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On the Partial Derivatives of Thetafunctions

by HENRIK H. MARTENS

Let $\Pi = (\pi_k^j)$ be a matrix of $2n$ **R**-linearly independent column vectors $\pi_1, \pi_2, \ldots, \pi_{2n}$ in $\mathbf{C}^n$, let G be the group of translations generated by these vectors, and let T be the complex torus $T = \mathbf{C}/G$.

Let $\Lambda = (\lambda_k^j)$ be a matrix of $2n$ column vectors in $\mathbf{C}^n$, and let γ be a column vector in $\mathbf{C}^{2n}$.

A holomorphic function F, defined on $\mathbf{C}^n$ is said to be *multiplicative of type* (Π, Λ, γ) *over* a subset S of T if it satisfies the relations

$$F(u + \pi_k) = F(u) \exp 2\pi i\,({}^t\lambda_k u + \gamma^k)$$

with $k = 1, 2, \ldots, 2n$, for every $u \in \mathbf{C}^n$ whose projection lies in S. Here the presuperscript t denotes matrix transposition, and u is to be thought of as a column vector.

We refer to Conforto [1] for the standard theory of multiplicative functions. This theory is normally concerned with functions which are multiplicative over all of T, and the presence of the subset S in the preceding definition is unorthodox. The motivation for introducing S, and indeed for writing this paper, is the following simple observation:

0.1 *Let F be multiplicative of type (Π, Λ, γ) over T. Let $\partial_1, \partial_2, \ldots, \partial_k$ be first order partial differential operators with constant coefficients on $\mathbf{C}^n$. Then the* kth *order partial derivative $\partial_1 \ldots \partial_k F$ is multiplicative of type (Π, Λ, γ) over the projection in T of the set of common zeros of F and all its partial derivatives of the form $\partial_{j_1} \ldots \partial_{j_r} F$ where. $j_1 < j_2 < \cdots < j_r$ and $r < k$.*

The result is an immediate consequence of the formula obtained by differentiating both sides of the defining relations.

At the Vth Nordic Summer School in Mathematics in Oslo 1970 I outlined how this observation may be used in the proof of Riemann's vanishing theorems for the thetafunction of a jacobian variety [3]. In this paper I shall give a more careful exposition of the argument which will permit the derivation of additional information about the partial derivatives that appears relevant to the study of jacobian varieties.

My objective is to provide a reasonably elementary analytic approach to the vanishing theorems. An approach using different ideas will be found in J. Lewittes, *Riemann Surfaces and the Theta Function,* Acta Math. *111* (1964), 37–61, and a brief and elegant treatment is given by A. Mayer, *Special Divisors and the Jacobian Variety,* Math. Ann. *153* (1964), 163–167.

I am grateful to A. Mayer for pointing out to me that the present approach to some extent was anticipated by E. B. Christoffel, *Vollständige Theorie der Riemann' schen θ – Function,* Math. Ann. *54* (1901) 347–399.

All of these proofs deal with the case of characteristic zero, i.e. the classical case of Riemann surfaces. It should be emphasised that the validity of the vanishing theorems by no means is restricted to this case. D. Mumford has an unpublished proof of the theorem for all characteristics, and a striking generalization has been obtained by G. Kempf in his Ph.D. thesis at Columbia (see Séminaire Bourbaki, Exposé 417).

1. Jacobian Varieties

If Π is a period matrix for a closed Riemann surface X of genus $g \geqslant 2$, the complex torus constructed with the column vectors of Π is denoted $J(X)$ and referred to as the *jacobian variety* of X.

We then have a *canonical map*

$$\kappa : X \to J(X)$$

determined up to a translation in $J(X)$ by the requirements that κ be holomorphic and that the coordinate differentials $du^1, \ldots, du^g$ of $\mathbf{C}^g$ pull back via κ to a basis $w^1, \ldots, w^g$ for the abelian differentials on X such that

$$\pi_k^j = \int_{\alpha_k} w^j$$

where $\alpha_1, \ldots, \alpha_{2n}$ is a basis for the first integral homology group on X.

The map κ can be extended to a map of divisors by setting

$$\kappa(D) = \sum m_i \kappa(Q_i)$$

when $D = \sum m_i Q_i$. According to Abel's theorem, two positive divisors D_1 and D_2 of the same degree are linearly equivalent if and only if $\kappa(D_1) = \kappa(D_2)$.

We shall rely on [3] for proofs of certain basic results from the theory of jacobian varieties. It will be convenient here, however, to review how properties of linear series on X are reflected in the structure of certain subvarieties of $J(X)$.

We introduce some notation for operations on subsets A and B of a complex torus T. For $v \in T$, the *translate of A by v* will be denoted

$$A_v = \{a + v : a \in A\}.$$

We define

$$A \oplus B = \{a + b : a \in A,\ b \in B\}$$
$$= \bigcup \{A_b : b \in B\}$$

and

$$A \ominus B = \bigcap \{A_{-b} : b \in B\}.$$

Since $u \in A_{-b}$ if and only if $u + b \in A$ we get

$$A \ominus B = \{v : B_v \subset A\}.$$

Clearly

$$(A \ominus B) \ominus C = A \ominus (B \oplus C) = (A \ominus C) \ominus B.$$

We assume chosen, once and for all, a point $P \in X$ and normalize κ by setting $\kappa(P) = 0$. We denote by W^r the set of points in $J(X)$ which are images of positive divisors of degree r, and take W^0 to be $\{0\}$. A positive divisor D of degree $s < r$ has the same image as the divisor $D + (r - s)P$, and hence $W^s \subset W^r$. Translates of W^r are denoted by $W_a^r = (W^r)_a$, and the image of W_a^r under the involution $u \to -u$ will be denoted $-W_a^r$.

We further denote by G_n^r the set of points in $J(X)$ which are images of positive divisors of degree n and (projective) dimension $\geqslant r (= l(D) - 1)$.

To any positive divisor D of degree $g - 1$ there is a positive divisor D' of degree $g - 1$ such that $D + D'$ is canonical. The image of canonical divisors (which are linearly equivalent) will be denoted by K. Thus

$$\kappa(D) = -\kappa(D') + K$$

where, as D varies over all positive divisors of degree $g - 1$, so does D'. Hence

$$G_{g-1}^0 = W^{g-1} = -W_{-K}^{g-1}$$

More generally, if D and D' are positive divisors such that $D + D'$ is canonical, then the Riemann-Roch theorem in Brill-Noethers symmetric version states that

$$\deg(D) - 2l(D) = \deg(D') - 2l(D')$$

Hence, in the equation

$$\kappa(D) = -\kappa(D') + K$$

as D varies over positive divisors of degree $g - 1 + r$ and dimension $\geqslant s \geqslant r$, D' varies over positive divisors of degree $g - 1 - r$ and dimension $\geqslant (s - r)$, and conversely.

Hence, for $0 \leqslant r \leqslant s$,

$$G^s_{g-1+r} = (-G^{s-r}_{g-1-r})_K$$

In particular if $s = r$ we have $G^0_{g-1-r} = w^{g-1-r}$ whence

$$G^r_{g-1+r} = -W^{g-1-r}_{-K}$$

Now, let D be a positive divisor of degree $r+s$. Then a necessary and sufficient condition for D to be of dimension $\geqslant r$ is that every positive divisor D' of degree r determines at least one positive divisor D'' of degree s such that $D' + D''$ is linearly equivalent to D. Thus, in the equation

$$-\kappa(D') + \kappa(D) = \kappa(D'')$$

as D' varies over positive divisors of degree r, the right hand side takes values in W^s. If we set $d = \kappa(D)$, the condition may then be rephrased

$$d \in G^r_{r+s} \Leftrightarrow (-W^r)_d \subset W^s$$

whence

$$G^r_{r+s} = W^s \ominus (-W^r)$$

Comparing with previous formulas we have

$$W^{g-1} \ominus (-W^r) = G^r_{g-1+r} = -W^{g-1-r}_{-K}$$

Using the equation $W^{g-1} = -W^{g-1}_{-K}$ we get

$$W^{g-1} \ominus W^r = W^{g-1-r}$$

which may be rephrased as

$$W^r_a \subset W^{g-1} \Leftrightarrow a \in W^{g-1-r}$$

Since clearly $W^r \oplus W^s = W^{r+s}$, we see that for $0 \leqslant r \leqslant s \leqslant g-1$

$$W^r_a \subset W^s \Leftrightarrow W^{r+t}_a \subset W^{g-1}$$

where $t = g-1-s$. Hence $a \subset W^{s-r}$, and since the inclusion $W^r_a \subset W^s$ is trivial for $a \in W^{s-r}$ we have

$$W^r_a \subset W^s \Leftrightarrow a \in W^{s-r}.$$

From the preceding it follows immediately that if $r \leqslant g-1$ then $W_a^r = W^r$ only if $a=0$, and hence all translates of such W^r are distinct. In particular, if $W_a^{g-1} = -W_a^{g-1}$ then we have $W^{g-1} = -W_{2a}^{g-1} = W_{-K}^{g-1}$ whence $2a = -K$. Thus

$$W_a^{g-1} = -W_a^{g-1} \Leftrightarrow 2a = -K$$

The right hand side equation has 2^{2n} solutions.

2. Riemann's Vanishing Theorems

Riemann's vanishing theorems show how some of the subsets discussed in the preceding section can be described by means of the thetafunction of $J(X)$. A fundamental theorem in this connection is the following, which goes back to Riemann:

2.1. *Let Π be a period matrix formed with a canonical homology basis of the closed Riemann surface X of genus $g \geqslant 2$. Let F_1 and F_2 be multiplicative over W^1 in $J(X)$ of types (Π, Λ, γ_1) and (Π, Λ, γ_2), respectively. Assuming that F_1 and F_2 do not vanish identically over W^1, they induce non-negative divisors D_1 and D_2 on X via κ such that*

$$\kappa(D_1) - \kappa(D_2) = p\Pi J(\gamma_1 - \gamma_2)$$

and

$$\deg(D_1) = \deg(D_2) = \tfrac{1}{2}\mathrm{Tr}(JN)$$

where

p *is the projection* $C^g \to J(X)$

J *is the matrix* $\begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$

E *is the unit* $g \times g$ *matrix*

$N = {}^t\Pi\Lambda - {}^t\Lambda\Pi$ *is the characteristic matrix of* F_1 *and* F_2.

A proof of this theorem is given in [3].

When dealing with jacobian varieties it is convenient to assume that Π is in normal form

$$\Pi = (\pi i E \; A)$$

where A is symmetric with negative definite real part (Riemann's relations). This can be arranged by a proper choice of coordinates for $J(X)$ (i. e. by a proper choice of the basis $w^1, \ldots, w^g$ for the abelian differentials). One then defines the thetafunction

$$\theta(u, A) = \sum_{m \in \mathbf{Z}^g} \exp {}^t m(Am + 2u)$$

which can be shown to be multiplicative over $J(X)$ of type (Π, Λ, γ) with

$$\Lambda = \left(0 \frac{i}{\pi} E\right)$$

$$\gamma^k = \begin{cases} 0 & k = 1, \dots, g \\ a_l^l & k = g + l \end{cases}$$

This gives $N = -J$ and $\frac{1}{2}\mathrm{Tr}(JN) = g$.

Let $\hat{a}$ be a point in $\mathbf{C}^g$ with projection $a \in J(X)$. Let $\theta_{\hat{a}}$ denote the translate of θ by $\hat{a}$ defined by

$$\theta_{\hat{a}}(u; A) = \theta(u - \hat{a}, A).$$

Then an easy calculation shows that $\theta_{\hat{a}}$ is multiplicative over $J(X)$ of type $(\Pi, \Lambda, \gamma - {}^t\Lambda\hat{a})$, where Λ and γ are as above.

Let now $\theta_{\hat{a}}$ and $\theta_{\hat{b}}$ be two translates of θ which do not vanish identically over W^1. They then induce divisors on X via κ which only depend on a and $b \in J(X)$. Denoting these divisors by D_a and D_b, theorem 2.1 gives

$$\begin{aligned} \kappa(D_a) - \kappa(D_b) &= p\Pi J^t\Lambda(\hat{b} - \hat{a}) = p(\hat{a} - \hat{b}) \\ &= a - b \end{aligned}$$

since $\Pi J^t \Lambda = -E$. The degree of the divisors is g.

We may note that the thetafunction as defined is not identically zero. This can easily be seen by substituting $e^{2u^j} = z_j$ and observing that we then get a Laurent series in the z_j with non-vanishing coefficients. The fact that the thetafunction does have zeros follows from the above result since $g \neq 0$. It is then clear that almost all translates of θ do not vanish identically over W^1.

Selecting a translate of θ which does not vanish identically over W^1, we can use it as a reference and prove

2.2. *There exists a constant $c \in J(X)$ such that if $\theta_{\hat{a}}$ does not vanish identically over W^1, then it induces a divisor D_a on X of degree g such that*

$$\kappa(D_a) = a + c.$$

If $\theta_{\hat{a}}$ and all its partial derivatives of order $< r$ vanish identically over W^1, then any r-th order partial derivative of $\theta_{\hat{a}}$ which does not vanish identically over W^1 induces a divisor D_a on X of degree g such that

$$\kappa(D_a) = a + c.$$

The statement about the partial derivatives follows from 0.1 and 2.1, since the assumption insures that the derivatives are multiplicative over W^1 of the same type as $\theta_{\hat{a}}$.

Selecting an arbitrary point $u \in J(X)$ and setting $a = u - c$, we get a divisor of degree g whose images is u, for it cannot happen that $\theta_{\hat{a}}$ and all its partial derivatives of all orders vanish identically over W^1. This shows that $W^g = J(X)$ and is Riemann's solution to the Jacobi inversion problem.

In the case where $\theta_{\hat{a}}$ vanishes identically over W^1, the divisor induced by the partial derivatives will not be unique, but any two such divisors must be linearly equivalent. This agrees with the fact that the quotient of two such derivatives defines a meromorphic function on X. We shall get a more precise result on this point:

Let x be any point of W^1. Then $x = \kappa(Q)$ for some $Q \in X$. Let z be a local coordinate near Q with $z(Q) = 0$. The coordinate functions $u^1, \ldots, u^g$ of $\mathbf{C}^g$ serve as local coordinates near $x \in J(X)$ and via κ can be pulled back as holomorphic functions $\hat{u}^j$ of z. The differentials

$$\frac{d\hat{u}^j}{dz} dz \qquad j = 1, 2, \ldots, g$$

then give a local representation of the basis $w^1, \ldots, w^g$. The partial differential operator

$$\partial_x = \sum \frac{d\hat{u}^j}{dz}(0) \frac{\partial}{\partial u^j}$$

can also be viewed as an operator on $J(X)$ and will then be tangent to W^1 at x. If $x_1, \ldots, x_g$ are points of W^1 corresponding to distinct points $Q_1, \ldots, Q_g$ on X, let $z_1, \ldots, z_g$ be local parameters centered at the Q_j and let $\hat{u}_j^k$ be the pullbacks of the coordinate functions at Q_j. Then the operators $\partial_{x_1}, \ldots, \partial_{x_g}$ will form a basis for the first order partial differential operators on $\mathbf{C}^g$ provided the determinant

$$\left| \frac{\partial \hat{u}_j^k}{dz_j}(0) \right|$$

is different from zero. Since the entries are local representations of the basis for the abelian differentials on X, the vanishing of the determinant i precisely the condition that the divisor $Q_1 + \cdots + Q_g$ have positive dimension. Thus the operators form a basis for almost all g-tuples $x_1, \ldots, x_g$, i.e. for all g-tuples $Q_1, \ldots, Q_g$ except for a set of positive co-dimension in $X \times \cdots \times X$ (g times).

Let u be an arbitrary point in $\mathbf{C}^g$, and let f be holomorphic in a neighborhood of u. Let $x_1, \ldots, x_r$ be points of W^1 corresponding to points $Q_1, \ldots, Q_r$ of X. With a simplified notation, we have

$$\partial_{x_1} \ldots \partial_{x_r} f(u) = \sum_k \frac{d\hat{u}^{k_1}}{dz_1}(0) \ldots \frac{d\hat{u}^{k_r}}{dz_r}(0) \frac{\partial}{\partial u^{k_1}} \cdots \frac{\partial}{\partial u^{k_r}} f(u).$$

The expression

$$\sum_k \frac{d\hat{u}^{k_1}}{dz_1}(z_1)\cdots\frac{d\hat{u}^{k_r}}{dz_r}(z_r)\frac{\partial}{\partial u^{k_1}}\cdots\frac{\partial}{\partial u^{k_r}}f(u)$$

defines, for fixed u, a holomorphic function of $(z_1,\ldots,z_r)$. Hence, if $\partial_{x_1}\ldots\partial_{x_r}f(u)\neq 0$, the expression remains different from zero for all r-tuples $(z_1,\ldots,z_r)$ in a neighborhood of $(0,\ldots,0)$. The expression, however, corresponds to a partial derivative of the form

$$\partial_{x'_1}\cdots\partial_{x'_r}f(u)$$

where $x'_1,\ldots,x'_r$ are points in W^1 corresponding to an r-tuple $(Q'_1,\ldots,Q'_r)$ near $(Q_1,\ldots,Q_r)$ in $X\times\cdots\times X$ (r times). If not all partial derivatives of f of order r vanish at u, we can obviously find some r-tuple of points $x_1,\ldots,x_r$ such that

$$\partial_{x_1}\cdots\partial_{x_r}f(u)\neq 0$$

since operators of this type form a basis. Then the above argument and analytic continuation on $X\times\cdots\times X$ (r times) shows

2.3. *If f is holomorphic near $u\in\mathbf{C}^n$, and if some* r-th *order partial derivative of f is different from zero at u then*

$$\partial_{x_1}\cdots\partial_{x_r}f(u)\neq 0$$

except for r-tuples $(x_1,\ldots,x_r)$ corresponding to a subset of positive codimension in $X\times\cdots\times X$ (r times).

Consider now the case when $\theta_{\hat{a}}$ vanishes identically over W^1. Assume that some first order partial derivative of $\theta_{\hat{a}}$ does not vanish identically over W^1. Then $\partial_x\theta_{\hat{a}}$ will not vanish identically over W^1 except possibly for a finite number of $x\in W^1$, and $\partial_x\theta_{\hat{a}}$ will be multiplicative over W^1 of the same type as $\theta_{\hat{a}}$. Since ∂_x is tangent to W^1 at x, $\partial_x\theta_{\hat{a}}$ will have a zero over x, and hence will induce a divisor on X of the form $D+Q$ where D is of degree $g-1$ and $\kappa(Q)=x$. Then

$$a+c=\kappa(D)+\kappa(Q)$$

and since Q can vary over an infinite subset of X, we must have

$$a+c\in G_g^1.$$

2.4. *If $\theta_{\hat{a}}$ and all its partial derivatives of order $<r$ vanish identically over W^1, then*

$$a+c\in G_g^r.$$

To see this, we assume that some partial derivative of order r does not vanish identically over W^1. If this assumption is violated, the assumption of the theorem will hold for $s > r$, and if we can prove the theorem for s, it will hold a fortiori for r, since $G_g^s \subset G_g^r$ when $s > r$.

By assumption we then conclude that $\partial_{x_1} \dots \partial_{x_r} \theta_{\hat{a}}$ is multiplicative over W^1 of the same type as $\theta_{\hat{a}}$ and fails to vanish identically over W^1 for almost all r-tuples $(x_1, \dots, x_r)$ of points in W^1. But $\partial_{x_2} \dots \partial_{x_2} \theta_{\hat{a}}$ is multiplicative and vanishes identically over W^1. Since ∂_{x_1} is tangent to W^1 at x_1, $\partial_{x_1}(\partial_{x_2} \dots \partial_{x_r} \theta_{\hat{a}})$ must be zero over x_1. Since the partial differential operators may be permuted arbitrarily, we conclude that $\partial_{x_1} \dots \partial_{x_r} \theta_{\hat{a}}$ is zero over x_k for each k, and hence induces a divisor on X of the form $D + Q_1 + + \dots + Q_r$ where D is of degree $g - r$. It is a corollary of the argument that $r \leqslant g$.

We then have

$$a + c = \kappa(D) + \kappa(Q_1 + \dots + Q_r)$$

and conclude that $a + c - w \in W^{g-r}$ where $w = \kappa(Q_1 + \dots + Q_r)$. We have already said enough to insure that $a + c \in G_g^1$. If we knew that W^r is an analytic subvariety and irreducible, we could conclude

$$(-W^r)_{a+c} \subset W^{g-r}$$

and hence $a + c \in G_g^r$, from the fact that w can vary over all of W^r except a subset of positive codimension by 2.3. This argument will be justified by our next proof.

From the definition of θ, we see that it is symmetric, i.e. $\theta(u; A) = \theta(-u; A)$. Hence if $\theta(\hat{a}; A) = 0$, then $\theta_{\hat{a}}$ is zero over the origin. Assuming this to be the case, we conclude that if $\theta_{\hat{a}}$ does not vanish identically over W^1 it induces a divisor on X of degree g of the form $D + P$ with $\deg(D) = g - 1$. Hence

$$a + c = \kappa(D + P) \in W^{g-1}.$$

If $\theta_{\hat{a}}$ vanishes identically over W^1, then $a + c \in G^1 g \subset W^{g-1}$ by the first part of the proof of 2.4.

Assume on the other hand that a is such that $a + c \in W^{g-1}$ and $a + c \notin G_g^1$. Then $a + c$ is the image of a unique positive divisor of the form $D + P$, with $\deg(D) = g - 1$, and hence this must be the divisor induced by $\theta_{\hat{a}}$ on X. $\theta_{\hat{a}}$ cannot vanish identically over W^1 since this would imply $a + c \in G_g^1$. But then $\theta_{\hat{a}}$ must be zero over $\kappa(P) = 0$, and hence $\theta(\hat{a}; A) = 0$. W^{g-1} and in general W^r is the image under a holomorphic map of the $(g-1)$-fold (r-fold) cartesian product of X with itself, and hence irreducible. The preimage of G_g^1 is a set of positive codimension in the product, and hence we have shown that $\theta(; A) = 0$ for all a such that $a + c \in W^{g-1}$.

2.5. *The divisor induced by the thetafunction on $J(X)$ is precisely a translate W^{g-1}_{-c} of W^{g-1}, where $2c = K$.*

We have in fact proved that the divisor is W^{g-1}_{-c}, and it only remains to observe that the symmetry of the thetafunction implies $W^{g-1}_{-c} = -W^{g-1}_{-c}$, whence $2c = K$.

With 2.5. W^{g-1} is established as an analytic subvariety of $J(X)$. All the sets W^r and G^r_n can be obtained by intersecting suitable translates of W^{g-1}, and are therefore analytic subvarieties defined as common zeros of translates of the thetafunction. This justifies the final part of the proof of 2.4.

Let $S^r(\theta)$ denote the set of points of $J(X)$ over which θ and all its partial derivatives of order $\leqslant r$ vanish. Then 2.5 expresses the equality

$$S^0(\theta) = W^{g-1}_{-c}$$

or

$$S^0(\theta_{\hat{c}}) = W^{g-1}$$

where $\hat{c}$ is a point in $\mathbf{C}^g$ with projection c. The statement of 2.4 may then be expressed as

$$W^1 \subset S^r(\theta_{\hat{a}}) \Rightarrow a + c \in G^{r+1}_g$$

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$$S^r(\theta_{\hat{c}}) = G^r_{g-1}$$

Proof. For $r = 0$ this is a restatement of 2.5. We proceed by induction on r. We note first that

$$\begin{aligned} a \in G^{r+1}_{g-1} &= W^{g-2-r} \ominus (-W^{r+1}) \\ &\Leftrightarrow W^1_a \subset W^{g-1-r} \theta (-W^{r+1}) \\ &\Leftrightarrow W^1_a \subset (W^{g-1-r} \theta (-W^r)) \, \theta (-W^1) \\ &\Leftrightarrow W^1_{a-x} \subset G^r_{g-1} \quad \text{for all} \quad x \in W^1 . \end{aligned}$$

By the induction hypothesis $W^1_{a-x} \subset S^r(\theta_{\hat{c}})$ for all $x \in W^1$. Then all partial derivatives of $\theta_{\hat{c}}$ of order $\leqslant r$ vanish identically over W^1_{a-x}. Any partial derivative of $\theta_{\hat{c}}$ of order $r+1$ is a sum of terms of the form $\partial_x \theta^r_{\hat{c}}$ where $\theta^r_{\hat{c}}$ is a partial derivative of order r and $x \in W^1$. But since ∂_x is tangent to W^1_{a-x} at a, it follows that $\partial_x \theta^r_{\hat{c}}$ vanishes over a. Hence, x being arbitrary

$$a \in S^{r+1}(\theta_{\hat{c}})$$

This proves the inclusion

$$G^r_{g-1} \subset S^r(\theta_{\hat{c}})$$

for all $r \geqslant 0$.

Assume now that $a \in S^1(\theta_{\hat{c}})$. Then, for any $x \in W^1$ the translate $\theta_{\hat{c}-\hat{a}+x}$ will vanish together with its first derivatives at x. Hence, unless it vanishes identically over W^1 it induces a divisor on X of degree g of the form

$$D + 2Q$$

where $\kappa(Q) = x$. Then we have

$$c + c - a + x = K - a + x = \kappa(D) + 2x$$

whence

$$K - a = \kappa(D) + x \in W_x^{g-2}.$$

If this happens for infinitely many $x \in W^1$ we have

$$K - a \in W^{g-2}\theta(-W^1) = G_{g-1}^1$$

and hence

$$a \in G_{g-1}^1$$

by the Brill-Noether formula. If $\theta_{\hat{c}-\hat{a}+x}$ vanishes identically over W^1 for all but a finite number of $x \in W^1$, it vanishes for all x, by continuity. But then

$$W^1 \subset W_{-a+x}^{g-1}$$

and hence

$$a \in W_x^{g-2}$$

for all x, which implies

$$a \in G_{g-1}^1.$$

Thus the theorem holds for $r = 0, 1$.

Suppose now that $a \in S^{r+1}(\theta_{\hat{c}})$. Then a fortiori $a \in S^r(\theta_{\hat{c}})$ and by the induction hypothesis $a \in G_{g-1}^r$. By the argument of the first part of the proof and by induction hypothesis

$$W_{a-x}^1 \subset G_{g-1}^{r-1} = S^{r-1}(\theta_{\hat{c}})$$

for all $x \in W^1$. Hence all partial derivatives of order $\leqslant r-1$ of $\theta_{\hat{c}}$ vanish identically over W_{a-x}^1, and all partial derivatives of order r of $\theta_{\hat{c}}$ are multiplicative over W_{a-x}^1

If, for all $x \in W^1$ all r-th order derivatives vanish identically over W_{a-x}^1, then

$$W_{a-x}^1 \subset S^r(\theta_{\hat{c}}) = G_{g-1}^r$$

by induction hypothesis, and by the first part of the proof

$$a \in G_{g-1}^{r+1}.$$

If $W_{a-x}^1 \not\subset S^r(\theta_{\hat{c}})$ for some $x \in W^1$, then the inclusion can hold for only finitely many x. For then there is a point $y \in W^1$ such that $y + a - x \notin S^r(\theta_{\hat{c}})$, and for fixed y the condition $y + a - x \in S^r(\theta_{\hat{c}})$ is an analytic condition on x. Hence it either holds for all $x \in W^1$ or for at most finitely many. Hence, for all but a finite number of $x \in W^1$ some partial derivative of order r of $\theta_{\hat{c}}$ will not vanish identically over W_{a-x}^1. Then, by the proof of 2.4.

$$W_{a-x}^1 \subset S^{r-1}(\theta_{\hat{c}}) \Leftrightarrow W^1 \subset S^{r-1}(\theta_{\hat{c}-\hat{a}+x})$$
$$\Rightarrow c + c - a + x = K - a + x \in G_g^r.$$

But $a \in S^{r+1}(\theta_{\hat{c}})$, hence any r-th order partial derivative of $\theta_{\hat{c}-\hat{a}+x}$ will induce a multiple zero on W^1 at x. Hence the divisor induced by such a partial derivative on X is of the form $D + 2Q$, and this is independent of the choice of partial differential operator. Hence $2Q$ is a fixed divisor in the linear series of $(D+2Q)$ and hence the dimension of D is r.

We therefore have

$$c + c - a + x = K - a + x = \kappa(D) + 2x$$

whence

$$K - a - x = \kappa(D) \in G_{g-2}^r$$

for all but finitely many, and hence for all $x \in W^1$. But then

$$W_{K-a-x}^1 \subset G_{g-1}^r \qquad \forall x \in W^1$$

whence

$$K - a \in G_{g-1}^{r+1}$$

and

$$a \in G_{g-1}^{r+1}$$

by the Brill-Noether formula. This completes the proof of the vanishing theorems.

We may observe in retrospect that $a \in G_{g-1}^{r+1}$ implies $W_{a-x}^1 \subset G_{g-1}^r = S^r(\theta_{\hat{c}})$ for all x, and hence all partial derivatives of $\theta_{\hat{c}}$ of order $\leqslant r$ do indeed vanish over W_{a-x}^1. This gives the argument an interesting constructive aspect. Assume namely that D is a positive divisor of degree g and dimension $r \geqslant 1$. Then, setting $d = \kappa(D)$, we get $d \in G_g^r$ and hence

$$a = K - d \in G_{g-2}^{r-1}$$

by the Brill-Noether formula. Then

$$W_a^1 \subset G_{g-1}^{r-1} = S^{r-1}(\theta_{\hat{c}})$$

and hence all partial derivatives of $\theta_{\hat{c}}$ of order $\leqslant r-1$ vanish identically over W_a^1. Then the r-th order partial derivatives are multiplicative over W_a^1, and we claim that they do not all vanish identically over W_a^1. For in that case we should have

$$W_a^1 \subset S^r(\theta_{\hat{c}})$$

which implies $W^1 \subset S^r(\theta_{\hat{c}-\hat{a}})$, and hence

$$c + c - a = K - a = d \in G_g^{r+1}$$

by 2.4. But this contradicts the assumption that the dimension of D is precisely r. Thus, for almost all r-tuples $x_1, \ldots, x_r$ of points in W^1, $\partial_{x_1} \ldots \partial_{x_r} \theta_{\hat{c}-\hat{a}}$ does not vanish identically over W^1, and induces a divisor D' on X of degree g such that

$$\kappa(D') = c + c - a = K - a = d$$

Hence D' lies in the linear equivalence class of D, and by varying the r-tuple $(x_1, \ldots, x_r)$ we obtain all divisors D' in the linear equivalence class except those containing the exceptional r-tuples. We claim that the set of r-th order partial derivatives must contain $r+1$ linearly independent elements which do not vanish identically over W^1, for otherwise the equations

$$\sum_j a_j \psi_j(x_k) = 0$$

with r-th order derivatives $\psi_1, \ldots, \psi_s s \leqslant r$ could not be solved simultaneously for r-tuples $(x_1, \ldots, x_r)$ in an r-dimensional neighborhood of $W^1 \times \cdots \times W^1$ (r times). On the other hand, since all these partial derivatives are multiplicative of the same type over W^1, their quotients define meromorphic functions on X, and the quotients of the derivatives in a basis set by any one of them generate $L(D')$ for some D' in the linear equivalence class of D. Thus the restrictions of the partial derivatives to the points over W^1 is a linear space of dimension $r+1$. Hence

2.6. *If D is a positive divisor of degree g and dimension $r \geqslant 1$ on X then the space*

*$L(D)$ of meromorphic functions on X is generated by the functions induced by quotients of r-*th *order partial derivatives of $\theta_{\hat{c}-\hat{a}}$ where $a = K - \kappa(D)$.*

A positive divisor D on X is said to be *special* if there is a positive divisor D' such that $D + D'$ is canonical. If D is of dimension $r \geqslant 1$, then D can be written as

$$D'' + Q_1 + \cdots + Q_r$$

where the dimension of D'' is 0, and the dimension of $D'' + Q_k$ is 1. Moreover $\deg(D'') \leqslant g - 1$. Adding suitable positive divisor to $D'' + Q_k$ we can produce a positive divisor of degree g and dimension 1 whose linear series has the added divisor as a fixed divisor. Using 2.6 we see that the space $L(D'' + Q_k)$ will be generated by the constant function and the meromorphic function induced on X by the quotient of two first order partial derivatives of a suitable translate of the thetafunction. Hence

2.7. *The functions on X associated with special linear series are precisely those that can be expressed by means of linear combinations of functions induced by quotients of first order partial derivatives of translates of the thetafunction.*

We may note that if X is a hyperelliptic Riemann surface, then the functions of the special linear series do not generate the function field of X, by a theorem of Noether (see e.g. [2]). The consequences of this for the theteafunction of $J(X)$ appears unknown.

Riemann's vanishing theorems are concerned with the common zeros of partial derivatives of a given order. Since $\theta_{\hat{c}}$ is the translate of θ which induces the divisor W^{g-1} on $J(X)$, it follows from the theorems that $\theta_{\hat{c}}$ has simple zeros over the nonsingular points of W^{g-1}, and hence that G^r_{g-1} may be characterized as the set of singularities of multiplicity $r+1$ of W^{g-1}. It is inherent in the arguments that there are limitations on the multiplicity of these singularities, since by Clifford's theorem G^r_{g-1} is empty if $2r > g - 1$ and also if $2r = g - 1$ except in the hyperelliptic case (see e.g. [2]). We shall not go into the consequences of the vanishing theorems here, but shall prove one characterization of the zeros of first order derivatives.

2.8. *If $x \in W^1$, then the divisor induced by $\partial_x \theta_{\hat{c}}$ on W^{g-1} is*

$$W^{g-2}_x \cup - W^{g-2}_{x-K}$$

Proof. Let $w = x_1 + \cdots + x_{g-1}$ be a nonsingular point of W^{g-1}. From

$$0 = d\theta_{\hat{c}} = \sum_1^g \frac{\partial \theta_{\hat{c}}}{\partial u^k} du^k$$

which holds on all of W^{g-1} we see that if $\partial\theta_{\hat{c}}/\partial u^1 = 0$ at w, then the differential

$$\sum_{2}^{g} \frac{\partial\theta_{\hat{c}}}{\partial u^k}(w)\, du^k$$

vanishes at w. It is known (see e.g. [3]) that this differential by pull-back then corresponds to the abelian differential on X with zeros at the points corresponding to $x_1, \ldots, x_{g-1}$ (including multiplicities).

If, by a linear change of coordinates, we choose $u^1, \ldots, u^g$ such that $du^2, \ldots, du^g$ correspond to a basis for the abelian differentials vanishing at a point $x \in W^1$, then ∂_x and $\partial/\partial u^1$ coincide except for a constant factor. But then the differential

$$\sum_{2}^{g} \frac{\partial\hat{\theta}_{\hat{c}}}{du^k}(w)\, du^k$$

corresponds to an abelian differential vanishing at the point corresponding to x. Hence its divisor is a $2g-2$ tuple of points one of which corresponds to x, and since w is the image of a $(g-1)$-tuple of these, either w or $K - w$ must lie in W_x^{g-2}. If w is a singular point of W^{g-1} then $w \in W_x^{g-2}$, and the proof is complete.

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