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§0. Introduction

The span $\sigma(M)$ of a differentiable manifold M^n is defined as the maximum number of linearly independent vector fields on M. Let Φ_X^k denote the trivial vector bundle of rank k over X. Let T_M denote the tangent vector bundle of M. The span of M can then be identified with the largest integer k such that $T_M \simeq \Phi_M^k \oplus \eta$ for some vector bundle η over M. The span σ_n of the sphere has been completely determined by J. F. Adams [1]. If $n+1=2^{c(n)}16^{d(n)}b_n$ with $0 \leqslant c(n) \leqslant 3$, $d(n) \geqslant 0$ and b_n odd, then $\sigma_n = 2^{c(n)} + 8d(n) - 1$.

A differentiable manifold M is called a π-manifold if the tangent vector bundle T_M of M is stably trivial. In [2] Bredon and Kosinski proved that if M^n is a closed (i.e. compact without boundary) π-manifold of dimension n then either M is parallelizable or the span of $M^n = \sigma_n$.

Let N be a topological manifold. Denoting the tangent microbundle of N by τ_N and the trivial microbundle of rank k over N by ε_N^k, we can define the span of N as the largest integer k such that $\tau_N \simeq \varepsilon_N^k \oplus \beta$ (as microbundles for some microbundle β over N. In case N is also a differentiable manifold the span of N considered as a topological manifold will be referred to as the topological span of N. It is easy to give examples of differentiable manifolds for which the topological span and the differentiable span are different. Actually in [5] Milnor has shown that on some open set M^n of R^n for some n there exists a differentiable structure with respect to which the integral pontrjagin class $p(M)$ of M is different from 1. With this differentiable structure on M^n, we have span $M^n < n$ as a differentiable manifold, whereas the topological span of M^n is n. A topological manifold N will be called a topological π-manifold if the tangent microbundle τ_N is stably trivial.

In this paper we first show that the topological span of S^n is σ_n. Then we also prove the analogue of the result of Bredon and Kosinski in the case of topological π-manifolds of dimensions $\neq 4$ and 15. Namely, we prove that in case $n \neq 4$, 15 a closed topological π-manifold M of dimension n is either topologically parallelizable (i. e. $\tau_M \simeq \varepsilon_M^n$) or the topological span of $M^n = \sigma_n$.

Finally the author wishes to thank his colleague Mr. Gopal Prasad for some very profitable discussions he had with him and also record his indebtedness to the referee for simplifying the proof of Theorem 2.1 considerably. Actually the proof given here is due to the referee.

§1. Topological Span of S^n

The main result to be proved in this section can be stated as follows:

THEOREM 1.1. *The topological span of S^n is σ_n.*

Before taking up the proof of this we will explain the notations used. Top (n) will denote the group of homeomorphisms of $\mathbf{R}^n$ fixing the origin, $H(n)$ the space of homotopy equivalences of S^{n-1} and $F(n)$ the space of homotopy equivalences of (S^n, x_0) where x_0 is some chosen base point in S^n. Unreduced suspension gives rise to an injection $H(n) \subset F(n)$. Let $BH(n)$, $BF(n)$ denote Stasheff's classifying spaces [6] of the H-spaces $H(n)$ and $F(n)$ respectively Let $B0(n)$ and $B\,\mathrm{Top}(n)$ denote the classifying spaces of the topological groups $0(n)$ and $\mathrm{Top}(n)$. The inclusions $0(n) \subset \mathrm{Top}(n)$ and $H(n) \subset F(n)$ give rise to maps $B0(n) \to B\,\mathrm{Top}(n)$, $BH(n) \to BF(n)$. Also if $i: S^{n-1} \to \mathbf{R}^n - 0$, $r: \mathbf{R} - 0 \to S^{n-1}$ respectively denote the inclusion of S^{n-1} in $\mathbf{R} - 0$ and radial projection of $\mathbf{R}^n - 0 \to S^{n-1}$ then $\phi m \to ro\{\phi \mid (\mathbf{R}^n - 0)$ **io** gives a H-map of $\mathrm{Top}(n)$ in $H(n)$ and this induces a map $B\,\mathrm{Top}(n) \to BH(n)$.

Let K be a finite dimensional complex and α a microbundle of rank n over K. We denote by $G(\alpha)$ the element in $[K, BF(n)]$ represented by the composite of $f: K \to B\,\mathrm{Top}(n)$ with the obvious map $B\,\mathrm{Top}(n) \to BF(n)$ where f is classifying map for α. If $\alpha \simeq \varepsilon_K^k \oplus \beta$ for some microbundle β over K then $G(\alpha)$ will be in the image of the map $[K, BF \times (n-k)] \to [K, BF(n)]$. For proving Theorem 1.1 we need a consequence of a classical result of I. M. James. [3]

PROPOSITION 1.2. (I. M. James) *The map*

$$\pi_i(0(n), 0(n-k)) \to \pi_i(F(n), F(n-k))$$

is an isomorphism for $i \leqslant 2(n-k) - 2$.

By using obstruction theory one gets as an immediate consequence of proposition 1.2 the following

PROPOSITION 1.3. *Let K be a CW complex of dimension n. Let ξ be a vector bundle of rank n over K and α the underlying microbundle of ξ. Let k be any integer $\leqslant (n-1)/2$. Then $\xi \simeq \Phi_K^k \oplus \eta$ for some vector bundle η of rank $(n-k)$ over K if and only if $\alpha \simeq \varepsilon_K^k \oplus \beta$ for some microbundle β of rank $(n-k)$ over K.*

Proof of Theorem 1.1. Let T_{S^n} denote the tangent vector bundle of S^n and τ_{S^n} the tangent microbundle of S^n. Then τ_{S^n} is the underlying microbundle of T_{S^n}. If $T_{S^n} \simeq \simeq \Phi_{S^n}^k \oplus \eta$ for some vector bundle η then clearly $\tau_{S^n} \simeq \varepsilon_{S^n}^k \oplus \beta$ where β is the microbundle underlying η. From this it follows that the topological span of $S^n \geqslant \sigma_n$. Thus for proving Theorem 1.1 we have only to show that the topological span of $S^n \leqslant \sigma_n$. For

$n=1, 3, 7$ there is nothing to prove since $\sigma_n=n$ in these cases. Also since $0(2)\to \mathrm{Top}(2)$ is a homotopy equivalence there is no distinction between vector bundles and microbundles in rank 2. If $n\neq 1, 3, 7, 2$ and 15 one can directly check that $\sigma_n+1\leqslant(n-1)/2$. In this case if $\tau_{S^n}\simeq\varepsilon^k_{S^n}\oplus\beta$ with $k=\sigma_n+1$ for some microbundle β it follows from Proposition 1.3 that $T_{S^n}\simeq\Phi^{k^n}_S\oplus\eta$ for some vector bundle η over S^n. This contradicts the definition of σ_n.

Now let us take up the case $n=15$. We have $\sigma_{15}=8$. If possible let $\tau_{S^{15}}\simeq\varepsilon^9_{S^{15}}\oplus\beta$. Let $x\in\pi_{15}(B\,\mathrm{Top}(15))\simeq\pi_{14}(\mathrm{Top}(15))$ represent the tangent microbundle $\tau_{S^{15}}$ of S^{15} and $y\in\pi_{14}(F(15))$ be the image of x under the obvious map $\pi_{14}(\mathrm{Top}(15))\to\pi_{14}(F(15))$. If $\tau_{S^{15}}\simeq\varepsilon_{S^{15}}\oplus\beta$ there should exist an element $z\in\pi_{14}(\mathrm{Top}(6))$ getting mapped into x under $\pi_{14}(\mathrm{Top}(6))\to\pi_{14}(\mathrm{Top}(15))$. From the commutativity of diagram 1

$$\begin{array}{ccc}\pi_{14}(\mathrm{Top}(6)) & \to & \pi_{14}(\mathrm{Top}(15))\\ \downarrow & & \downarrow\\ \pi_{14}(F(6)) & \to & \pi_{14}(F(15))\end{array}$$

Diagram 1

we see immediately that there exists an element $\omega\in\pi_{14}(F(6))$ getting mapped into y under $\pi_{14}(F(6))\to\pi_{14}(F(15))$. One knows that $\pi_{14}(F(6))\simeq\pi_{20}(S^6)$, $\pi_{14}(F(15))\simeq\pi_{29}\times(S^{15})$ under isomorphisms making diagram 2 commutative up to sign.

$$\begin{array}{ccc}\pi_{14}(F(6)) & \to & \pi_{14}(F(15))\\ \downarrow\simeq & & \downarrow\simeq\\ \pi_{20}(S^6) & \overset{\Sigma^9}{\to} & \pi_{29}(S^{15})\end{array}$$

Diagram 2

Here Σ^9 is the 9 fold suspension. Since the element $x\in\pi_{14}$ $(\mathrm{Top}(15))$ lies in the kernel of $\pi_{14}(\mathrm{Top}(15))\to\pi_{14}(\mathrm{Top}(16))$ and since diagram 3 below is such that the upper rectangle commutes and the lower rectangle commutes up to sign,

$$\begin{array}{ccc}\pi_{14}(\mathrm{Top}(15)) & \to & \pi_{14}(\mathrm{Top}(16))\\ \downarrow & & \downarrow\\ \pi_{14}(F(15)) & \to & \pi_{14}(F(16))\\ \downarrow & & \downarrow\\ \pi_{29}(S^{15}) & \overset{\Sigma}{\to} & \pi_{30}(S^{16})\end{array}$$

Diagram 3

it follows that the element $y'\in\pi_{29}(S^{15})$ which corresponds to y under the isomorphism $\pi_{14}(F(15))\simeq\pi_{29}(S^{15})$ lies in the kernel of Σ. From Toda's calculations [8] we have $\pi_{29}(S^{15})\simeq\mathbf{Z}_4\oplus\mathbf{Z}_2$ and $\pi_{30}(S^{16})\simeq\mathbf{Z}_2\oplus\mathbf{Z}_2$ and $\Sigma:\mathbf{Z}_4\oplus\mathbf{Z}_2\to\mathbf{Z}_2\oplus\mathbf{Z}_2$ is an epimorphism carrying the second fractor $\mathbf{Z}_2$ isomorphically. Denoting the generators of $\mathbf{Z}_4$ and $\mathbf{Z}_2$ by λ and μ respectively we see that $y'=2\lambda$ in $\pi_{29}(S^{15})\simeq\mathbf{Z}_4\oplus\mathbf{Z}_2$.

Now, Σ^9 can be written as the composite $\pi_{20}(S^6)\xrightarrow{\Sigma^5}\pi_{25}(S^{11})\xrightarrow{\Sigma^4}\pi_{29}(S^{15})$.
Again from Toda's calculations [8] we have $\pi_{20}(S^6)\simeq\mathbf{Z}_{12}\oplus\mathbf{Z}_2$; $\pi_{25}(S^{11})\simeq\mathbf{Z}_{16}\oplus\mathbf{Z}_2$. Moreover Σ^4 carries the factor $\mathbf{Z}_2$ of $\pi_{25}(S^{11})\simeq\mathbf{Z}_{16}\oplus\mathbf{Z}_2$. into the factor $\mathbf{Z}_2$ of $\pi_{29}\times(S^{15})\simeq\mathbf{Z}_4\oplus\mathbf{Z}_2$. Let c, d denote the generators of $\mathbf{Z}_{12}$ and $\mathbf{Z}_{16}$. The given arbitrary homomorphisms $\theta:\mathbf{Z}\to\mathbf{Z}_{16}$, $\theta:\mathbf{Z}_2\to\mathbf{Z}_{16}$, it is possible to find integers k and l such that $\theta(c)=4kd$, $\varphi(\mu)=8ld$. From this it follows that whatever be the homomorphism $\psi=\mathbf{Z}_{16}\to\mathbf{Z}_4$ we will have $\psi\circ\theta=0=\psi\circ\varphi$. It follows from these comments that there does not exist an element in $\pi_{20}(S^6)$ getting mapped onto $y'=2\lambda\neq 0$ in $\pi_{29}(S^{15})$ by Σ^9. Hence it is not possible to have $\tau_{S^{15}}\simeq\varepsilon^9\oplus\beta$. This completes the proof of Theorem 1.1.

COROLLARY 1.4. *The topological span of the real projective space P^n is σ_n.*

This is an immediate consequence of Theorem 1.1 and the following known facts

(a) Differentiable span of P^n is σ_n.

(b) If $\tilde{N}^p\to N$ is a covering space with N a topological manifold then $\tau_{\tilde{N}}\simeq p^*(\tau_N)$.

§2. Topological π-manifolds

The main result proved here is

THEOREM 2.1. *If M^n is a closed topological π- manifold of dimension $n\neq 4$ and 15 then either M^n is topologically parallelizable or the topological span of M^n is σ_n.*

For the proof of Theorem 2.1 we need the following

PROPOSITION 2.2. *Let M^n be any closed topological π- manifold of dimension $\pi\neq 4$. Then M^n carries a differentiable structure under which it is a differentiable π- manifolds.*

Proof. For $n>4$ it is an immediate consequence of the product structure theorem (Theorem 8 of [4]) of Kirby-Siebenmann. For $n\leqslant 2$ it is well-known. For $n=3$ it follows from the result of Stiefel [7] that any orientable closed 3-dimensional manifold is parallelizable.

Theorem 2.1 follows immediately from the theorem of Bredon and Kosinski for closed differentiable π-manifolds [2] and Propositions 1.3 and 2.2 of this paper, together with the observation that $\sigma_n+1\leqslant(n-1)/2$ whenever $n\neq 15$.

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