Quadratic Spaces with Few Isometries (Quadratic Forms and Linear Topologies VI)

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HERBERT GROSS AND ERWIN OGG

Herrn Professor Dr. Alexander M. Ostrowski zum 80. Geburtstag gewidmet.

Introduction

What sort of metric automorphisms do always exist on infinite dimensional quadratic spaces? Clearly, we always have the symmetries about (nondegenerate) hyperplanes, the identity **1** of the space, **−1**, and of course finite products of these isometries; they form an invariant subgroup $\mathfrak{J}$ in the full orthogonal group of the space. In the finite dimensional case $\mathfrak{J}$ is already the full orthogonal group. In the infinite case however, $\mathfrak{J}$ usually represents only a negligible part of the orthogonal group associated with the space. In this note we shall show that there are quadratic spaces of arbitrarily large dimension whose full orthogonal groups equal $\mathfrak{J}$. In § 1 we shall describe how to define such spaces over prescribed (non denumerable) base fields.

The spaces E which we shall investigate below share the following property on subspaces F,

$$F \subset E \ \&\ \dim F \geqslant \aleph_0 \rightarrow \dim F^{\perp} < \dim E. \tag{*}$$

In particular, if such a space E is decomposed orthogonally, $E = E_1 \oplus E_2$, then one of the summands E_i necessarily is of finite dimension. Spaces with such few orthogonal splittings are an extreme counterpart to quadratic spaces admitting orthogonal bases. For subspaces F of spaces wich admit orthogonal bases we invariably have $\dim E/F^{\perp} = \dim F$ which sharply contrasts (*). We see in particular that $\dim E \neq \aleph_0$ for all E satisfying (*). The construction given in § 1 yields spaces which actually satisfy the stronger property on subspaces F,

$$F \subset E \ \&\ \dim F \geqslant \aleph_0 \rightarrow \dim F^{\perp} \leqslant \aleph_0. \tag{**}$$

The notion which stands in the center of our discussion of spaces with small orthogonal group $\mathfrak{O}$ in the sense indicated above ($\mathfrak{O} = \mathfrak{J}$) is that of a locally algebraic isometry (§ 2). An isometry T on E is called locally algebraic if T admits for every $x \in E$ a polynomial $f_x(T)$ (with coefficients in the base field of E) that annihilates x, $f_x(T)x = 0$. If f_x does not depend on x we call T algebraic. Theorem 3 of § 2 says that the spaces constructed in § 1 admit locally algebraic isometries only; in other words, there are infinite dimensional (**)-spaces E with property (λ): 'Every isometry on E is locally algebraic'. By means of somewhat complicated examples one can however show that (**) does not, in general, imply (λ) (the converse implication is seen not to be true either by Theorem 3 of §2). Spaces with property (λ) and which, in addition, satisfy (*) absolutely (i.e. which preserve (*) under extensions of the base field) are seen to have trivial quotient $\mathfrak{O}/\mathfrak{J}$ (Corollary 1 of Theorem 3 in §2).

In [3] it is shown that certain spaces constructed in §1 satisfy Witt's cancellation theorem: If $E=E_1\oplus E_2=F_1\oplus F_2$ are orthogonal decompositions of E with E_1 and F_1 isometric, then E_2 and F_2 must be isometric; a rare thing indeed to happen in the infinite dimensional case.

Notations.

Generally speaking, forms $\Phi: E\times E\to k$ are additive in each argument and satisfy $\Phi(\lambda x, y)=\lambda\Phi(x, y)$, $\Phi(x, \lambda y)=\Phi(x, y)\lambda^{\alpha}$ with respect to some fixed involution α (=antiautomorphism of period 2) of the division ring k. We shall however always assume below that k is *commutative*. We shall furthermore assume Φ to be ε-*hermitean*, i.e. $\Phi(y, x)=\varepsilon\Phi(x, y)^{\alpha}$ with $\varepsilon=+1$ (hermitean) or $\varepsilon=-1$ (antihermitean). If α is the identity, then k is necessarily commutative and we speak of symmetric and antisymmetric forms respectively. In any case, '$x\perp y$', defined as usual to be '$\Phi(x, y)=0$', is a symmetric relation. $E^{\perp}$ is called the *radical* of E (rad E). If rad $E=(0)$ we call Φ non-degenerate and – in analogy with algebras – the space (E, Φ) *semisimple*. Φ is said to be *tracevalued* if for every $x\in E$ there is a $\xi\in k$ such that $\Phi(x, x)=\xi+\varepsilon\xi^{\alpha}$. We shall always assume Φ to be tracevalued, a non trivial requirement only when chark $k=2$ ([1] §4, No. 2). We shall make use of Witt's theorem in §2 below ([1] §4, No. 3): Let E be a space with a non degenerate form Φ which is hermitean or anti-hermitean, and tracevalued if it is hermitean. Then any isometry (=vectorspace isomorphism that preserves Φ) between finite-dimensional subspaces can be extended to a isometric automorphism of E.

Let (E, Φ) be an ε-hermitean k-vectorspace with respect to the involution α. Assume that the division ring k' contains k and admits an extension (involution) of α to k'. We know that the abelian group $E'=k'\otimes_k E$ may be regarded as a vectorspace over k and as a vectorspace over k'. The form $\Phi': E'\times E'\to k'$, defined by $\Phi'(\sum\lambda_i\otimes x_i, \sum\mu_j\otimes y_i)=\sum\lambda_i\Phi(x_i, y_j)\mu_j^{\alpha}$ for $\lambda_i, \mu_j\in k'$ is ε-hermitean. We say that Φ satisfies (*), or (**), *absolutely*, if the form Φ' possesses these properties for all extensions k' of k. (E', Φ') is called the k'-*ification* of (E, Φ) or the space obtained from (E, Φ) by extending the ring of scalars.

A space (E, Φ) is called anisotropic if it contains no isotropic elements, i. e. no vectors $x\neq 0$ with $\Phi(x, x)=0$.

Unless stated otherwise, (E, Φ) will be assumed to be of infinite dimension.

§1. The Existence of Spaces with Property **

In this short section we shall describe the construction of infinite dimensional spaces (E, Φ) where Φ is an ε-hermitean form satisfying (**) absolutely.

Let α be an involution of the commutative field k, $\operatorname{card} k > \aleph_0$. Let $X \subset k$ be a maximal subset of algebraically independent elements over the prime field k_0 so that k is an algebraic extension of $k_0(X)$. Let $\varepsilon = +1$ or -1. Since α is of period 2, there is a subset $Y \subset X$ with $\operatorname{card} Y = \operatorname{card} X$ $(=\operatorname{card} k)$ and for every $\eta \in Y$ either $\varepsilon\eta^\alpha = \eta$ or $\varepsilon\eta^\alpha \notin Y$. Let then $(e_\iota)_{\iota \in I}$ be a basis of a k-vectorspace with $\operatorname{card} k \geqslant \operatorname{card} I > \aleph_0$. We define an ε-hermitean form Φ on $E \times E$ as follows: Pick an ordering on I. For all $\iota < \kappa$ in I set $\Phi(e_\iota, e_\kappa) = \varepsilon\Phi(e_\kappa, e_\iota)^\alpha = \eta_{\iota\kappa} \in Y$ such that all elements $\eta_{\iota\kappa}$ $(\iota < \kappa)$ are different. Furthermore $\Phi(e_\iota, e_\iota) = \varepsilon\Phi(e_\iota, e_\iota)^\alpha \in k$ such that no $\Phi(e_\iota, e_\iota)$ equals a $\Phi(e_\iota, e_\kappa)$ with $\iota \neq \kappa$. *We assert that Φ satisfies* (**).

Proof. Let U and V be subspaces of E with $\dim V > \dim U = \aleph_0$, $(u_i)_{i \in \mathbb{N}}$ and $(v_\iota)_{\iota \in J}$ bases of U and V respectively. $u_i = \sum \alpha_{i\kappa} e_\kappa$, $v_\iota = \sum \beta_{\iota\kappa} e_\kappa$ where the first sum extends over the finite set $M_i = \{\kappa \in I \mid \alpha_{i\kappa} \neq 0\}$, the second over the finite set $N_\iota = \{\kappa \in I \mid \beta_{\iota\kappa} \neq 0\}$. Set $M = \bigcup_{\mathbb{N}} M_i$, $N = \bigcup_J N_\iota$. Thus $\operatorname{card} N > \operatorname{card} M = \aleph_0$. Our assertion is proved if we can exhibit a pair $u, v \in U \times V$ with $\Phi(u, v) \neq 0$. Such a pair is found as follows.

(i) X contains a denumerable subset A such that $\{\alpha_{i\kappa} \mid i \in \mathbb{N}, \kappa \in M_i\}$ is contained in the algebraic closure in k of the subfield $k_0(A)$.

(ii) There is a $\varrho_0 \in N \setminus M$ such that

$$A \cap \{\Phi(e_\nu, e_{\varrho_0}), \Phi(e_{\varrho_0}, e_\nu) \mid \nu \in I \setminus \{\varrho_0\}\} = \emptyset .$$

Let $\varrho_0 \in N_{\nu_0}$.

(iii) X contains a finite subset B such that $\{\beta_{\nu_0\mu} \mid \mu \in N_{\nu_0}\}$ is contained in the algebraic closure in k of $k_0(B)$. Since M is infinite, there is a $\kappa_0 \in M$ such that $\Phi(e_{\kappa_0}, e_{\varrho_0}), \Phi(e_{\varrho_0}, e_{\kappa_0}) \notin B$. Let $\kappa_0 \in M_{\iota_0}$.

(iv) Notice that $\kappa_0 \neq \varrho_0$. If $\kappa_0 < \varrho_0$ we let

$$C = \{\Phi(e_\kappa, e_\varrho) \mid (\kappa, \varrho) \in M_{\iota_0} \times N_{\nu_0} \setminus \{(\kappa_0, \varrho_0)\}\};$$

if $\varrho_0 < \kappa_0$ we let

$$C = \{\Phi(e_\varrho, e_\kappa) \mid (\varrho, \kappa) \in N_{\nu_0} \times M_{\iota_0} \setminus \{(\varrho_0, \kappa_0)\}\} .$$

Thus, if $\kappa_0 < \varrho_0$ we see by (ii), (iii), (iv) that $\eta_{\kappa_0\varrho_0} = \Phi(e_{\kappa_0}, e_{\varrho_0}) \notin A \cup B \cup C$; similarly, if $\varrho_0 < \kappa_0$ we have $\eta_{\varrho_0\kappa_0} = \Phi(e_{\varrho_0}, e_{\kappa_0}) \notin A \cup B \cup C$. Thus, if k_1 is the algebraic closure in k of $k_0(A \cup B \cup C)$ we see that $\eta_{\kappa_0\varrho_0} \notin k_1$ if $\kappa_0 < \varrho_0$ and $\eta_{\varrho_0\kappa_0} \notin k_1$ when $\varrho_0 < \kappa_0$. In the first case we consider

$$\Phi(u_{\iota_0}, v_{\nu_0}) = \sum_{(\kappa,\rho)\neq(\kappa_0,\rho_0)} \alpha_{\iota_0\kappa}\beta_{\nu_0\varrho}\,\phi(e_\kappa, e_\varrho) + \alpha_{\iota_0\kappa_0}\,\beta_{\nu_0\varrho_0}\,\eta_{\kappa_0\varrho_0}.$$

If we had $\Phi(u_{\iota_0}, v_{\nu_0})=0$ then we had a nontrivial linear equation for $\eta_{\kappa_0\varrho_0}$ with coefficients in k_1, so $\eta_{\kappa_0\varrho_0}\in k_1$. If $\varrho_0<\kappa_0$ we conclude in the same manner that $\Phi(v_{\nu_0},u_{\iota_0})\neq 0$. Clearly our proof remains valid if we pass to the form Φ' on the k'-ification $E'=k'\otimes_k E$ of E with respect to some overfield k' of k (admitting an extension of α). This proves our assertions. We note our result as

THEOREM 1. *For $\varepsilon=+1$ and for $\varepsilon=-1$ there exist ε-hermitean forms Φ over any commutative field k with given involution and* $\operatorname{card} k>\aleph_0$ *which satisfy* (**) *absolutely; we may choose the dimension of Φ to be* card k.

We had $\operatorname{card} k\geqslant \dim E$ for the spaces E in the above construction. We do not know if this is necessarily so for spaces with property (**). It is easy to see that (**) does imply $(\operatorname{card} k)^{\aleph_0}\geqslant\dim E$. Thus, at least in the special cases where $\operatorname{card} k$ is a beth (e.g. when $k=\mathbb{R}$ or $\mathbb{C}$), (**) does imply $\operatorname{card} k\geqslant\dim E$.

THEOREM 2. *Let $k=k_0(X)$ be a purely transcendental extension of k_0 and* $\operatorname{card} X>\aleph_0$. *If – in the notation of the preceding construction – Φ is chosen symmetric with $\Phi(e_\iota, e_\kappa)=\xi_{\iota\kappa}\in X(\iota, \kappa\in I$ and* $\operatorname{card} I>\aleph_0)$ *such that $\xi_{\iota\kappa}=\xi_{\nu\mu}$ if and only if $\{\iota,\kappa\}=\{\nu,\mu\}$, then $\Phi(x, x)$ is a square in k only when $x=0$.*

This result is proved in [3]; it guarantees the existence of *anisotropic* forms with property (**) over all fields of a certain type. In the special case where k_0 is assumed orderable the part of theorem 2 ruling out isotropic vectors follows directly from Jacobi's diagonalization formula (for finite spaces). It is clear that after extending the base field Φ may admit isotropic vectors. The fact that k is a purely transcendental extension of some k_0 is not however crucial for the existence of an anisotropic Φ over k satisfying (**). We give an example of such a form over $\mathbb{R}$ by specifying a subspace of an infinite separable Hilbertspace (H, Φ) over the reals: Note that the collection of all sets M of linearly independent vectors $x, y, \cdots$ with $\{\Phi(x,y)\mid x,y\in M\}$ algebraically independent over $\mathbb{Q}$ is inductively ordered by inclusion. Let M_0 be a maximal element by Zorn's lemma. If $\operatorname{card} M_0>\aleph_0$, then the restriction of Φ to the span of M_0 satisfies (**) as we have demonstrated above. Assume by way of contradiction that $\operatorname{card} M_0\leqslant\aleph_0$. Let $(x_i)_{i\in J}$ be the elements of M_0 in some ordering, and let $A=\{\Phi(x_i, x_j)\mid i,j\in J\}$. Introduce an orthonormal basis $(e_i)_{i\in J}$ in the span X of the x_i $(i\in J)$, $e_i=\sum\alpha_{ij}x_j$ with (α_{ij}) triangular. Then $(\alpha_{ij})^{-1}=(\beta_{ij})$ is triangular and $\alpha_{ij}, \beta_{ij}\in\overline{\mathbb{Q}(A)}$ (real closure). Since $\operatorname{card} A\leqslant\aleph_0$ we can pick a family $(t_i)_{i\in J}$, the t_i in $\mathbb{R}$ and algebraically independent over $\overline{\mathbb{Q}(A)}$ with $\sum_J t_i^2=t<\infty$. The closure $\bar{X}$ of X in H (in the normtopology of Φ) contains a vector x with $\Phi(x, e_i)=\lambda_i t_i$ for any choice of λ_i with, say, $0<\lambda_i<1$. We have $\Phi(x, x_i)=\sum \beta_{ij}\lambda_j t_j$. It follows that the set $\{\Phi(x, x_j) \mid j\in J\}$ is algebraically independent over $(\mathbb{Q}A)$ for λ_i rational. If we can arrange for $\Phi(x, x)=\sum (\lambda_i t_i)^2$ to be outside $\overline{\mathbb{Q}(A\cup(t_i)_J)}$ we have the desired contradiction: $M_0\cup\{x\}$ contradicting the maximality of M_0. Now if J should be finite, then $\bar{X}=X$ and we may, if necessary, pass from x to a vector $x+y$ with $y\in X^\perp$ and $\Phi(y, y)=\alpha-\Phi(x, x)$ and suitably chosen α. If $\operatorname{card} J=\aleph_0$, then by varying the *rational* λ_i in the open unit interval we can arrange for $\Phi(x, x)$ to be any real number of the open interval $[0, t]$. Clearly then, there is a choice with $\Phi(x, x)$ outside the denumerable $\overline{\mathbb{Q}(A\cup(t_i)_J)}$. Q.E.D. We can do the same for hermitean forms over a complex Hilbert space. Thus

THEOREM 3. *There exist (infinite) positive definite symmetric (hermitean) forms over* $\mathbb{R}(\mathbb{C})$ *which satisfy* (**) *absolutely.*

Remark. We briefly indicate how to construct spaces which satisfy (**) but not absolutely so. Let k be nondenumerably infinite. Let $(f_\iota)_{\iota\in I}$, $(g_\iota)_{\iota\in I}$ be bases of k-vectorspaces F and G respectively, $\operatorname{card} I=\operatorname{card} k$. Choose subsets X and Y of k with $X\cap Y=\emptyset$ and $X\cup Y$ algebraically independent over the primefield k_0 of k. Define a symmetric bilinear form Φ on $E=F\oplus G$ as follows: $\Phi(f_\iota, f_\kappa)=-\Phi(g_\iota, g_\kappa)=\xi_{\iota\kappa}$. $\Phi(f_\iota, g_\kappa)=\Phi(f_\kappa, g_\iota)=\eta_{\iota\kappa}$ with $\xi_{\iota\kappa}\in X$, $\eta_{\iota\kappa}\in Y$ and $\xi_{\iota\kappa}=\xi_{\nu\mu}$ and $\eta_{\iota\kappa}=\eta_{\nu\mu}$ if and only if $\{\iota, \kappa\}=\{\nu, \mu\}$. If k is assumed orderable, then the reader proves by the method illustrated above that $E=F\oplus G$ satisfies (**). However, over the extension $k(\sqrt{-1})$ E decomposes orthogonally, $E=H\oplus L$ with H spanned by all $f_\iota+\sqrt{-1}\cdot g_\iota(\iota\in I)$ and L spanned by all $f_\iota-\sqrt{-1}\cdot g_\iota(\iota\in I)$.

§2. The Orthogonal Group

In this section we study the orthogonal group $\mathfrak{O}$ associated with certain infinite dimensional spaces (E, Φ) which satisfy (*). Here Φ will always be symmetric or anti-symmetric and tracevalued if it is symmetric.

Consider an isometry T such that there is an orthogonal decomposition $E=E_0\oplus E_1$ with $\dim E_1<\infty$ and $T=\pm\mathbf{1}$ on E_0. Any isometry T with $\operatorname{Ker}(T-\mathbf{1})$ or $\operatorname{Ker}(T+\mathbf{1})$ of finite codimension in E admits such an orthogonal decomposition of E. The set $\mathfrak{J}$ of all such isometries T is an invariant subgroup of the orthogonal group $\mathfrak{O}$ associated with the space E; it contains the subgroup $\mathfrak{J}_0$ of index $\leqslant 2$ of all T which are the identity on almost all of E. For symmetric Φ and $\operatorname{char} k \neq 2$ [2] gives a detailed account of $\mathfrak{J}_0$; in that case $\mathfrak{J}_0$ is generated by all symmetries about semisimple hyperplanes. We shall show that for prescribed natural $n>1$ there are infinite spaces (E, Φ) with $\mathfrak{O}/\mathfrak{J}_0$ isomorphic to a product of n copies of $\mathbb{Z}_2$ (characteristic not 2).

It is natural to expect, that spaces with few orthogonal splittings in the sense of (**) admit 'few' isometries. A confirmation of this expection is provided by the first two theorems.

THEOREM 1. *If (E, Φ) satisfies (*), then every isometry on E is determined modulo a factor from $\mathfrak{J}_0$ by its action on any subspace of denumerably infinite dimension.*

THEOREM 2. *If (E, Φ) satisfies (*), and this absolutely so when the base field is not algebraically closed, then every locally algebraic isometry belongs to the group $\mathfrak{J}$ associated with (E, Φ).*

Proof of Theorem 1. Assume first that E is semisimple. For $\lambda \neq 0$ an element of the basefield k let $X(\lambda)$ be the eigenspace $\ker(T-\lambda 1)$ of the isometry T of E. $X(\lambda) \perp X(\mu)$ if $\lambda\mu \neq 1$. Thus we cannot have $\dim X(\lambda) = \dim E$ unless $\lambda^2 = 1$ by (*). $\mathrm{Im}(\lambda T - 1) \subset \subset X(\lambda)^{\perp}$ and $\mathrm{Ker}(\lambda T - 1) = X(\lambda^{-1})$ so

$$\dim E/X(\lambda^{-1}) \leqslant \dim X(\lambda)^{\perp}. \tag{1}$$

Assume that for some subspace U of E we have $T|_U = \mathbf{1}_U$, $\dim U = \aleph_0$. Since T preserves Φ we conclude that $\mathrm{Im}(T - \mathbf{1})$ is contained in $U^{\perp}$ and thus of dimension smaller than $\dim E$. Hence we must have $\dim X(1) = \dim E$ and therefore $\dim X(1)^{\perp} < \infty$ by (*). Hence $\dim E/X(1) < \infty$ by (1) and therefore $\dim X(1)^{\perp} \leqslant \dim E/X(1)$ as E is semisimple. Together with (1) $\dim X(1)^{\perp} = \dim E/X(1) < \infty$. From this we conclude that there exists a subspace $H \subset X(1)$ of finite codimension in E with $E = H \oplus H^{\perp}$. Since T is the identity on H we have $T \in \mathfrak{J}_0$. If E is not semisimple, then $\mathrm{rad}\, E$ is of finite dimension. Let E_0 be a linear complement of $\mathrm{rad}\, E$ in E. We can find T_0 in $\mathfrak{J}_0$ such that $T_0 T(E_0)) \subset E_0$. Since radicals are mapped onto themselves under isometries we must have $T_0 T(E_0) = E_0$. By what we have already proved it follows that the restriction of $T_0 T$ to E_0 is determined modulo $\mathfrak{J}_0$ by its action on $U \cap E_0 \cap \mathrm{Ker}(T_0 - 1)$. Hence the same holds for T. Q.E.D.

Proof of Theorem 2. *Case* 1: there is a λ with $\dim X(\lambda) = \dim E$. Hence $\lambda^2 = 1$ and $T \in \mathfrak{J}$ by Theorem 1.

Case 2: $\dim X(\lambda^{-1}) < \dim E$ for all $\lambda \in k \setminus \{0\}$. Thus $\dim X(\lambda)^{\perp} = \dim E$ by (1) and so $\dim X(\lambda) < \infty$ for all $\lambda \in k \setminus \{0\}$ by (*). For every member x of a Basis $\mathscr{B}$ of E we let f_x be the annihilating polynomial. f_x splits into linear factors over the algebraic closure k' of k, $f_x = \prod (Z - \lambda_i)$. Every linear factor provides an eigenvalue $\lambda_i \in k'$ of $T' : E' = k' \otimes E \to E'$. Since E' satisfies (*) by the assumptions of the theorem we see that the number l of different λ_i must be less than $\dim E$. Hence there are only $l < \dim E$ different annihilating polynomials $f_x (x \in \mathscr{B})$. We conclude that there is at least one f_x annihilating a subspace $G \subset E$ of dimension $\dim G = \dim E$. Let $f_x = \prod (Z - \lambda_i)$ be the splitting of this very polynomial. If some of the λ_i equal ± 1 we let G_0 be the image of G under the map $\prod_{\lambda_i = \pm 1} (T - \lambda_i 1)$. We have $\dim G_0 = \dim G$ in the present case. Let g be the product of the remaining linear factors $(Z - \lambda)$. Since $\dim G_0 = \dim E$ and since $g(T)$ annihilates G_0 and hence also $G_0' = k' \otimes G_0$, we conclude that the dimension of $\ker(T - \lambda)$ must equal $\dim E$ for at least one $\lambda \neq \pm 1$. This is a contradiction as G_0' satisfies property (*).

COROLLARY. *If* (E, Φ) *is as in Theorem* 2, *then the set of all locally algebraic isometries on E is a group. It coincides with the set of all algebraic isometries on E and it is generated by all* $T\in\mathfrak{O}$ *with* $E/\mathrm{Ker}(T-1)$ *or* $E/\mathrm{Ker}(T+1)$ *finite dimensional; hence it is a normal subgroup of* $\mathfrak{O}$.

LEMMA. *Assume that* $E_1,\ldots,E_n$ *all satisfy* (*) *and that* $\dim E_i>\dim E_{i+1}$ $(i=1,\ldots,n-1)$. *If T is any endomorphism of the orthogonal sum* $E=E_1\oplus\cdots\oplus E_n$ *that preserves orthogonality then the* E_i *are left almost invariant under* T: $\dim(E_i+T(E_i))/E_i$ *is finite for all i.*

Proof. Let $F_1=E_2\oplus\cdots\oplus E_n$. $\mathrm{Dim} E_1>\dim F_1$ so that there is a subspace V_1 of E_1 with $T(V_1)\subset E_1$ and $\dim V_1=\dim E_1$. By the assumptions of the lemma $\dim T(V_1)=\dim E_1$. Call K_1 the projection of $T(F_1)$ onto E_1 (for the decomposition $E=E_1\oplus F_1$). $T(V_1)\perp K_1$ hence K_1 and $(F_1+T(F_1))/F_1$ are finite dimensional. Setting $F_2=E_3\oplus\cdots\oplus E_n$ we have $\dim E_2>\dim F_2$. As $F_1+T(F_1))/F_1$ is finite dimensional we conclude that there exists $V_2\subset E_2$ with $T(V_2)\subset E_2$ and $\dim V_2=\dim E_2$. It is now clear how the argument may be repeated in order to conclude that there exist spaces $V_i\subset E_i$ with $T(V_i)\subset E_i$ and $\dim V_i=\dim E_i$. Let then K_{ij} be the projection of $T(E_i)$ on E_j. $K_{ij}\perp V_j$ for all $i\neq j$. Since $\dim T(V_i)=\dim E_i$ by the choice of the V_i and by the assumptions of the lemma, we conclude that K_{ij} is finite dimensional for all pairs $i\neq j$. This is what the lemma asserts.

We now consider the orthogonal sum of finitely many spaces (E_i, Φ_i) of the kind constructed in §1. For the sake of simplicity we choose Φ_i symmetric: For $i=1,2,\ldots n$ let $(e^i_\iota)_{\iota\in J(i)}$ be a basis of E_i, $\Phi_i(e^i_\iota, e^i_\nu)=\xi^i_{\iota\nu}$ where $\xi^i_{\iota\nu}=\xi^i_{\mu\kappa}$ if and only if $\{\iota,\nu\}=\{\mu,\kappa\}$ and where, for every fixed i, the set X^i of all $\xi^i_{\iota\nu}$ $\{\iota, \nu\in J(i)\}$ is algebraically independent over the prime field k_0 of the basefield k. We shall *not* assume that the sets $X^1,\ldots,X^n$ are disjoint. For these symmetric spaces we prove

THEOREM 3. *Assume that* $\dim E_i>\dim E_{i+1}>\aleph_0$ $(i=1,\ldots,n-1)$. *Then every isometry of the orthogonal sum* $E=E_1\oplus\cdots\oplus E_n$ *is locally algebraic.*

Proof. For the sake of simplicity we omit the superscript 1 when mentioning e^1_ι and $x^1_{\iota\nu}$; furthermore let $J(1)=J$. Let us study the action of T on E_1 for T an isometry of E:

$$Te_\iota=\sum_J \alpha_{\iota\mu}e_\mu+g_\iota,\quad \text{where}\quad g_\iota\in E_2\oplus\cdots\oplus E_n$$

By the previous lemma, $G=k(g_\iota)_{\iota\in J}$ is of finite dimension. Let $Q\subset J$ be such that $(g_\iota)_{\iota\in Q}$ is a basis of G. We introduce the finite sets $M(\iota)=\{\mu\in J\mid \alpha_{\iota\mu}\neq 0\}$. Let $M=\bigcup_{\iota\in J}[M(\iota)\backslash\{\iota\}]$. We show that M is finite. Assume by way of contradiction that M is infinite. There is a denumerably infinite subset $S\subset J$ and a map κ that assigns to every $\iota\in S$ a $\kappa(\iota)\in J$ with $\kappa(\iota)\in M(\iota)\backslash\{\iota\}$ and $\kappa(\iota)\neq\kappa(\nu)$ for all $\iota\neq\nu$ in S. There is a subset A of $X^1 \cup \cdots \cup X^n$ with card $A \leqslant \aleph_0$ such that $\alpha_{\iota\kappa} \in \overline{k_0(A)}$ (the algebraic closure of $k_0(A)$ in k) for all $\kappa \in M(\iota)$, $\iota \in S$ and $\Phi(g_\iota, g_\kappa) \in \overline{k_0(A)}$ for all $\iota \in S$, $\kappa \in Q$. Let $N = \bigcup_{\iota \in S} M(\iota)$. card $N = \aleph_0$. There is a $\nu \in J$ and for it a $\mu_0 \in M_\nu$ such that $\mu_0 \notin N \cup S$ and

$$\{\xi_{\iota\mu_0} \in X \mid \iota \in N\} \cap A = \emptyset \tag{2}$$

For $Te_\nu = \sum_{\mu \in M(\nu)} \alpha_{\nu\mu} \cdot e_\mu + \sum_{\kappa \in Q} \beta_{\nu\kappa} g_\kappa$ we have

$$(Te_\iota, Te_\nu) = \xi_{\iota\nu} = \alpha_{\iota\kappa(\iota)} \alpha_{\nu\mu_0} \xi_{\kappa(\iota)\mu_0} + \sum_{\kappa\mu} \alpha_{\iota\kappa} \alpha_{\nu\mu} \xi_{\kappa\mu} + \sum_{\kappa \in Q} \beta_{\mu\kappa} \Phi(g_\iota, g_\kappa) \tag{3}$$

The first sum in (3) extends over the set $[M(\iota) \times M(\nu)] \setminus \{\kappa(\iota), \mu_0\}$. There is a finite subset B of $X^1 \cup \cdots \cup X^n$ such that $\alpha_{\nu\mu} \in \overline{k_0(B)}$ for all $\mu \in M(\nu)$ and $\beta_{\nu\kappa} \in \overline{k_0(B)}$ for all $\kappa \in Q$. Since S is infinite, there is a $\sigma \in S$ with $\xi_{\kappa(\sigma)\mu_0} \notin B$. As $\kappa(\sigma) \neq \sigma$ by the choice of the map κ and since $\mu_0 \neq \sigma$ we have $\xi_{\sigma\nu} \neq \xi_{\kappa(\sigma)\mu_0}$. Let $C = A \cup B \cup \{\xi_{\sigma\nu}, \xi_{\kappa\mu} \mid (\kappa, \mu) \in [M(\sigma) \times M(\nu)] \setminus (\kappa(\sigma), \mu_0)\}$. By (2) we have $\xi_{\kappa(\sigma)\mu_0} \notin A$, hence $\xi_{\kappa(\sigma)\mu_0} \notin C$. All quantities in equation (3) equated for $\iota = \sigma$ are contained in $\overline{k_0(C)}$ with the exception of $\xi_{\kappa(\sigma)\mu_0}$. The coefficient of $\xi_{\kappa(\sigma)\mu_0}$ in (3) is not zero. Hence we should have $\xi_{\kappa(\sigma)\mu_0} \in \overline{k_0(C)}$; so $\xi_{\kappa(\sigma)\mu_0}$ is algebraically dependent over C which is a contradiction. We have thus shown that M is finite. G being finite dimensional, there is a subspace F_1 of E, spanned by finitely many e^i_ι, $\iota \in J(i)$, $i = 1, \ldots, n$ such that $Te^1_\nu \in k(e^1_\nu) + F_1$ for all $\nu \in J(1)$. In the same manner we find for $i = 2, \ldots, n$ finite dimensional spaces F_i such that $Te^i_\nu \in k(e^i_\nu) + F_i$. Set $F = \sum_{i=1}^n F_i$. We have $Te^i_\mu \in k(e^i_\mu) + F$ for all $\mu \in J(i)$ and all $i = 1, \ldots, n$. In particular $T(F) \subset F$. Since F is finite dimensional we conclude that T is locally algebraic on all basis vectors e_μ and hence locally algebraic on each $x \in E$. Q.E.D.

Let us look at the proof for one more moment. We have shown that there is a subspace F of E, spanned by finitely many of the basisvectors e^i_ι such that $Te \in k(e) + F$ for all basis vectors $e = e^i_\iota$. Hence F is the orthogonal sum of its projections onto the summands E_i in the decomposition $E = E_1 + \cdots + E_n$. These projections, say G_i, are semissimple (as are all spans of collections of basisvectors of our particular bases $(e^i_\iota)_{\iota \in J(i)}$, $(i = 1, \ldots, n)$). Therefore $E_i = G_i \oplus (G_i^\perp \cap E_i)$. Since $T(F) = F$ it follows that the spaces $G_i^\perp \cap E_i$ are left invariant under T. If we extend $T^{-1}|_F$ to an isometry T_0 on E by letting T_0 act as the identity on $F^\perp$ we have $T_0 \in \mathfrak{J}_0(E)$ and $T_0 \circ T$ leaves each summand E_i of E invariant. The restriction of $T_0 \circ T$ to E_i is locally algebraic. Hence if char $k \neq 2$ then we see by Theorem 2 that these restrictions are, up to a factor ± 1, a product of finitely many symmetries. We have thus shown that we can find altogether finitely many symmetries S on E such that $T_0 \circ T \circ \prod S$ acts on each E_i as $\mathbf{1}_{E_i}$ or $-\mathbf{1}_{E_i}$. Since $T_0 \in \mathfrak{J}_0(E)$ we obtain the

COROLLARY. *Let* $E = E_1 \oplus \cdots \oplus E_n$ *be as in Theorem* 3 *and* char $k \neq 2$. *The* *quotient group* $\mathfrak{O}/\mathfrak{J}_0$ *of the full orthogonal group of E modulo the invariant subgroup* $\mathfrak{J}_0$ *is isomorphic to the direct product of n copies of* $\mathbb{Z}_2$. *In particular, if* $n=1$, *then* $\mathfrak{O}/\mathfrak{J}$ *is trivial.*

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