Some Global Theorems on Non-Complete Surfaces

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§1. Introduction

The first theorem we want to prove in this paper is about the non-immersibility in euclidean three-space of some non-compact surfaces with non-positive curvature. If a surface is compact, the proof of non-immersibility is well-known and easy. It depends on the fact that the euclidean norm attains its maximum on the surface. For non-compact surface, this fact is no longer valid. In fact there are numerous complete surfaces with non-positive curvature in euclidean space. Our theorem shows that if the surface has no more than three ends and if it is complete and of finite volume, it cannot be immersed in euclidean space R^3 with non-positive curvature. The number three is best possible. Furthermore, completeness can be replaced by the requirement that it be parabolic and non-flat.

In section two, we discuss the total curvature of an open surface. In general we assume some finiteness condition at the ends of the surface. But we do not assume completeness. If the surface is parabolic, we prove that the total curvature is not less than $2\pi\chi$ where χ is the Euler number of the surface. This may be compared with the Cohn-Vossen inequality which states that for complete surfaces, the total curvature is not greater than $2\pi\chi$. We also discuss some other results under the above finiteness condition.

In section three, we discuss strongly positively curved surfaces. Bonnet's theorem says that a complete strongly positively curved surface is compact. We replace completeness by the finiteness condition mentioned above and prove that the surface is bounded. Obviously without any finiteness assumption, the assertion need not be true. Finally, we give an upper bound for the diameter of a ball which contains a compact hypersurface with positive curvature. The upper bound is given in terms of the sectional curvature of the hypersurface. Unfortunately, we can only do this for hypersurfaces of dimension not less than three. We note that a consequence of Theorem 8 is that the only compact constantly curved hypersurface in euclidean space is the sphere.

Results of sections one and three can be generalized to higher dimension and to the case where the ambient manifold has constant curvature.

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I also want to thank the referee and R. Osserman for their comments. R. Osserman informed me that the example of Ogawa is also discussed in [7].

§2. Immersions of Surfaces with Non-Positive Curvature

Throughout this paper, we shall assume that our surface is finitely connected, i.e., M is diffeomorphic to some compact surface $\tilde{M}$ minus a finite number of points $\{p_1, p_2, \ldots, p_n\}$. The number n is called the number of ends of M.

It is well known that every riemannian metric defines a conformal structure on M. Each p_i has a neighborhood U_i in $\tilde{M}$ such that $U_i\backslash\{p_i\}$ is conformally equal to an annulus in the complex plane. Let the annulus be $0<R_1\leqslant|z|\leqslant R_2\leqslant\infty$. It is known [1] that M is parabolic iff $R_2=\infty$ and that M is hyperbolic iff $R_2<\infty$ for each p_i. Recall that a Riemann surface is called parabolic if there is no negative subharmonic function on it. Throughout this paper when we say that a sequence of closed curves converges to an end p_i, we assume that these curves are not homotopic to zero in the neighborhood $U_i\backslash\{p_i\}$. Now let us first prove the following.

PROPOSITION 1. *Let M be a parabolic riemannian surface with finite volume. Then for each end p_i, we can find a sequence of closed curves $\{\sigma^i_j\}$ which converges to p_i and whose lengths approach to zero.*

Proof. Assume $i=1$. Let $U_1\backslash\{p_1\}$ be the neighborhood which is conformally equivalent to the complex plane minus a disc. Let the radius of this disc be R_1. We claim that there is a sequence of circles with radii $\{r_i\}$ going to infinity and lengths approaching to zero.

Since $U_1\backslash\{p_1\}$ is conformally equivalent to the plane minus a disc, we may assume the metric has the form

$$dx^2 = e^{2u}(dx^2 + dy^2). \tag{1}$$

In terms of polar coordinates, the circle with radius r has length

$$l(r) = \int_0^{2\pi} e^u r \, d\theta. \tag{2}$$

If our assertion is false, we may assume

$$\liminf_{r\to\infty} l(r) \geqslant \varepsilon > 0 \tag{3}$$

for some constant ε. We claim that this is impossible. In fact, from (2) and (3), we would have

$$\int_0^{2\pi} e^u \, d\theta \geqslant \frac{\varepsilon}{r}. \tag{4}$$

This gives the inequality

$$2\pi \int_{R_1}^{R} \int_0^{2\pi} e^{2u} r \, d\theta \, dr \geqslant \int_{R_1}^{R} r \left(\int_0^{2\pi} e^u \, d\theta \right)^2 dr \geqslant \int_{R_1}^{R} r \left(\frac{\varepsilon}{r} \right)^2 dr = \varepsilon^2 \log \frac{R}{R_1}. \tag{5}$$

The left hand side of (5) is – up to a factor 2π – the area of the annulus between the circles of radius R_1 and R. By hypothesis, this term is bounded by a constant independent of R which contradicts (5).

COROLLARY. *The conclusion of Proposition* 1 *still holds if we replace M by a complete riemannian surface with finite volume.*

Proof. It is easy to see [1] that complete surface with finite volume is parabolic.

The following theorem of R. Osserman [3] will be used.

THEOREM 1. *Let M be a surface with non-positive curvature in euclidean three-space. Let D be a compact domain in M. Then D lies in the convex hull of its boundary in M, the convex hull being taken in the euclidean space.*

Now let us prove

THEOREM 2. *Let M be a non-positively curved surface in euclidean three-space, with finite volume. Suppose it is parabolic and has no more than three ends. Then it must lie on a plane.*

Proof. Without loss of generality, we may assume that M has three ends. Using the notations of §2, we write these three ends as $\{p_1, p_2, p_3\}$.

By Proposition 1, for each p_i, there is a sequence of closed curves $\{\sigma_j^i\}$ which decrease to p_i and whose length approach to zero. Let $\{q_j^i\}$ be arbitrary points on these curves $\{\sigma_j^i\}$. Then for each j, $\{q_j^i\}$ determine a plane. Let P_j be the convex body which is obtained by thickening this plane and which just contains the curves σ_j^i. Since the lengths of these curves$_j^i$ σ approach zero, the width of P_j approaches zero.

Now let D_j be the compact domain bounded by the curves σ_j^i. By Theorem 1, D_j is contained in P_j for each j. We claim that P_j approaches some fixed plane. In fact, fix three non-collinear points in D_k for some k. Then they are contained in P_j for j large enough, and the distance between these three points and the boundary of P_j

approaches zero. Hence P_j converges to the plane determined by these three points. It is then clear that the whole surface lies on this plane.

As in Proposition 1, we have the following

COROLLARY. *Let M be a complete non-positively curved surface with finite volume. If M has no more than three ends, then M cannot be isometrically immersed in euclidean three space.*

R. Ogawa pointed out to us that the number three is the best possible, i.e., there is a complete non-positively curved surface with finite volume and four ends in euclidean three space. Such a surface can be constructed in the following way. Consider the tetrahedron in euclidean three space. Delete one point from each of the four faces and pull the faces to infinity at these points. Smoothing the surface along the edges, we obtain the example required.

§3. The Curvature Restriction on an Open Surface

The purpose of this section is to discuss the total curvature of an open surface under some conformal restriction. The total curvature is by definition the integral $\int_M K\,dA$ where K is the Gauss curvature and dA is the volume element of the surface. We shall denote the Euler number of the surface by χ.

THEOREM 3. *Let M be a finitely connected parabolic surface. Suppose there is a sequence of closed curves of uniformly bounded length converging to each end. Then*

$$\int_M K\,dA \geqslant 2\pi\chi. \tag{7}$$

Proof. Let p_i be the ends and U_i be the neighborhoods such that $U_i\backslash\{p_i\}$ is conformally equivalent to the plane minus a disc with radius R_i. For each i, let $\{\sigma_j^i\}$ be the sequence of closed curves of uniformly bounded length which converges to p_i. Without loss of generality, we may assume that each σ_j^i is a simple, closed curve. In view of the Gauss-Bonnet Theorem, we need only construct a sequence of closed curves converging to each end such that the total geodesic curvature of this sequence tends to a non-positive number.

Since we are only interested in the behavior at infinity, we may assume M is in fact conformally equivalent to the plane. (One can extend the metric outside the annulus.) Let us first prove a special case of our theorem. Namely, we first assume all the σ_j's are concentric circles in the plane. Suppose the metric on the plane is given by

$$ds^2 = e^{2u}(dx^2 + dy^2). \tag{8}$$

Then the total geodesic curvature of the circle with radius r is given by

$$\int_0^{2\pi} r \frac{\partial u}{\partial r} d\theta + 2\pi. \tag{9}$$

If we cannot find a sequence of closed curves going to infinity such that the total geodesic curvature is tending to a non-positive number, we may assume there is a positive number $\varepsilon > 0$ such that

$$\int_0^{2\pi} r \frac{\partial u}{\partial r} d\theta + 2\pi \geqslant 2\pi\varepsilon \tag{10}$$

for r large enough.

Suppose the σ_j's are circles with radius r_j. Then by hypothesis, there is a positive constant $M > 0$ such that

$$\int_0^{2\pi} e^u r_j \, d\theta \leqslant M \tag{11}$$

for all j.

Integrating (10), we obtain

$$\frac{1}{2\pi} \int^{2\pi} u \, d\theta \geqslant (\varepsilon - 1) \log r + C \tag{12}$$

where C is a constant independent of r.

Now by the convexity of exponential function, we have

$$\frac{1}{2\pi} \int_0^{2\pi} e^u \, d\theta \geqslant \exp\left(\frac{1}{2\pi} \int_0^{2\pi} u \, d\theta\right). \tag{13}$$

From (11), (12), and (13), we have

$$e^c \cdot r_j^{\varepsilon - 1} \leqslant \frac{M}{2\pi r_j} \tag{14}$$

for each j. Such an inequality is impossible if r_j goes to infinity. Hence we have proved the theorem for our special case.

To prove the general case, we have to use the Riemann mapping theorem. We assume that σ_i is a simple, closed curve such that σ_i is inside σ_{i+1} for all i and that 0 lies in the region bounded by σ_i for all i. For each i, let f_i be the biholomorphic transformation mapping the region bounded by the curve σ_i onto the disk $D(r_i)$ with radius r_i such that $f_i(0) = 0$ and $f_i'(0) = 1$.

Recall that Bloch's theorem says that there exists a universal constant b such that for any holomorphic function f defined on the disc of radius r and satisfying $f'(0)=1$, the image of f contains a disk with radius rb. Applying Bloch's theorem to the above situation, we see that $r_i \to \infty$. It is also easy to see $r_{i+1} > r_i$. (Otherwise we can define $f_i f_{i+1}^{-1}$ which maps $D(r_{i+1})$ into $D(r_i)$ with $f_i f_{i+1}^{-1}(0)=0$ and $(f_i f_{i+1}^{-1})'(0)=1$, a contradiction to the Schwartz Lemma.)

For each r_i, consider the family of functions $f_j^{-1} \mid D(r_i)$, $j>i$. It is well-known (see [5]) that this is a normal family. Hence by a diagonal process, a subsequence of $\{f_j^{-1}\}$ converges locally uniformly to a holomorphic function f^{-1}. Similarly $\{f_j\}$ converges locally uniformly to a holomorphic function which must be the inverse of f^{-1}. In particular, f^{-1} is biholomorphic and $(f^{-1})' \neq 0$ everywhere. An immediate consequence is that $\{|(f_i^{-1})'|\}$ is uniformly bounded from below by a positive constant on any disk $D(r)$.

For each $R>0$, let $S(R)$ be the circle of radius R. We assert that if $\{R_j\} \to \infty$ and if i is a function of j, then $\{f_i^{-1}\ S(R_j)\}$ diverges to infinity. In fact, if this is not true, we can find a sequence $\{z_j\} \to \infty$ such that $f_{i(j)}^{-1}(z_j)$ converges to some point $f^{-1}(z_0)$. This is impossible because $f_{i(j)}$ converges uniformly near $f^{-1}(z_0)$.

Now for each i, consider the induced metric $(f_i^{-1})_* ds^2$ on the disk $D(r_i)$. Let it be

$$e^{2u_i}(dx^2 + dy^2). \tag{15}$$

The fact that $|(f_i^{-1})'|$ is uniformly bounded from below by a positive constant on compact sets implies that $\{u_i\}$ is uniformly bounded from below on compact sets.

For each $R>0$, consider the number

$$g(R) = \inf_{r_j \geqslant R} \left[\int_0^{2\pi} R \frac{\partial u_j}{\partial r} d\theta + 2\pi \right]. \tag{16}$$

This is the infinnum of the total geodesic curvatures of the circle $S(R)$ with respect to the metric $(f_i^{-1})_* ds^2$. If we can find a sequence $\{R_j\} \to \infty$ such that $g(R_j)$ tends to a non-positive number, then we can find a sequence $\{i(j)\}$ such that the curves $f_{i(j)}^{-1}\{S(R_j)\}$ diverge to infinity and the total geodesic curvatures of $f_{i(j)}^{-1}\{S(R_j)\}$ tend to a non-positive number. This will prove our theorem.

Hence, we may assume that there are numbers $R>0$, $\varepsilon>0$ such that for all $r \geqslant R$, $g(r) \geqslant 2\pi\varepsilon$, i.e.,

$$\int_0^{2\pi} r \frac{\partial u_j}{\partial r} d\theta + 2\pi \geqslant 2\pi\varepsilon \tag{17}$$

for all j with $r_j \geqslant R$.

Let

$$C_i = \frac{1}{2\pi} \int_0^{2\pi} u_i(R, \theta)\, d\theta - (\varepsilon - 1) \log R. \tag{18}$$

Then as in the special case

$$\exp(C_i) \cdot r_i^{\varepsilon - 1} \leqslant \frac{M}{2\pi r_i}. \tag{19}$$

Since C_i is uniformly bounded from below, (19) is impossible when r_i goes to infinity.

COROLLARY. *Let M be a parabolic surface with finite volume, then* (7) *holds.*
Proof. This follows from Proposition 1 and the theorem.

COROLLARY. *Let M be a complete surface with non-positive curvature. If M is diffeomorphic to the plane or the plane minus a point, then M has infinite volume.*

Proof. If M has finite volume, it is parabolic and the above corollary says (7) is valid. Therefore M is conformally the plane minus a point and the curvature is identically zero. This manifold has infinite volume as will be seen in Theorem 4.

Remark. Theorem 3 was proved by Cohn-Vossen and Huber [2] under the assumption that the metric is complete.

THEOREM 4. *Let M be a parabolic surface with finite volume. Then there is a sequence of closed curves σ_i^j converging to each end p_j such that, if B_i is the compact domain bounded by the σ_i^j, we have*

$$\int_{B_i} K\, dA > 2\pi\chi. \tag{20}$$

Proof. As in Theorem 3, we may assume M is conformally equivalent to the plane. Then by Proposition 1, there is a sequence of circles going to infinity with length approaching zero.

We claim that we can find a sequence of circles going to infinity such that their total geodesic curvature is negative. Otherwise (10) holds with $\varepsilon = 0$ and we deduce as in theorem 3 the following

$$2\pi e^C \leqslant M_j \tag{21}$$

where M_j is the length of the circle with radius r_j and C is a fixed constant. When j goes to infinity, we may assume $M_j \to 0$. This contradicts (21).

COROLLARY. *Let M be parabolic manifold with finite volume. Suppose that, except for a compact set, the curvature of M is non-negative. Then*

$$\int_M K\,dA > 2\pi\chi. \tag{22}$$

Proof. Because $\int_M K\,dA \geqslant \int_{Bi} K\,dA > 2\pi\chi$ when i is large enough.

Now let us combine theorems of Huber [1] and [2] to prove the following.

THEOREM 5. *Let M be a surface with $\int_M K^+\,dA < \infty$ where K^+ is the positive part of the Gaussian curvature. If there is a sequence of closed curves of uniformly bounded length converging to each end, then either*

i) $\int_M K\,dA = 2\pi\chi$ *or* ii) *M has finite volume.*

Proof. Without loss of generality, we may assume that M has one end and that $\{\sigma_i\}$ is the sequence of closed curves of uniformly bounded length going to this end. If M is complete, i) follows by the theorem of Cohn-Vossen and Huber [2]. Otherwise there is a curve Γ of finite length, going to infinity.

As in Theorem 3, we may assume that the σ_i's are simple, closed curves and that σ_i is inside σ_{i+1} for all i. Let i be large enough so that

$$\int_{B_i} K^+\,dA < 2\pi \tag{23}$$

where B_i is the complement of the region bounded by σ_i.

Let Γ_n be a component of the curve Γ joining the closed curves σ_i and σ_n. In [1], Huber proved the following isoperimetric inequality. If N is a compact simply connected surface with boundary, then

$$L^2 \geqslant 2\left(2\pi - \int_N K^+\,dA\right)A \tag{24}$$

where L is the length of the boundary and A is the volume of the surface N. Using (23), (24) and the fact that σ_i, σ_n and Γ have uniformly bounded length, one sees that the area of B_i is uniformly bounded and hence M has finite area. This completes the proof of the theorem.

If one does not assume $\int_M K^+\,dA < \infty$, one has the following theorem, which is a direct consequence of Huber's inequality.

THEOREM 6. *Let M be a simply connected surface with infinite volume. Suppose* *there is a sequence of closed curves of uniformly bounded length, going to infinity; then*

$$\int_M K^+ \, dA \geqslant 2\pi . \tag{25}$$

§4. Surfaces with Strongly Positive Curvature

It is well known that if a complete surface has curvature bounded from below by a positive constant, then it is bounded. We shall replace completeness by some finiteness condition at infinity.

THEOREM 7. *Let M be a surface with strongly positive curvature. Suppose there is a sequence of closed curves of uniformly bounded length going to each end. Then M is bounded.*

Proof. Let $\{q_1, \ldots, q_n\}$ be the ends of M. Let p be an arbitrary point in M. We shall call the distance between p and q_i the infinmum of the lengths of all the curves from p to q_i. If these distances are infinite, M is complete and hence compact, by Bonnet's theorem. Therefore, assume at least one of these distances is finite. We are going to prove that the infinma of these distances are actually uniformly bounded (independent of the point p). Using the hypothesis, it is then easy to conclude the proof of the theorem.

To prove the assertion, let us assume q_1 is the end such that the distance between p and q_1 is the infinmum of the distances between p and the q_i. Let this distance be d. Let $\{\sigma_i\}$ be a sequence of curves from p to q_1 with lengths approaching d. Let $[0, d']$ be an interval parametrizing the curves $\{\sigma_i\}$ such that the length of the tangent of each curve is not greater than one. For any integer n, one can see easily that when i is large enough, all the curves $\{\sigma_i \mid [0, d'-(1/n)]\}$ have length not greater than $d-\varepsilon$ for some positive $\varepsilon>0$. Hence, by the choice of q_1 and d, all these curves lie in a compact subset of M and they have a subsequence converging to a curve defined on $[0, d'-(1/n)]$. Call this subsequence $\{\sigma_i^n\}$. Repeating the same procedure as above, one chooses a subsequence $\{\sigma_i^{n+1}\}$ of $\{\sigma_i^n\}$ converging to a curve defined on $[0, d'-(1/n+1)]$. Continuing this process and picking the sequence $\{\sigma_i^i\}$, we find the limit of this sequence of curves is a curve defined on $[0, d')$. Furthermore, it is easy to see that this curve has length d. Therefore, this curve minimizes the distance from p to q_1 and a standard argument shows that it is actually a minimizing geodesic. A minimizing geodesic cannot have a critical point and the argument of Bonnet's theorem shows $d \leqslant \pi/\sqrt{K}$. This proves our assertion.

We note that the hypothesis in Theorem 7 is essential. For example, let R^2 be the universal cover of the sphere minus two points. Then R^2 has a metric induced from the sphere. Its curvature is a positive constant. It is not hard to see that such a surface is not bounded. On the other hand, it is quite possible that any surface with constant positive curvature in euclidean three-space is bounded.

Let us consider the last statement. Let M be a hypersurface in eudlidean space. We shall call the smallest diameter of balls containing M the outer diameter of M and the largest diameter of balls lying inside M the inner diameter of M.

THEOREM 8. *Let M be a compact hypersurface in R^{n-1} with curvatures bounded between two positive constants K_1 and K_2. If $n \geqslant 3$, then the outer diameter of M is not greater than $2\sqrt{K_2/K_1}$ and the inner diameter of M is not less than $2\sqrt{K_1/K_2}$.*

Proof. We first note the following theorem of [4]. (R. Ogawa has another proof of this fact.) If σ is a closed convex curve in the plane with curvature not less than k, then the outer diameter of σ is not greater than $2/k$.

To prove Theorem 8, we need the following generalization of the above statement. Let M be a compact convex hypersurface of euclidean space. Let k be the infinmum of all principal curvatures of M. Then the outer diameter of M is not greater than $2/k$. In fact, let p be a point in M such that one of the principal curvatures at p attains the value k. The tangent plane at p cuts the euclidean space into two halves, one of which contains M completely. Let S be the sphere of radius $1/k$ contained in this half space and tangent to the hyperplane at p. It suffices to prove M is contained inside S.

Let q be any point in M. Through p and q draw a two-dimensional plane which is perpendicular to the tangent plane at p. This plane cuts the manifold M into a curve. It is not hard to prove that the geodesic curvature of this curve in the two-dimensional plane is not less than $1/k$. By the theorem quoted above, this curve lies entirely in S. This proves our assertion.

Now by the Gauss equation and the assumption, we have

$$K_1 \leqslant \lambda_i \lambda_j \leqslant K_2 \tag{26}$$

for all principal curvatures λ_i, λ_j with different principal directions.

If $n \geqslant 3$, it is easy to see from (26) that

$$\frac{K_1}{\sqrt{K_2}} \leqslant k. \tag{27}$$

The first assertion of Theorem 8 then follows from (27) and the above generalization. The second assertion can be proved similarly.

Remark 1. We do not know whether Theorem 8 is true for $n=2$ or not. If it is true, it would imply the famous theorem that the only compact constant curved surface in euclidean three space is the sphere.

Remark 2. After this paper was submitted for publication, we learned that Pogorelov [6] had studied an analogous question for $n=2$. He calls a surface almost spherical if the ratio λ_1/λ_2, two of the principal curvatures, is close to 1. In this case, he obtains a theorem like Theorem 8. Our methods of proof give this result also.

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