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## Geodesics Satisfying General Boundary Conditions

by KARSTEN GROVE<sup>1)</sup>

The existence of infinitely many geodesics joining orthogonal two submanifolds  $V_1$  and  $V_2$  of a Riemannian manifold  $M$  has been studied in Morse [4] and in Serre [5] under the assumptions  $V_1$  compact and  $V_1 \cap V_2 = \emptyset$ . Existence of such geodesics is in that case clear. If however  $V_1 \cap V_2 \neq \emptyset$  this is not in general the case (see §2). It is the purpose of this paper to examine that situation.

In §1 we shall study geodesics satisfying a general boundary condition. Special cases of this boundary condition are satisfied by  $V_1 - V_2$ -connecting geodesics, closed geodesics and isometry-invariant geodesics (see Grove [2], [3]). In §2 we concentrate on  $V_1 - V_2$ -connecting geodesics. A typical result in that section is that if  $V_1$  and  $V_2$  are compact and  $M$  is contractible then there exists  $V_1 - V_2$ -connecting geodesics.

The main tools in our approach are critical point theory on infinite dimensional manifolds and elementary homotopy theory.

### §1. $N$ -Geodesics

Throughout this paper  $M$  will be a connected complete Riemannian manifold and  $N \subset M \times M$  a closed submanifold of  $M \times M$ . We shall say that a geodesic  $\gamma: [0,1] \rightarrow M$  is a  $N$ -geodesic if it satisfies the boundary condition

$$(\gamma(0), \gamma(1)) \in N \quad \text{and} \quad (\dot{\gamma}(0), -\dot{\gamma}(1)) \text{ is normal to } N, \quad (1.1)$$

where  $\dot{\gamma}(t)$  denotes the velocity vector of  $\gamma$  at  $t$  and  $M \times M$  is endowed with the product metric.

EXAMPLES. (1)  $\Delta(M)$ -geodesics are closed geodesics on  $M$ .

(2)  $\text{graph}(A)$ -geodesics,  $A: M \rightarrow M$  an isometry, are  $A$ -invariant geodesics (see [2] and [3]).

(3)  $V_1 \times V_2$ -geodesics are  $V_1 - V_2$ -connecting geodesics.

Let  $L_1^2(I, M)$  denote the complete Riemannian Hilbertmanifold consisting of absolutely continuous curves  $\sigma: I = [0, 1] \rightarrow M$  with  $\dot{\sigma}$  square integrable (see Flaschel and Klingenberg [1]). From the propositions I.1.4 and I.1.5 of [2] it follows easily that

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*N*-geodesics are in one to one correspondence with critical points for the energy integral  $E: \Lambda_N(M) \rightarrow \mathbf{R}$ ,  $\sigma \mapsto \frac{1}{2} \int_0^1 \|\dot{\sigma}\|^2$ , where  $\Lambda_N(M) = \{\sigma \in L_1^2(I, M) \mid (\sigma(0), \sigma(1)) \in N\}$  is a submanifold of  $L_1^2(I, M)$ . Furthermore  $E: \Lambda_N(M) \rightarrow \mathbf{R}$  satisfies condition (C) of Palais and Smale if projection on the first,  $P_1(N) \subset M$  or the second factor  $P_2(N) \subset M$  is compact (Theorem I.2.4 of [2]). We shall therefore assume that e.g.  $P_1(N) \subset M$  is compact. To make successful use of critical point theory for Hilbert-manifolds we assume furthermore that  $N \cap \Delta$  is a union of closed submanifolds, – here  $\Delta$  denotes the diagonal  $\Delta(M)$  of  $M$  in  $M \times M$ .

With  $N$  as above we have,

LEMMA 1.2. *If there are no non-trivial N-geodesics on M, then the inclusion*

$$e: N \cap \Delta \rightarrow \Lambda_N(M), \quad e(x, x)(t) = x \quad \forall t \in I$$

is a homotopy equivalence.

*Proof (Sketch).* First we observe that each component of  $N \cap \Delta$  is a compact non-degenerate critical submanifold of  $\Lambda_N(M)$  of index 0. To see this we just note that the Hessian of  $E: \Lambda_N(M) \rightarrow \mathbf{R}$  at a constant curve  $\bar{p}: I \rightarrow M$ ,  $\bar{p}(t) = p$  for all  $t \in I$  is given by

$$H(E)_{\bar{p}}(X, Y) = \int_0^1 \langle X'(t), Y'(t) \rangle_p dt$$

for all  $X, Y \in T_{\bar{p}}\Lambda_N(M)$ . From this and  $(X(0), X(1)) \in TN$  for  $X \in T\Lambda_N(M)$  it easily follows that each component of  $N \cap \Delta$  is a non-degenerate critical submanifold of index 0. We can now argue exactly as in the proofs of Corollary II.3 and Lemma II.4 of [2], i.e. by the generalized Morse Lemma and condition (C) prove that there exists an  $\varepsilon > 0$  such that  $N \cap \Delta$  is a strong deformation retract of  $\Lambda_N(M)^\varepsilon := \{\sigma \in \Lambda_N(M) \mid E(\sigma) < \varepsilon\}$ . Assuming that there are no critical values  $> 0$  (no non-trivial *N*-geodesics) we obtain from this, completeness of  $\Lambda_N(M)$  and condition (C) that the inclusion  $e: N \cap \Delta \rightarrow \Lambda_N(M)$  is a weak homotopy equivalence and hence a homotopy equivalence.

We are now ready to prove the main result of this section.

THEOREM 1.3. *If there are no non-trivial N-geodesics on M, then there is an exact sequence of homotopy groups*

$$\dots - \pi_{*+1}(N) \xrightarrow{(P_1)^{*+1} - (P_2)^{*+1}} \pi_{*+1}(M) \rightarrow \pi_*(N \cap \Delta) \xrightarrow{i_*} \pi_*(N), \quad * \geq 0.$$

Exactness at  $\pi_{*-1}(N \cap \Delta)$  implies that

$$\forall [k] \in \pi_*(M) \exists [g] \in \pi_*(N) \exists [h] \in \pi_*(N, N \cap \Delta) \text{ s.t.}$$

$$[k] = ((P_1)_*([h]) - (P_2)_*([h])) - ((P_1)_*([g]) - (P_2)_*([g])) \quad * \geq 1.$$

(for  $*=1$  read multiplicative).

*Proof.* The inclusion  $\Lambda_N(M) \rightarrow C_N^0(M) := \{f \in C^0(I, M) \mid (f(0), f(1)) \in N\}$  is a homotopy equivalence (Theorem I.1.3 of [2]) so instead of  $\Lambda_N(M)$  we consider  $C_N^0(M)$ . Consider now the commutative diagram,

$$\begin{array}{ccccccc} & \rightarrow & \pi_{*+1}(C_N^0 M, \Omega M) & & & & \\ & & \downarrow P_{*+1} \cong & \searrow \delta & & & \\ & \rightarrow & \pi_{*+1}(N) & \xrightarrow{\partial} & \pi_*(\Omega M) & \xrightarrow{j_*} & \pi_*(C_N^0 M) & \xrightarrow{P_*} & \pi_*(N) \\ & & \searrow H \circ \partial & & \downarrow H \cong & & \uparrow e_* \cong & & \nearrow i_* \\ & & & & \pi_{*+1}(M) & \rightarrow & \pi_*(N \cap \Delta) & & \end{array}$$

where the mid-sequence is the exact sequence for the fibration  $P: C_N^0(M) \rightarrow N$ ,  $P(f) = (f(0), f(1))$  with fiber the loop space  $\Omega M$  of  $M$  (Serre [5]), where  $\delta$  is the boundary map in the exact sequence for the pair  $(C_N^0(M), \Omega M)$  and  $H$  is the Hurewicz map. We shall compute the maps in cupic homotopi.

Let  $\alpha: I^q \times I \rightarrow C_N^0(M)$  represent an element  $[\alpha]$  in  $\pi_{*+1}(C_N^0 M, \Omega M)$ . Evaluation of  $\alpha_i: I^q \times \{0\} \rightarrow \Omega M$ ,  $\hat{\alpha}_i$  represents  $H \circ \partial[\alpha] = H \circ \partial(P_*[\alpha])$ . From this we see that evaluation of  $\alpha: I^{q+1} \rightarrow C_N^0 M$ , is a homotopy between  $-P_1 \circ P \circ \alpha + \hat{\alpha}_i + P_2 \circ P \circ \alpha$  and the constant map, thus  $H \circ \partial([\beta]) = (P_1)_*([\beta]) - (P_2)_*([\beta])$  for all  $[\beta] = P_*[\alpha] \in \pi_{*+1}(N)$ . Since we assume that there are no non-trivial  $N$ -geodesics  $e_*: \pi_*(N \cap \Delta) \rightarrow \pi_*(C_N^0 M)$  is an isomorphism by Lemma 1.2 and the lower sequence is the desired sequence.

We will now examine in detail what exactness at  $\pi_*(N \cap \Delta)$  i.e.  $e_*(\ker i_*) = \text{im } j_*$  means.

Let  $[f] \in \pi_q(\Omega M)$  be represented by  $f: I^q \rightarrow \Omega M$ . Since  $j_*([f]) \in \text{im } e_*$  there is a homotopy

$$G_1: I^q \times I \rightarrow C_N^0(M)$$

with  $G_1(\cdot, 0) = j \circ f$  and  $G_1(\cdot, 1): I^q \rightarrow e(N \cap \Delta)$ . Identifying  $e(N \cap \Delta)$  with  $N \cap \Delta$  we get from  $[G_1(\cdot, 1)] \in e_*(\ker i_*)$  a homotopy

$$G_2: I^q \times I \rightarrow N$$

with  $G_2(\cdot, 0) = G_1(\cdot, 1)$  and  $G_2(\cdot, 1) = \text{base point}$ . The homotopies  $P \circ G_1: I^q \times I \rightarrow N$  and  $G_2: I^q \times I \rightarrow N$  combines to an element  $[g] \in \pi_{q+1}(N)$  and  $G_2$  itself represents an element  $[h] \in \pi_{q+1}(N, N \cap \Delta)$ . Evaluation of  $G_1, \hat{G}_1: I^q \times I \times I \rightarrow M$  give rise to a homotopy between  $P_1 \circ g + H(f) - P_2 \circ g: I^{q+1} \rightarrow M$  and  $P_1 \circ h - P_2 \circ h: I^{q+1} \rightarrow M$ , note that  $\hat{G}_1(\cdot, 0, \cdot) = H(f)$  i.e.  $H[f] = (P_{1*+1}[h] - P_{2*+1}[h]) - (P_{1*+1}[g] - P_{2*+1}[g])$ . Q.E.D.

*Remark.* From theorem 1.3 we can derive the following theorems. *If  $M$  is compact there exist closed geodesics on  $M$ . If  $V$  is a compact submanifold of  $M$  and if there are no  $V - V$ -connecting geodesics then the inclusion  $N \rightarrow M$  is a homotopy equivalence.* We can also obtain some partial results from [2] and [3] by this theorem.

**§2.  $V_1 - V_2$ -Connecting Geodesics.<sup>2)</sup>**

In this paragraph  $N = V_1 \times V_2$ , where  $V_1$  and  $V_2$  are closed connected submanifold of  $M$ ,  $V_1$  is compact and  $V_1 \cap V_2$  is a union of closed submanifolds of  $M$  (may be of different dimensions). As mentioned in §1 is a  $V_1 \times V_2$ -geodesic  $\gamma: I \rightarrow M$  with  $\gamma(0) \in V_1, \gamma(1) \in V_2, \dot{\gamma}(0) \perp V_1$  and  $\dot{\gamma}(1) \perp V_2$ .

We shall derive all our conclusions from the exact sequence of Theorem 1.3, which in the case  $N = V_1 \times V_2$  can be written as,

$$\rightarrow \pi_{*+1}(V_1 \times V_2) \xrightarrow{(i_1)_{*+1} - (i_2)_{*+1}} \pi_{*+1}(M) \rightarrow \pi_*(V_1 \cap V_2) \rightarrow \pi_*(V_1 \times V_2) \quad (2.1)$$

We get immediately

**COROLLARY 2.2.** *Suppose that  $\dim(V_1 \cap V_2) = 0$  and that  $\pi_1(M) = 0$  or  $V_1 \cap V_2 = \{pt\}$ . If there are no non-trivial  $V_1 - V_2$ -connecting geodesics on  $M$  then all the homotopy groups of  $M$  are isomorphic to those of  $V_1 \times V_2$ , - in fact  $(i_1)_* - (i_2)_* : \pi_*(V_1 \times V_2) \rightarrow \pi_*(M)$  is an isomorphism.*

The following example illustrates this corollary.

**EXAMPLE.** Let  $V$  and  $W$  be Riemannian manifolds and let  $M = V \times W$  be endowed with the product metric. For a fixed  $(v, w) \in M$  put  $V_1 = V \times \{w\}$  and  $V_2 = \{v\} \times W$ , then there are no non-trivial  $V_1 - V_2$ -connecting geodesics on  $M$ . Other immediate consequences of (2.1) are

**COROLLARY 2.3.** *If  $V_1 \cap V_2$  is not connected and  $M$  is 1-connected then there exists non-trivial  $V_1 - V_2$ -connecting geodesics.*

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<sup>2)</sup> Note that  $N$ -geodesics are in 1-1-correspondence with  $N - \Delta(M)$ -connecting geodesics in  $M \times M$  with product metric (see e.g. L. N. Pattersen, On the index theorem, Amer. J. Math. 85 (1963), 271-297).

**COROLLARY 2.4.** *Suppose that  $M$  is a  $K(\pi, 1)$  (e.g.  $M$  has negative curvature) and that there are no non-trivial  $V_1 - V_2$ -connecting geodesics on  $M$ . Then we have:*

(1) *If  $\pi_1(V_1) = \pi_1(V_2) = 0$  then  $\pi_1(V_1 \cap V_2) = 0$  and  $\pi_1(M)$  is finite (impossible if  $M$  has neg. curvature).*

(2) *If  $V_1 \cap V_2$  consists only of isolated points then  $V_1 \times V_2$  is a  $K(\pi', 1)$ .*

**COROLLARY 2.5.** *Suppose that  $M$  is contractible. Then we have:*

(1) *If  $V_1$  and  $V_2$  are compact there exists non-trivial  $V_1 - V_2$ -connecting geodesics. ( $V_1 = V_2 = \{pt\}$  not included).*

(2) *If  $V_2$  is contractible and there are no non-trivial  $V_1 - V_2$ -connecting geodesics on  $M$  then  $V_1 \subset V_2$ .*

*Proof.* (1) From (2.1) we get that

$$i: V_1 \cap V_2 \rightarrow V_1 \times V_2, \quad i(p) = (p, p) \quad p \in V_1 \cap V_2$$

is a homotopy equivalence if there are no non-trivial  $V_1 - V_2$ -connecting geodesics. Since now  $V_1 \times V_2$  is compact we get especially that  $\dim(V_1 \cap V_2) = \dim(V_1 \times V_2)$  which is impossible except for the case  $V_1 = V_2 = \text{point}$ .

(2) In the case  $V_2$  contractible we obtain that

$$V_1 \cap V_2 \rightarrow V_1$$

is a homotopy equivalence if there are no non-trivial  $V_1 - V_2$ -connecting geodesics on  $M$ . By compactness  $\dim(V_1 \cap V_2) = \dim V_1$  which then implies that  $V_1 \cap V_2 = V_1$  or equivalently  $V_1 \subset V_2$ .

*Remark.*  $M = \mathbf{R}^3 \supset \mathbf{R}^2 = V_2 \supset V_1 = S^1$  gives an example of (2) in corollary 2.5. – It is clearly difficult to get more general results from (2.1). In concrete situations however where one knows more about homotopy groups of  $M$ ,  $V_1$ , and  $V_2$  (2.1) is useful in deciding whether there exists  $V_1 - V_2$ -connecting geodesics on  $M$ .

Let us finish with some remarks on the case where  $M = \Sigma^n$  is a homotopy sphere.

**COROLLARY 2.6.** *Let  $M = \Sigma^n$  be a homotopy sphere. Then*

(1) *If  $V_1 \cap V_2$  consists only of isolated points there exists non-trivial  $V_1 - V_2$ -connecting geodesics on  $M$ .*

(2) *If there is a  $q < n - 1$  so that  $\pi_q(V_1) \neq 0$  and  $\pi_q(V_2) \neq 0$  then there exists non-trivial  $V_1 - V_2$ -connecting geodesics on  $M$  (more general  $q < l - 1$  if  $M$  is  $(l - 1)$ -connected).*

*Proof.* Since  $\max(\dim V_1, \dim V_2) < n$  we have that  $\pi_q(V_1 \times V_2) \neq 0$  for some  $q < n$ . This together with corollary 2.2 proves (1). To prove (2) we see from (2.1) that  $i_*: \pi_*(V_1 \cap V_2) \rightarrow \pi_*(V_1 \times V_2)$  is an isomorphism for  $* < n - 1$  if there are no non-

trivial  $V_1 - V_2$ -connecting geodesics. If there is a  $q < n - 1$  with  $\pi_q(V_1) \neq 0$  and  $\pi_q(V_2) \neq 0$  this is clearly impossible.

To illustrate that no reasonable general existence results, besides those already mentioned, can be expected, let us give one more example on non-existence of  $V_1 - V_2$ -connecting geodesics.

EXAMPLE. Let  $M = S^3 \subset \mathbf{R}^4$  with standard metric of constant curvature 1 and similar  $V_2 = S^2$  the equator of  $S^3$ . Then for any  $V_1 = S^1$  embedded in  $S^3$  such that  $V_1 \cap (-V_1) = \emptyset$  there are no non-trivial  $V_1 - V_2$ -connecting geodesics. On the other hand we note that if  $V_1 = S^k$ ,  $V_2 = S^l$  and  $V_i \setminus V_1 \cap V_2 \neq \emptyset$ ,  $i = 1, 2$  then there exists non-trivial  $V_1 - V_2$ -connecting geodesics on  $M = S^n$ .

#### REFERENCES

- [1] FLASCHEL, P. and KLINGENBERG, W., *Riemannsche Hilbertmannigfaltigkeiten. Periodische, Geodötische* Lecture Notes in Mathematics 282 (1972, Springer Verlag, Berlin-Heidelberg-NeyYork).
- [2] GROVE, K., *Condition (C) for the energy integral on certain path-spaces and applications to the theory of geodesics*, to appear in J. Diff. Geom.
- [3] —, *Isometry-invariant geodesics*, to appear.
- [4] MORSE, M., *Calculus of Variations in the Large*, Amer. Math. Soc. Colloquium Publications, Vol. XVIII (1934).
- [5] SERRE, J.-P., *Homologie singulière des espace fibrés*, Ann. of Math. 54 (1951), 425–505.

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