp(e)$ *is unique.*

Proof. See [8]. A proof is also given in [6] p. 218. For the relation between Lie triple systems and totally geodesic submanifolds see [6] p. 189.

(3.4) COROLLARY. *In the notation already established, we have the following unique topological decompositions*:

a) $G = K \cdot \exp(r \cap p) \cdot \exp(q \cap p)$ b) $G = K \cdot \exp(q \cap p) \cdot \exp(r \cap p)$

c) $U^* = R^* \cdot \exp(i(k \cap q)) \cdot \exp(p \cap q)$ d) $U^* = R^* \cdot \exp(p \cap q) \cdot \exp(i(k \cap q))$.

Now we can give the proof of (3.2)

Proof. Let k_1 be the semi-simple part of k. Then $(k_1, r \cap k_1)$ is a (Riemannian) Hermitian pair, since $r \cap k_1$ contains the canonical central element z of r and z acts non-singularly on $q_1 = q \cap k_1$. Let q_1^- be the $-i$ eigenspace of ad z in $(q_1)_C$. Let B_1 be the connected subgroup corresponding to $(r \cap k_1)_C \oplus q_1^-$, K_1 the subgroup corresponding to k_1, L the subgroup corresponding to $r \cap k_1$, $B = Q^- R_C$, and K_1^* the subgroup corresponding to $k^* = i(k_1 \cap q) + (k_1 \cap r)$.

If k has a center, it must be contained in $r \cap k$. For $r = \text{centralizer}_g(z)$, so if z_1 is central in k then $[z_1, z] = 0$ so z_1 is in r. Hence $q \cap k = q \cap k_1$. Now we claim that

(3.5) $\exp i(q \cap k) \subset K \cdot B$.

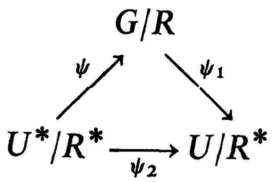
The Borel embedding theorem applied to K_1^*/L tells us that as subspaces of $(K_1)_C/B_1$, K_1^*/L is contained in K_1/L, i.e., $K_1^* \cdot L \subset K_1 \cdot L \cdot B_1$. Hence $K_1^* \subset K_1 \cdot L \cdot B_1 \cdot L$. Now by the Cartan decomposition for a semi-simple group of non-compact type $K_1^* = \exp i(q \cap k_1) \cdot L$. So we obtain

$$\exp i(q \cap k_1) \subset K_1 \cdot L \cdot B_1 \cdot L \subset K \cdot B$$

since $L \subset R_C \cdot G_B$ and $B_1 \subset B$. Now $\exp i(q \cap k) = \exp i(q \cap k_1) \subset K \cdot B$.

Finally, $U^* = \exp(q \cap p) \exp i(q \cap k) \cdot R^*$ by Corollary (3.4c). So $U^* \cdot B = \exp(q \cap p) \exp i(q \cap k) \cdot B$ since $R^* B = B$ and by (3.5) this is contained in $\exp(q \cap p) \cdot K \cdot B$. By (3.4) $G = \exp(q \cap p) \exp(r \cap p) \cdot K$. So $G \cdot B = \exp(q \cap p) \exp(r \cap p) \cdot K \cdot B$. This clearly shows that $U^* \cdot B \subset G \cdot B$. But $\psi_2(U^*/R^*) = U^* \cdot B/B$ and $\psi_1(G/R) = G \cdot B/B$ so $\psi_2(U^*/R^*) \subset \psi_1(G/R)$. Q.E.D.

We may describe this theorem by the following diagram



where $\psi = \psi_1^{-1} \cdot \psi_2$, which shows how the Borel embedding of U^*/R^* into U/R^* "factors" through G/R.

§4. We will begin by stating and proving a theorem of Berger on the fibering of an affine symmetric space, paying special attention to the specific features arising in the case of a pseudo-Hermitian symmetric space [2], [7].

Recall that if H is any closed subgroup of G, then $G \to G/H$ is a principal G-bundle with structure group H. If F is any manifold on which H acts smoothly, then we may form the associated bundle with fiber F. This is denoted $G \otimes_H F \to G/H$. It is a fiber bundle with base G/H, fiber F and structural group H. $G \otimes_H F$ is realized as the quotient of the Cartesian product $G \times F$ under the equivalence relation $(g, f) \sim (g \cdot h, h^* f)$ where $h^* f$ denotes the action of h on f and $g \cdot h$ is just group multiplication.

(4.1) THEOREM (Berger fibering). *The irreducible pseudo-Hermitian symmetric space G/R is a C^∞ fiber bundle. The base is a Hermitian symmetric space of compact type, the fiber a Hermitian symmetric space of non-compact type and at each point the symmetry commutes with the projection on the base.*

Proof. In view of (2.4) we may work entirely with subgroups of the complex group G_C. Consider the subalgebra $g_1 = p \cap q + k \cap r$. This is an involutive Lie algebra which in general need not be semi-simple. However if g_1 has a center then as in the proof of (3.2) we see that it must be contained in $k \cap r$. So with respect to the symmetric space structure we lose nothing if we assume g_1 semi-simple. Let G_1 and L be the connected Lie subgroups of G_C corresponding to g_1 and $k \cap r$ respectively. Then $G_1/L = F$ is an Hermitian symmetric space of non-compact type. Note that since the polar decomposition $G_1 = \exp(p \cap q) \cdot L$ is unique, F is diffeomorphic to $\exp(p \cap q)$.

Let K be the connected Lie subgroup of G_C corresponding to k. Then K must be compact. By the same reasoning as above we may assume that k is semi-simple and K/L is an Hermitian symmetric space of compact type.

Consider now the principal bundle $K \to K/L$ and the associated bundle with fiber F. This may be written $K \otimes_L \exp(p \cap q)$ where the action of L on $\exp(p \cap q)$ is given by conjugation. This action makes sense because $r \cap k$ normalizes $p \cap q$.

Now we show that this associated bundle is diffeomorphic to G/R. Consider the mapping

$$\begin{array}{ccc} K \times \exp(p \cap q) & \to & G/R \\ a \quad\quad b & & abR \end{array}$$

Since $(ax, x^{-1}bx) \to axx^{-1}bxR = abxR = abR$ for all $x \in L$, this map is constant on equivalence classes and hence yields a map $\chi : K \otimes_L \exp(p \cap q) \to G/R$ which is an epimorphism by (3.4b). We now show that χ is a monomorphism. Let (a_1, b_1) and (a_2, b_2) be two representatives of equivalence classes in $K \otimes_L \exp(p \cap q)$ and assume $\chi(a_1, b_1) = \chi(a_2, b_2)$. Then $a_1 \cdot b_1 = a_2 \cdot b_2 \cdot c$ for some $c \in R$. Since R/L is a Riemannian symmetric space, $R = L \cdot \exp(p \cap r)$, $c = x^{-1} \cdot d$ with $x^{-1} \in L$ and $d \in \exp(p \cap r)$. Hence $a_1 \cdot b_1 = a_2 \cdot b_2 x^{-1} \cdot d = (a_2 x^{-1})(x b_2 x^{-1}) \cdot d$. Since $L \subset K$ and $\exp(p \cap q)$ is stable under conjugation by L, the uniqueness of the decomposition (3.4) shows that $a_1 = a_2 x^{-1}$ and $b_1 = x b_2 x^{-1}$ and d=identity. Hence (a_1, b_1) and (a_2, b_2) represent the same element of $K \otimes_L \exp(p \cap q)$ and so χ is indeed a monomorphism.

Finally, let $\pi: G/R \to K/L$ be the projection of the bundle G/R onto the base space. Let's prove that π commutes with the symmetry Θ. First let us describe π. If $x \in G$ has the unique decomposition $x = k \cdot a \cdot b$, $a \in \exp(p \cap q)$, $b \in \exp(p \cap r)$ given by (3.4) then $\pi(x, R) = kL$. Now Σ is given by conjugation with the element $s = \exp \pi z$ and since $\exp(p \cap q)$ and $\exp(p \cap r)$ are stable under this conjugation, $\Theta(x) = sxs = (sks)(sas)(sbs)$ is the unique decomposition of $\Sigma(x)$ given by

Hence $\pi\Sigma(x) = sksL = skL = \Sigma\pi(x)$. Q.E.D.

In the general affine symmetric case the fiber need not be a symmetric space. All that is known is that topologically it is Euclidean. Also, in general there is a further complication because K need not be compact, for we have no reason to limit ourselves to subgroups of the complex group G_C. In this case the non-compact part of K is factored out and put in with the fiber.

(4.2) THEOREM. *Each pseudo-Hermitian symmetric space G/R induces the structure of a fiber bundle in the associated Riemannian symmetric space of non-compact type. Both the base and fiber are Hermitian symmetric spaces of non-compact type. The fiber is the same as in* (4.1).

Proof. We will show U^*/R^* is diffeomorphic to the bundle $K^* \otimes_L F \to K^*/L$ the notation being the same as in (§3). As before, define $\chi^*: K^* \otimes_L F \to U^* \to U^*/R^*$ which is onto because of (3.4). The fact that χ^* is a monomorphism is proved from the uniqueness of the decomposition (3.4) exactly as was done for χ in the last Theorem. Q.E.D.

Consider the maps $K \to G \to G/R$ and $K^* \to G^* \to G^*/R^*$. Since $L \subset R$ and R^* they induce maps $\alpha: K/L \to G/R$ and $\beta: K^*/L \to U^*/R^*$.

(4.3) PROPOSITION. *α and β are monomorphisms.*

Proof. Since L, $K^* \cap R^*$ and $K \cap R$ all have the same Lie algebra $r \cap k$ and L is connected, it will suffice to show that the latter two groups are connected, but this follows by (2.4) applied to the Hermitian pairs (K^*, L) and (K, L). Q.E.D.

This proposition is no surprise, for α and β are just the injections of the base spaces K/L and K^*/L into the fiber bundles as the zero sections.

We have a map $i: K^* \otimes_L F \to K \otimes_L F$ defined by $i = \chi^{-1} \psi \chi_1: K^* \otimes_L F \to U^*/R^* \to G/R \to K \otimes_L F$. When restricted to the fiber over the origin i is just the identity and when restricted to the base i is just the Borel embedding of K^*/L into its dual K/L. Thus we have interpreted the inclusion of Theorem (3.2) in terms of the fiber structure.

We will now give a simple but interesting generalization of Harish-Chandra's realization of an Hermitian symmetric space of non-compact type as a bounded symmetric domain.

(4.4) THEOREM. *G/R can be holomorphically embedded in a holomorphic vector bundle over the compact Hermitian symmetric space K/L. In terms of the fibering given by* (4.1) *each fiber is mapped holomorphically into a bounded symmetric domain in the corresponding fiber of the vector bundle, and on the base K/L the mapping is the identity.*

Proof. Consider the fibering $K\otimes_L F$ of G/R given by (4.1). Harish-Chandra's results applied to $F=G_1/L$ may be described as follows. Every element of G_1 can be uniquely written as an element of $(G_1)_C$ as x^-yx^+ where $x^-\in\exp(p\cap q)^-$, $y\in L_C$ and $x^+\in\exp(p\cap q)^+$. Here, as usual, $(p\cap q)^+$ denotes the $+i$ eigenspace of ad$z$ in $(p\cap q)_C$ etc. Then $\lambda: G_1/L\to(p\cap q)^-$ defined by $\lambda(x^-yx^+L)=\log(x^-)$ is the realization of G_1/L as a bounded symmetric domain in $(p\cap q)^-$. Also, λ is holomorphic [6]. L acts on $(p\cap q)^-$ by conjugation and on G_1/L by left multiplication. It is clear from the description of λ that λ commutes with this action so L, and indeed even L_C, leaves the image of λ in $(p\cap q)^-$ stable. Now form the complex vector bundle $K\otimes_L(p\cap q)^-$ associated to the principal bundle $K\to K/L$. Since λ commutes with the action L, the map $\Lambda: K\times F\to K\times(p\cap q)^-$ defined by $\Lambda(a,b)=(a,\lambda(b))$ induces a map $\Lambda: K\otimes_L F\to$ $\to K\otimes_L(p\cap q)^-$ which is easily seen to satisfy all the requirements except the assertion of holomorphicity. Of course we may regard Λ as defined on G/R by means of χ^{-1}.

We induce a holomorphic structure on our complex vector bundle by means of the C^∞ bundle isomorphism

$$\begin{array}{ccc} K\otimes_L(p\cap q)^- & \to & K_C\otimes_{B_1}(p\cap q)^- \\ \downarrow & & \downarrow \\ K/L & \longrightarrow & K_C/B_1 \end{array}$$

where B_1 is the parabolic subgroup $L_C\exp(p\cap q)^-$. Since $\exp(p\cap q)^-$ acts trivially on $(p\cap q)^-$ it is clear that B_1 leaves $\lambda(F)$ stable.

Since λ is holomorphic on $\exp(p\cap q)$ and the almost complex structure on $p\cap q$ is the restriction of that on q it follows that J and $d\Lambda$ commute at all points yR with $y\in\exp(p\cap q)$. Now all elements of K commute with Λ so $dL_x\cdot d\Lambda_y=d\Lambda_{xy}\cdot dL_x$ for $x\in K$. From $G=K\exp(p\cap q)\cdot R$ it follows that J and $d\Lambda$ commute at all points of G/R and so Λ is holomorphic. Q.E.D.

Let us now consider the case of a reducible pseudo-Hermitian symmetric space G_C/R_C. Consider the projection $G_C/R_C\to G_C/R_CQ^-$. This is a holomorphic fibering whose base is the associated Riemannian space of compact type and whose fiber is Q^- which is analytically diffeomorphic to the complex Euclidean space q^- under log. We will examine this fibering in greater detail in the following theorem.

(4.5) THEOREM. *The fibering $G_{\mathbf{C}}/R_{\mathbf{C}} \to G_{\mathbf{C}}/R_{\mathbf{C}}Q^-$ is analytically equivalent to the cotangent bundle of the associated compact space $G_{\mathbf{C}}/R_{\mathbf{C}}Q^-$.*

Proof. For convenience of notation let us denote the semi-direct product $R_{\mathbf{C}}Q^-$ by B, and its Lie algebra by b. Now the tangent bundle of $G_{\mathbf{C}}/B$ is the bundle $G_{\mathbf{C}} \otimes_B (g_{\mathbf{C}}/b)$ where B acts on $g_{\mathbf{C}}/b$ by Ad. This is the same as $G_{\mathbf{C}} \otimes_B q^+$ where the action of Q^- on q^+ is trivial. Then by the duality of q^+ and q^- under the Killing form we may identify $G_{\mathbf{C}} \otimes_B q^-$ with the cotangent bundle, and since exp is an analytic diffeomorphism of q^- with Q^- we may identify the cotangent bundle with $G_{\mathbf{C}} \otimes_B Q^-$ where now B acts on Q^- by conjugation.

Now $G_{\mathbf{C}} = U \cdot B = U \cdot R_{\mathbf{C}} Q^-$. This decomposition is not unique, but if $ub = u_1 b_1$ then $u_1^{-1} u = b_1 b^{-1} \in U \cap B = R^*$ so for some $r \in R^*$ we have $u = u_1 r$ and $b = r^{-1} b_1 r$. We can write b uniquely as $x \cdot y$ with $x \in R_{\mathbf{C}}$ and $y \in Q^-$, so b_1 can be uniquely written in this form as $b_1 = (r^{-1} x r)(r^{-1} y r)$. From this discussion it is clear that we get a well defined map of $G_{\mathbf{C}}/R_{\mathbf{C}}$ into $G_{\mathbf{C}} \otimes_B Q^-$ by writing a representative g of a coset as $u \cdot x \cdot y$ with u, x, y as above and sending this to the equivalence class (ux, y). This map commutes with the projection onto the base space. It is also clear that this map is 1-1. Indeed if $(ux, y) \sim (u_1 x_1, y_1)$ then for some $b \in B$ $ux = u_1 x_1 b$ and so $u_1^{-1} u = x_1 b x^{-1} \in B$ so $x_1 b x^{-1} \in R^*$ so $b \in R_{\mathbf{C}}$ and $uxy = u_1 x_1 y_1 \bmod R_{\mathbf{C}}$. Finally, to show this map is onto consider a typical representative (g, w) with $g \in G_{\mathbf{C}}$ and $w \in Q^-$. Write $g = u \cdot x \cdot y$ as before. Then $(g, w) = (ux, y, w) \sim (uxw)$ so the coset uxw maps onto $(g, w) \bmod B$.

Q.E.D.

Remark. As C^∞ bundles the holomorphic cotangent bundle is $U \otimes_{R^*} q^- \to U/R^*$ which may also be considered the anti-holomorphic tangent bundle. The isomorphism between $U \otimes_{R^*} q^-$ and $G_{\mathbf{C}} \otimes_B q^-$ is realized by mapping a class $\bmod R^*$ into the corresponding class $\bmod B$.

The natural embeddings of U^*/R^* and G/R into $G_{\mathbf{C}}/R_{\mathbf{C}}$ still give monomorphisms when composed with the projection π to the base space; namely, the Borel embeddings. Hence they define sections S_1 and S_2 in the bundle $G_{\mathbf{C}} \otimes_B q^-$ over the open orbits of U^* and G in $G_{\mathbf{C}}/B$. Recall that these sections may be regarded as q^- valued functions defined on U^*B and GB respectively, which satisfy the relation

(4.6) $\quad S_i(xb) = \mathrm{Ad}\, b^{-1} S_i(x) \qquad i = 1, 2$

for all $b \in B$. We now relate these sections S_i to the Harish-Chandra embeddings. In order to do this we will have to recall Harish-Chandra's map in more detail than was done before. The reference for the next few paragraphs is [6], Chap. VIII.

Let h be a Cartan subalgebra of u^* contained in r^*. Then there exists a set of strongly orthogonal roots Δ_0 and corresponding root vectors $E_{\pm\alpha}$ with respect to h, such that $\{E_\alpha + E_{-\alpha} \mid \alpha \in \Delta_0\}$ is a maximal abelian subspace a of $q^* = p \cap q + i(k \cap q)$. By picking $\{E_{\pm\alpha}\}$ as part of a Weyl basis for $g_{\mathbf{C}}$ we may even assume that $\theta E_\alpha = -E_{-\alpha}$ and $\theta E_{-\alpha} = -E_\alpha$. Here θ is the Cartan involution on U^*, which is the canonical involution θ of G extended to $G_{\mathbf{C}}$ by conjugate linearity and then restricted to U^*. Let $H_\alpha=[E_\alpha E_{-\alpha}]$. Then $\theta H_\alpha=-H_\alpha$ so $H_\alpha\in ih\subset r_{\mathbf{C}}$.

(4.7) LEMMA. *The map* $Q^+\times R_{\mathbf{C}}\times Q^-\to Q^+R_{\mathbf{C}}Q^-$ *is an analytic diffeomorphism onto an open set of* $G_{\mathbf{C}}$.

Proof. [6] p. 317.

(4.8) LEMMA. $\exp\sum_i t_i(E_{\alpha_i}+E_{-\alpha_i})=\exp(\sum\tanh t_iE_{\alpha_i})\exp(\sum\log\cosh t_iH_{\alpha_i})$ $\exp(\sum\tanh t_iE_{-\alpha_i})$ *where the sums are over the set* Δ_0 *of strongly orthogonal roots.*

Proof. [6] p. 316.

Let $A=\exp a$. Any element of U^* may be written as $x\cdot a\cdot x'$ with $x, x'\in R^*$ and $a\in A$. Then Harish-Chandra's map λ sends the coset $x\cdot a\cdot R^*=x\exp\sum t_i(E_{\alpha_i}+$ $+E_{-\alpha_i})R^*$ to $\mathrm{Ad}\,x(\sum\tanh t_iE_{-\alpha})$ in $q^-=(q^*)^-$. It is clear that we do no harm if we replace this by the map which sends $x\cdot a\cdot R^*$ to $\mathrm{Ad}\,x(\sum\tanh 2t_iE_{-\alpha})$. We will now mean this stretched out map when we refer to λ.

Note that h is also a Cartan subalgebra of U so we can apply the complex Iwasawa decomposition to write $G_{\mathbf{C}}$ uniquely as $U\cdot\exp ih\cdot N^-$ where $N^-=\exp n^-$ and $n^-=\sum\mathbf{C}E_{-\alpha}$ is the sum of all the negative root spaces with respect to $h+ih$ and some ordering of the roots. If the roots are ordered in such a way that α is positive if $\alpha(z)=+i$ then n^- decomposes into a semi-direct sum $n^-=q^-+n_r$ where $n_r=\sum\mathbf{C}E_{-\alpha}$ where α positive and $\alpha(z)=0$. Since exp is a diffeomorphism on n^- this implies that $N^-=N_R\cdot Q^-$ is a semi-direct product, where $N_R=\exp(n_r)$.

Letting $N=\exp(ih)\cdot N_R$ we obtain the following

(4.9) LEMMA. *There is a subgroup* N *of* $R_{\mathbf{C}}$ *such that* $G_{\mathbf{C}}=U\cdot N\cdot Q^-$ *is a unique decomposition.*

Suppose $g=u\cdot n\cdot q$ is a decomposition of an element of $\exp q^*$ according to (4.9). Then $g^2=\theta(g^{-1})\,g=\theta(q^{-1})\,\theta(n^{-1})\,nq$ is a unique decomposition of g^2 with $\theta(q^{-1})\in Q^+$, $\theta(n^{-1})\,n\in R_{\mathbf{C}}$ and $q\in Q^-$ by Lemma (4.7). On the other hand there is some $x\in R^*$ such that $x^{-1}gx$ is in A and so for some set of real numbers $\{t_i\}$ we have $x^{-1}gx=\exp\sum t_i(E_{\alpha_i}+E_{-\alpha_i})$ so $x^{-1}g^2x=\exp\sum 2t_i(E_{\alpha_i}+E_{-\alpha})$

Using lemmas (4.8) and (4.7) and comparing the two expressions for g^2 we get that

$$(4.10)\quad q=\exp\mathrm{Ad}\,x\sum\tanh 2t_i E_{-\alpha_i}=\exp\lambda(g)$$

Under the isomorphism of theorem (4.5) the coset $gR_{\mathbf{C}}=u\cdot n\cdot qR_{\mathbf{C}}$ goes into $(u\cdot n,$ $\log q)\sim(u\cdot nq, \log q)=(g, \lambda(g))$ in $G_{\mathbf{C}}\otimes_B q^-$ since Q^- acts trivially on q^-. Hence we have proved the following

(4.11) THEOREM. *On the set* $\exp q^*$ *the section* S_1 *coincides with the stretched Harish-Chandra embedding* λ.

Remark. This is the best possible result along this line. For the orbit of U^* in G_C/B pulls back to $U^*B = \exp q^*B$ in G_C and so by (4.6) S_1 is completely determined by its values on $\exp q^*$.

(4.12) THEOREM. *On the set* $K\exp(p\cap q)\subset G_C$, S_2 *is given by* $S_2(Kg)=\lambda(g)$ *where* λ *is the stretched out version of the Harish-Chandra map of* G_1/L *into* $(p\cap q)^-$. *Furthermore, on the image of* Λ *there is defined an analytic monomorphism d such that* $d\circ\Lambda = S_2\circ\psi$ *where* S_2 *is now regarded as a bona-fide section instead of the coordinate function.*

Remark. Since $GB=K\exp(p\cap q)\cdot B$ this theorem completely describes S_2 via (4.6).

Proof. By 3.4b we can write an arbitrary coset of G/R as kgR with $k\in K$ and $g\in\exp(p\cap q)$. Now by lemma (4.9) applied to $(G_1)_C$ we can write $g=u\cdot n\cdot q$ where $u\in(G_1)_C\cap U$, $n\in(G_1)_C\cap N$ and $q\in\exp(p\cap q)^-$. This is a unique decomposition and $(ku)\cdot n\cdot q$ is thus the corresponding unique decomposition of kg given by (4.9) applied to G_C.

Under the diffeomorphism of theorem (4.5) kg goes into the class $(k\cdot u\cdot n, q)\sim$ $\sim(k\cdot u\cdot n\cdot q, q)=(k\cdot g, \lambda(g))$ by (4.10) applied to $\exp(p\cap q)$ in $(G_1)_C$. Hence $S_2(kg)=$ $=\lambda(g)$. Now $\Lambda(kgR)=(k, \lambda(g))$. Define $d(k, \lambda(g))=(kg, \lambda(g))$. Then $d\circ\Lambda=S_2\circ\psi$ and since Λ, S_2, and ψ are analytic monomorphisms, d must be also. It also follows that d is well defined on equivalence classes, a fact which may also be easily verified directly.

§5. Let $k=\dim_C K/L$. In this section we use a technique of Griffiths and Schmid [5] to show that G/R is $k+1$ complete in the sense of Andreotti and Grauert. In particular this means that $H^n(G/R, \mathscr{g})=0$ for all $n>k$ where $\mathscr{g}$ is any coherent analytic sheaf on G/R. [1]. A C^∞ real valued function φ on a complex manifold M is called an exhaustion function if $\varphi^{-1}(-\infty, c]$ is compact for every real number c. We say that M is k-complete if it has an exhaustion function φ whose Levi form $-\partial\bar{\partial}\varphi$ has $\dim_C M-k+1$ positive eigenvalues at every point of M. Denote the image of $\psi(G/R)$ in G_C/B by D. The idea is to give two different Hermitian metrics to the canonical line bundle over D and to obtain the exhaustion function as the ratio of these metrics. The Levi form will then be given as the difference of the metric curvature forms and thus may be directly calculated from the differential geometry of D.

We shall have need of the following special basis for q^+. Let h be a Cartan subalgebra of r which is stable under τ. Since $z\in h$ and $\operatorname{ad}z$ is non-singular on q, h must be a Cartan subalgebra of g. Let $\{E_{\pm\alpha}\}$ be a Weyl base for g_C with respect to h_C and τ. Recall that this means that $g_C=h_C+\sum CE_{\pm\alpha}$ is a root space decomposition of g_C and that (5.1) $\tau E_\alpha=-E_{-\alpha}$ and $(E_\alpha, E_{-\beta})=\delta_{\alpha\beta}$. Since $h_C\subset r_C$ any E_α is either in q_C or r_C so we can write $q^+=\sum_\Delta CE_\alpha$ where $\Delta=\{\text{all roots } \alpha \text{ such that } \alpha(z)=+i\}$.

The tangent bundle of G/R may be described as the associated bundle $G\times_R q\to$

$\to G/R$. Recall that the Killing form restricted to q gives the real part of the natural pseudo-Hermitian structure h_G on G/R. The tangent bundle of $G_{\mathbf{C}}/B$ is the associated bundle $G_{\mathbf{C}} \times_B g_{\mathbf{C}}/b$. Since $[q^-, q^+] \subset r$ this is bundle isomorphic to $G_{\mathbf{C}} \times_B q^+ = T$ where now $B = R_{\mathbf{C}}Q^-$ acts by $\mathrm{Ad}\, R_{\mathbf{C}}$ with Q^- acting trivially. The Borel embedding of G/R into $G_{\mathbf{C}}/B$ induces the inclusion of tangent bundles $G \times_R q \to G_{\mathbf{C}} \times_B q_{\mathbf{C}}/b$, where $x \in q \to x$ mod b. Under the isomorphism of $G_{\mathbf{C}} \times_B g_{\mathbf{C}}/b$ with T this is given on fibers by $X \to \frac{1}{2}(X - iJX)$. Denote this map of $G \times_R q$ onto $T|_b$ by $\mathfrak{M}$. Note that $\mathfrak{M}$ is a C^∞ bundle diffeomorphism onto the holomorphic bundle $T|_D$ which defines the holomorphic structure on $G \times_R q$.

We will now define two different Hermitian (indefinite) structures on T. The first, H_U, will be U-invariant and negative definite on the whole bundle and the second, H_G, will be G-invariant, indefinite and possibly degenerate; but on the restriction of T to D it will at least be non-degenerate. On $g_{\mathbf{C}}$ we define two Hermitian forms in terms of the complex Killing form. Here σ is the involution determined by the real form g.

$$H_\tau(x, y) = (x, \tau(y)), \quad H_\sigma(x, y) = (x, \sigma(y)).$$

Identify sections of T with functions $S: G_{\mathbf{C}} \to q^+$ such that $S(gx) = \mathrm{Ad}\, x^{-1} S(g)$ for $x \in B$. Then for any two sections S_1 and S_2 define

$$H_U(S_1, S_2)_g = H_\tau(\mathrm{Ad}\, g S_1(g), \mathrm{Ad}\, g S_2(g)),$$
$$H_G(S_1, S_2)_g = H_\sigma(\mathrm{Ad}\, g S_1(g), \mathrm{Ad}\, g S_2(g)).$$

If $x \in B$ then $H_U(S_1, S_2)_{gx} = H_U(S_1, S_2)_g$ and similarly for H_G. Thus we have two well-defined pseudo-Hermitian structures on T which are U-invariant and G-invariant respectively. Since U acts transitively on $G_{\mathbf{C}}/B$ it may easily be seen, by using (5.1) for example, that H_U is everywhere negative definite. On $T|_D$ H_G is non-degenerate since σ interchanges q^+ and q^- and these spaces are dually paired under the Killing form.

At each point $p \in G_{\mathbf{C}}/B$ there is a linear transformation of the complex vector space q^+ such that $H_G(S_1, S_2)_p = H_U(A_p S_1, S_2)_p$. Since A_p is given as an Hermitian matrix $\det A_p$ is real. Let us give a different interpretation of $\det A_p$. On the holomorphic line bundle $L = G_{\mathbf{C}} \times_B \Lambda^n q^+$ we have two Hermitian metrics γ_U and γ_G induced in the standard manner by H_U and H_G respectively. At any point p these two metrics differ by a scalar multiple which is just $\det A_p$. That is, given any section 5, $\gamma_G(S, S)_p = \det A_p \gamma_U(S, S)_p$. This proves that $\det A_p$ is C^∞ on $G_{\mathbf{C}}/B$. We may adjust the signs of γ_U and γ_G to make them both positive on D. With this done, we have $\gamma_G = +\det A \gamma_U$.

We claim that $\varphi(p) = -\log \pm \det A_p$ is an exhaustion function for D, where the sign is chosen to make $\pm \det A_p$ positive on D. This choice can be made because H_G is non-degenerate on the connected set D and so $\det A_p$ is never zero on D. Since $\det A$ is C^∞ on all of $G_{\mathbf{C}}/B$ it will suffice to show that $e^{-\varphi} = \pm \det A$ is zero on the topological boundary of D. We follow the line of proof given in [5, §8].

(5.2) PROPOSITION. *If p is on the topological boundary of D then* $\det A_p = 0$.

Proof. It will suffice to prove that H_G is degenerate at p. Let $p = xB$ and denote $(\mathrm{Ad}\,x)\,b = b'$ etc. If p is on the boundary then the orbit $G\cdot p$ cannot have an interior since the boundary is stable inder G. Hence the isotropy algebra at p must have strictly higher dimension than the isotropy algebra at the origin, so $\dim r < \dim g \cap b'$. Passing to the complexification and noting that $(g \cap b')_C = b' \cap \sigma(b')$ we get $\dim r_C < \dim b' \cap \sigma(b')$. Now lemma 2.10 of [12] p. 1133 says $\dim b' \cap \sigma(b') = \dim (r_C)' + \dim (q^+)' \cap \sigma(q^+)'$. Since $\dim r_C = \dim (r_C)'$ we are assured that $(q^+)' \cap \sigma(q^+)'$ is non-empty. Let u' and v' be elements of $q^{+\prime}$ such that $\sigma(v') = u'$. There exist u, v in q^+ s.t. $u' = x\cdot u$ and $v' = x \cdot v$. Then $H_G(q^+, v)_{xB} = (q^{+\prime}, \sigma(v')) = (q^{+\prime}, u') = (q^+, u) = 0$ since q^+ is isotropic. Q.E.D.

Now the Levi form of φ is $-\partial\bar\partial\varphi = \partial\bar\partial \log \gamma_G - \partial\bar\partial \log \gamma_U = -\Lambda_G + \Lambda_U$ where Λ_G and Λ_U are the curvature forms of the canonical Hermitian metric connections of γ_G and γ_U on the holomorphic line bundle $L|_D$. Note that for a line bundle we may identify the curvature tensors with the curvature forms.

Let ∇ be the canonical G-invariant symmetric space connection in the tangent bundle of G/R. Let ∇' be the connection induced on $T|_D$ by $\mathfrak{M}$, viz. $(\nabla'_x S)_{\mathfrak{M}_p} = (\mathfrak{M}\nabla_{\mathfrak{M}^{-1}(x)} \mathfrak{M}^{-1} S)_p$ where $p \in G/R$. Extend everything to the complex tangent spaces by complex linearity. We write $(q^+)_C$ as $q^+ + jq^+$ with $j^2 = -1$, being careful not to confuse j with the almost complex structure i of the real space q^+.

(5.3) PROPOSITION. ∇' *is the canonical connection determined by* H_G.

Proof. We must show that $\nabla' H_G = 0$ and that relative to a locally holomorphic frame field ∇' is given by a matrix of complex 1-forms of type $(1, 0)$. From [9] Theorem 15.6 we know that ∇ is the metric connection of any G-invariant metric on G/R. In particular it is the metric connection of the real part of h_G. Since h_G is Kähler we know that $\nabla h_G = 0$. A simple computation shows that $H_G(\mathfrak{M}X, \mathfrak{M}Y) = \frac{1}{2} h_G(X, Y)$ and so it follows that $\nabla' H_G = 0$. Let $\{S_i\}$ be a locally holomorphic frame field and $\{W_{ij}\}$ the matrix of 1-forms such that $\nabla_X S_i = \sum W_{ij}(X)\, S_j$. Since ∇ is Kähler the W_{ij} are all of type $(1, 0)$. The corresponding 1-forms $\{W'_{ij}\}$ for ∇' relative to the holomorphic frame field $\mathfrak{M} S_i \mathfrak{M}^{-1} = S'_i$ are given by $W'_{ij}(X) = \mathfrak{M} W_{ij}(\mathfrak{M}^{-1} X)\, \mathfrak{M}^{-1}$. Note that $\mathfrak{M}J = i\mathfrak{M}$ implies that $\mathfrak{M}$ preserves the type of complex vector fields. Hence W'_{ij} van nishes on all anti-holomorphic vector fields since W_{ij} does. Hence W'_{ij} is also of type $(1, 0)$. Q.E.D.

COROLLARY. *The extension of ∇' to $L|_D$ by the derivation rule is the Hermitian metric connection of γ_G and its curvature tensor is the extension of the curvature of ∇' by the derivation rule.*

(5.4) PROPOSITION. *In terms of the Killing form on g_C the G-invariant curvature* *form on $L|_D$ is given at the origin by*

$$\Lambda_G(x, y) = \tfrac{1}{2}(z, [\mathfrak{M}^{-1}X, \mathfrak{M}^{-1}Y]).$$

Proof. Let W and W' denote the curvature tensors of ∇ and ∇' respectively. We have $W'(X, Y)=\mathfrak{M}W(\mathfrak{M}^{-1}X, \mathfrak{M}^{-1}Y)\mathfrak{M}^{-1}$. For simplicity let $\mathfrak{M}^{-1}X=X'$, etc. Then at the origin we have $W(X'Y')=-\mathrm{ad}[X', Y']|_q$ and $W'(X, Y)=-\mathfrak{M}\,\mathrm{ad}[X', Y]|_q\mathfrak{M}^{-1}$, for $X, Y\in(q^+)_C$. Since $[X'Y']\in r_C$, $\mathrm{ad}[X', Y']$ commutes with $\mathfrak{M}$ and we obtain $W'(X, Y)=-\mathrm{ad}[X', Y']|_{q^+}$. Hence the curvature induced on $L|_D$ by W' is then $\Lambda_G(X, Y)=-\mathrm{trace}\,\mathrm{ad}[X', Y']|_{q^+}$. Now $0=\mathrm{tr}\,\mathrm{ad}[X', Y']|_{q_C}=\mathrm{tr}\,\mathrm{ad}[X', Y']|_{q_C}+\mathrm{tr}\,\mathrm{ad}[X', Y']_{r_C}$. But r_C is reductive so the latter trace is zero and we can conclude that $\mathrm{tr}\,\mathrm{ad}[X'Y']|_{q^+}=-\mathrm{tr}\,\mathrm{ad}[X'Y']|_{q^-}$. Finally $(z, [X', Y'])=\mathrm{tr}\,\mathrm{ad}\,z\,\mathrm{ad}[X'Y']|_{q_C}=i\,\mathrm{tr}\,\mathrm{ad}[X'Y']|_{q^+}-i\,\mathrm{tr}\,\mathrm{ad}[X', Y']|_{q^-}=2i\,\mathrm{tr}\,\mathrm{ad}[X', Y']|_{q^+}$. Q.E.D.

Now let ∇ be the U-invariant canonical connection on U/R^*. In the same manner as before we induce a U-invariant connection ∇' on T. Again we have a C^∞ vector bundle isomorphism $M: U\times_{R^*}q^*\to G_C\times_B q^+$ which is an isometry of h_U on each fiber and $\nabla'_X S=M\nabla_{M^{-1}(X)}M^{-1}S$.

As before we see that ∇' is the Hermitian metric connection of H_U on T. Calculating the U-invariant curvature tensor at the origin gives $W'_U(X, Y)=MW_U(M^{-1}X, M^{-1}Y)M^{-1}=-\mathrm{ad}[M^{-1}X, M^{-1}Y]|_{q^+}$ for $X, Y\in(q^+)_C$.

On L this gives (5.5) $\Lambda_U(X, Y)=\tfrac{1}{2}(z, [M^{-1}X, M^{-1}Y])$.

(5.6) LEMMA. *If X is in q^+ then*

$$\mathfrak{M}(X) = M(X) = (X - ijX) \quad\text{and}\quad \mathfrak{M}(\sigma(X)) = M(\tau(X)) = (X + ijX).$$

Proof. We will give the proof for $\mathfrak{M}$, the other case being similar. Write X as $(X+\sigma(X)\tfrac{1}{2})-i(iX+\sigma(iX)\tfrac{1}{2})$ which is of the form $A+iB$ with A and B in q. Apply $\mathfrak{M}$, remembering that σ interchanges q^+ and q^-, to get $\mathfrak{M}(X)=\mathfrak{M}(A)+j\mathfrak{M}(B)==\tfrac{1}{2}(A-iJA)+j\tfrac{1}{2}(B-iJB)=(X-ijX)$. Similarly write $\sigma(X)$ as $\tfrac{1}{2}(\sigma(X)+X)++i\tfrac{1}{2}(\sigma(iX)+iX)$ and apply $\mathfrak{M}$ to get $\mathfrak{M}\sigma(X)=X+ijX$. Q.E.D.

(5.7) PROPOSITION. *Λ_U is positive definite on D.*

Proof. By invariance it will suffice to check this at the origin, and since Λ_U is of type (1, 1) it will suffice to check this on a basis of holomorphic complex tangent vectors. Let $\{E_\alpha\}$ be a Weyl base as in (5.1). Then $\{E_\alpha \mid \alpha(z)=+i\}$ is a basis for q^+ over C. Let $X_\alpha=E_\alpha-ijE_\alpha$. Then by (5.5) and (5.6) we obtain

$$\Lambda_U(X_\alpha \bar{X}_\beta) = \frac{i}{2}(z, [E_\alpha, \tau E_\beta]) = \frac{-i}{2}(z, [E_\alpha, E_{-\beta}]) = +\tfrac{1}{2}(E_\alpha, E_{-\beta}) = \tfrac{1}{2}\delta_{\alpha\beta}.$$

Q.E.D.

(5.9) PROPOSITION. *On* $(q \cap k)_C^+ \subset (q^+)_C$ $\Lambda_G = \Lambda_U$ *and on* $(q \cap p)_C^+$ $\Lambda_G = -\Lambda_U$. *The spaces* $(q \cap k)_C^+$ and $(q \cap p)_C^+$ *are orthogonal under both* Λ_U *and* Λ_G.

Proof. The first two statements follow from the fact that $\sigma = \tau$ on $(q \cap k)_C \subset g_C$ and $\sigma = -\tau$ on $(q \cap p)_C \subset g_C$. Orthogonality follows because $(q \cap k)_C$ and $(q \cap p)_C$ are stable under σ and τ and these two spaces bracket together into $(p \cap r)_C$ which is orthogonal to $(k \cap r)_C$ under the Killing form of g_C. Q.E.D.

Let $p = \dim_C (q \cap p)^+$ and $k = \dim_C (q \cap k)^+$. From the preceding calculations it is evident that the Levi form of φ has at least p positive eigenvalues at every point of D. Now $\dim_C G/R = \dim_C q^+ = p + k$ and $\dim_C K/L = \frac{1}{2} \dim_R K/L = \frac{1}{2} \dim_R (q \cap k) = = \frac{1}{2} \dim_C (q \cap k)_C = \dim_C (q \cap k)^+ = k$ so D is $k+1$ complete as claimed.

COROLLARY. *If* $\mathcal{g}$ *is any coherent analytic sheaf on* G/R *then* $H^n(G/R, \mathcal{g}) = 0$ *for* $n > k$.

Proof. This follows from [1, pg. 250].

Appendix

The following is a list of pseudo-Hermitian symmetric spaces taken from Berger's classification of all the affine symmetric spaces [2]. These are the entires marked "$\frac{1}{2}$ Kähler", which indicates the presence of the canonical central element of r.

The letter T denotes the on-dimensional torus which is the non-discrete center of the isotropy subgroup and $\mathbf{C}^*$ denotes the complex torus $\mathbf{C} - \{0\}$. $S(A \times B)$ means the connected subgroup corresponding to the matrices

$$\left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right]$$

with the restriction trace A + trace $B = 0$. The other notation is standard, see [6] Chap. IX, for example. There the matrix groups are described explicitly.

Classical Simple Irreducible Pseudo-Hermitian Symmetric Spaces

1. $SL(2n, R)/SL(n, \mathbf{C}) \times T$
2. $SU^*(2n)/SL(n, \mathbf{C}) \times T$
3. $SU(n-i, i)/S(U(h, k) \times U(n-i-h, i-k))$
4. $SO^*(2n)/SO^*(2n-2) \times T$
5. $SO^*(2n)/U(n-k, K)$
6. $SO(n-k, k)/SO(n-k, k-2) \times T$
7. $SO(2(n-k), 2k)/U(n-k, k)$
8. $Sp(n-i, i)/U(n-i, i)$

Exceptional Simple Irreducible Pseudo-Hermitian Symmetric Spaces

1. $E_6^2/SO^*(10) \times T$
2. $E_6^2/SO(6,4) \times T$
3. $E_6^3/SO(8,2) \times T$
4. $E_6^3/SO^*(10) \times T$
5. $E_7^1/E_6^2 \times T$
6. $E_7^2/E_6^3 \times T$
7. $E_7^2/E_6^2 \times T$
8. $E_7^3/E_6^3 \times T$

Simple Reducible Pseudo-Hermitian Symmetric Spaces

1. $SL(n,\mathbf{C})/S(L(n-k,\mathbf{C}) \times L(k,\mathbf{C}))$
2. $SO(n,\mathbf{C})/SO(n-2,\mathbf{C}) \times SO(2,\mathbf{C})$
3. $SO(2n,\mathbf{C})/SL(n,\mathbf{C}) \times \mathbf{C}^*$
4. $Sp(n,\mathbf{C})/SL(n,\mathbf{C}) \times \mathbf{C}^*$
5. $E_6^{\mathbf{C}}/SO(10,\mathbf{C}) \times \mathbf{C}^*$
6. $E_7^{\mathbf{C}}/E_6^{\mathbf{C}} \times \mathbf{C}^*$

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