The Signature Defect and the Homotopy of BPL and PL/O

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1. Introduction

Several years ago we defined an invariant of framed cobordism and used it to obtain information about the homotopy groups of BPL and PL/O. Since the results on BPL and PL/O were obtained independently by Brumfiel and published in [3], we intended to include our results in a lengthy paper on almost-closed manifolds. As this paper has not appeared, and since the cobordism invariant has proved interesting in other contexts, we have decided to present these results separately.

First, then, we will define our invariant. Let Ω^F_{4k-1} be the cobordism group of framed $(4k-1)$-manifolds. Every element of Ω^F_{4k-1} can be represented by a framed exotic sphere (Σ, f). (If $k>1$, this is proved using surgery [11]. If $k=1$, every element of Ω^F_3 van be represented by a framing of the standard sphere S^3 because the J-homomorphism is surjective.) Moreover, since every framed manifold bounds in oriented cobordism, we can find a compact, oriented manifold M^{4k} with $\partial M=\Sigma$. Let M^* be the closed piecewise-linear manifold $M \cup \text{Cone } \Sigma$. Using the framing on Σ, we may extend the stable normal bundle ν_M to a vector bundle γ on M^*. Consider the expression

$$L_k(-\gamma)[M^*]-\sigma(M),$$

where L_k is the Hirzebruch L-genus, $(-\gamma)$ is the stable inverse to γ, $[M^*]$ is the orientation class of M^*, and σ is the signature. This rational number depends only on (Σ, f) and not upon M, for if N^{4k} is another manifold with $\partial N=\Sigma$, let $X=M \bigcup_\Sigma N$. Then the difference of the expression for M and that for N is

$$L_k(-\nu_X)-\sigma(X),$$

which is zero by the Hirzebruch Signature Theorem.

On the other hand, suppose we change (Σ, f) within its framed cobordism class to (Σ', f'). Let W^{4k} be a framed manifold with $\partial W=\Sigma \cup -\Sigma'$. Then the middle-dimension intersection pairing of W will be non-singular and even, hence $\sigma(W)$ is divisible by 8. (See [12].) It follows that

$$\tfrac{1}{8}(L_k(-\gamma)[M^*]-\sigma(M)) \bmod 1$$

is a well-defined function from Ω^F_{4k-1} into Q/Z. (For $k=1$, we require that Σ^3 be the

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standard sphere and replace 8 by 16.) This function, which is easily seen to be a homomorphism, will be denoted by $g: \Omega^F_{4k-1} \to Q/Z$.

Using the Thom-Pontrjagin construction, Ω^F_{4k-1} can be identified with the stable homotopy group of spheres π_{4k-1} $(=\pi_{4k-1+t}(S^t))$. Let $e: \pi_{4k-1} \to Q/Z$ be the Adams e-invariant. Also, let $a_k = 1$ if k is even and 2 if k is odd.

THEOREM 1. *If $k \geqslant 3$, then as homomorphisms from π_{4k-1} to Q/Z, $g = a_k 2^{2k-3} e$. If $k=2$, $g = -28e$. If $k=1$, $g=-e$.*

Now let us consider the classifying space BPL for stable piecewise-linear bundles. It is not difficult to show

THEOREM 2. *The group $\pi_{4k}(\mathrm{BPL})$ is isomorphic to $Z \oplus \ker g$.*

Using information about the e-invariant, we can calculate $\ker g$. Let $J: \pi_{4k-1}(\mathrm{SO}) \to \pi_{4k-1}$ be the J-homomorphism, let $(\mathrm{im} J)_2$ be the 2-primary component of the image of J, and let π'_{4k-1} be the cokernel of J. Then we find

PROPOSITION 3. *If $k \geqslant 3$, $\ker g = (\mathrm{im} J)_2 \oplus \pi'_{4k-1}$.* If $k=2$, $\ker g = Z_4 \subseteq \pi_7 = Z_{16} \oplus Z_{15} = \mathrm{im} J$. *If $k=1$, $\ker g = 0$.*

Thus we conclude

THEOREM 4. *If $k \geqslant 3$, $\pi_{4k}(\mathrm{BPL}) = Z \oplus (\mathrm{im} J)_2 \oplus \pi'_{4k-1}$; $\pi_8(\mathrm{BPL}) = Z \oplus Z_4$; $\pi_4(\mathrm{BPL}) = Z$.*

Finally, let Γ_{4k-1} be the group of exotic $(4k-1)$-spheres. There is an exact sequence (Kervaire-Milnor [11])

$$0 \to bP_{4k} \to \Gamma_{4k-1} \to \pi'_{4k-1} \to 0,$$

where bP_{4k} is the subgroup of those exotic spheres which bound π-manifolds. There is a natural map $h: \pi_{4k}(\mathrm{BPL}) \to \Gamma_{4k-1}$. Restricting h to the subgroup π'_{4k-1} of $\pi_{4k}(\mathrm{BPL})$ provides a splitting to the Kervaire-Milnor sequence, and we have

THEOREM 5. *The group Γ_{4k-1} is isomorphic to $bP_{4k} \oplus \pi'_{4k-1}$.*

2. On BPL and Exotic Spheres

In this section we will assume Theorem 1 and show how to obtain information on $\pi_{4k}(\mathrm{BPL})$ and Γ_{4k-1}. There is an exact sequence [7], [8], [9]

$$0 \to \pi_i(\mathrm{BSO}) \to \pi_i(\mathrm{BPL}) \to \Gamma_{i-1} \to 0.$$

Let Γ^F_{i-1} be the group of isomorphism classes of framed exotic $(i-1)$-spheres. Then the above sequence is isomorphic to the exact sequence

$$0 \to \pi_{i-1}(\mathrm{SO}) \xrightarrow{\varphi} \Gamma_{i-1}^F \xrightarrow{h} \Gamma_{i-1} \to 0,$$

where φ assigns to $\theta \in \pi_{i-1}(\mathrm{SO})$ the standard sphere with the framing θ and where h forgets the framing.

In particular, $\pi_i(\mathrm{BPL}) = \Gamma_{i-1}^F$.

There is also the exact sequence [10], [11]

$$0 \to Z \xrightarrow{\partial} \Gamma_{4k-1}^F \xrightarrow{j} \Omega_{4k-1}^F \to 0. \qquad (*)$$

Here the infinite cyclic group Z corresponds to the cobordism group P_{4k} of framed $4k$-manifolds with boundary an exotic sphere. It is generated by the Milnor manifold of signature 8. (If $k=1$, the infinite cyclic group is generated by a manifold of signature 16.) The map j assigns to a framed exotic sphere its framed cobordism class.

Let $(\Sigma, f) \in \Gamma_{4k-1}^F$. We noted in the introduction that the expression $L_k(-\gamma)[M^*] - \sigma(M)$ depends only on (Σ, f), where M^{4k} is a manifold with $\partial M = \Sigma$ and γ is the extension of ν_M. Thus this expression defines a homomorphism L from Γ_{4k-1}^F to Q.

From the exact sequence (*) we see that Γ_{4k-1}^F is isomorphic to $Z \oplus T$, where T is a torsion group and $T = \ker L$. Note that j maps T injectively into Ω_{4k-1}^F. Theorem 2 is contained in

PROPOSITION. *The function j maps T isomorphically onto* $\ker g$, *so* $\pi_{4k}(\mathrm{BPL}) = \Gamma_{4k-1}^F = Z \oplus \ker g$.

Proof. If $x \in \Gamma_{4k-1}^F$, then by definition $g(j(x)) = \frac{1}{8}L(x) \bmod 1$. If $x \in T$, then $L(x) = 0$ and $g(j(x)) = 0$. Thus $j(T) \subseteq \ker g$.

Conversely, if $g(j(x)) = 0$, then $L(x)$ is divisible by 8. Say $L(x) = 8n$. Let M be the Milnor manifold of signature $8n$. Then $j(x - \partial(M)) = j(x)$ and $L(x - \partial(M)) = 0$, so $x - \partial(M) \in T$. Thus $\ker g \subseteq j(T)$.

It remains only to calculate $\ker g$. From Adams' work on the e-invariant (together with work of Sullivan [16] and Quillan [14]), we know that $\pi_{4k-1} = (\mathrm{im}\, J) \oplus \pi'_{4k-1}$, where e is injective on the first summand and trivial on the second. Let $\mathrm{im}\, J = (\mathrm{im}\, J)_2 \oplus (\mathrm{im}\, J)_{\mathrm{odd}}$. Then for $k \geq 3$, $2^{2k-3}e$ annihilates $(\mathrm{im}\, J)_2$, so $\ker g = (\mathrm{im}\, J)_2 \oplus \pi'_{4k-1}$. If $k=2$, then $\pi_7 = \mathrm{im}\, J = Z_{16} \oplus Z_{15}$, where $g = -28e$. Thus $\ker g = Z_4$. This proves Proposition 3 and Theorem 4.

Finally, consider the exact sequence

$$0 \to bP_{4k} \to \Gamma_{4k-1} \xrightarrow{P} \pi'_{4k-1} \to 0.$$

The map P is defined as follows: if $\Sigma \in \Gamma_{4k-1}$, let $x \in \Gamma_{4k-1}^F$ be a pre-image. Then $j(x) \in \Omega_{4k-1}^F = \pi_{4k-1}$ and $P(\Sigma)$ is the coset of $j(x)$ in $\pi_{4k-1}/\mathrm{im}\, J = \pi'_{4k-1}$. Thus if

$h: \Gamma^F_{4k-1} \to \Gamma_{4k-1}$ is restricted to the subgroup π'_{4k-1}, it splits the sequence. This proves Theorem 5.

3. Proof of Theorem 1

Let $J: \pi_{4k-1}(\mathrm{SO}) \to \pi_{4k-1}$ be the J-homomorphism. It is enough to verify Theorem 1 for $\lambda \in \text{image } J$ and for $\lambda \in \text{kernel } e$, since every element of π_{4k-1} is a sum of elements of these two types. Suppose, then, that $\lambda = J(\theta)$, for some $\theta \in \pi_{4k-1}(\mathrm{SO})$. If we think of θ as a framing of the standard sphere S^{4k-1}, then (S^{4k-1}, θ) corresponds to λ in Ω^F_{4k-1}. Then for the manifold M^{4k} with $\partial M = S^{4k-1}$ we may choose the disk D^{4k}. Thus $M^* = S^{4k}$ and the vector bundle γ on S^{4k} is the bundle with characteristic map θ. Then

$$\begin{aligned} g(\lambda) &= \tfrac{1}{8}(L_k(-\gamma)[S^{4k}] - \sigma(S^{4k})) \bmod 1 \\ &= \tfrac{1}{8}(L_k(-\gamma)[S^{4k}]) \bmod 1 . \end{aligned}$$

(If $k=1$, replace 8 by 16.)

On the other hand, Adams shows [1]

$$e(\lambda) = \frac{\hat{A}(-\gamma)}{a_k}[S^{4k}] \bmod 1 .$$

Now

$$L_k(-\gamma) = \frac{2^{2k}(2^{2k-1}-1)B_k}{(2k)!} p_k(-\gamma)$$

and

$$\hat{A}(-\gamma) = \frac{-B_k}{2(2k)!} p_k(-\gamma),$$

where B_k is the k-th Bernoulli number. Hence

$$g(\lambda) = -a_k 2^{2k-2}(2^{2k-1}-1)e(\lambda), \qquad k \geqslant 2,$$

and $g = -e$ for $k=1$. But the e-invariant takes values in the cyclic subgroup of Q/Z having order j_k, where j_k is the denominator of $B_k/4k$. Moreover, $(2^{2k-1}-1)$ is relatively prime to the odd factor of j_k. (This is all we really need to know about g to prove Theorem 4.) In fact elementary number theory (using von Staudt's Theorem; compare [13]) shows that $2^{2k} \equiv 1$ mod odd factor of j_k. Thus $2(2^{2k-1}-1) = 2^{2k}-2 \equiv -1$ mod odd factor of j_k. Therefore $2^{2k-2}(2^{2k-1}-1) = 2^{2k-3}(2^{2k}-2) \equiv -2^{2k-3} \bmod j_k$, provided $2k-3 \geqslant$ largest power of 2 dividing j_k, which is the case for $k \geqslant 3$. This proves Theorem 1 when $\lambda \in \text{image } J$.

If $\lambda \in \ker e$, let (Σ, f) be a framed exotic sphere corresponding to λ in Ω^F_{4k-1}. Let M^{4k} be an oriented manifold with $\partial M = \Sigma$, and let γ be the extension of ν_M to M^*. We wish to show that $g(\lambda) = 0$ in Q/Z. That is, we want to show

$$\tfrac{1}{8}(L_k(-\gamma)[M^*] - \sigma(M))$$

is an integer. Let $T(\gamma)$ be the Thom complex of the bundle γ. Then our theory of almost closed manifolds ([6], [5, Theorem 1]) shows that

$$T(\gamma) = T(\nu_M) \bigcup_{\beta} D^{4k+t}$$

where $t = \dim \gamma$, $\beta: S^{4k+t-1} \to T(\nu_M)$ is the attaching map of the $(4k+t)$-cell, and β factors (up to homotopy) as the composite

$$S^{4k+t-1} \xrightarrow{\lambda} S^t = T(\nu_M \mid \text{point}) \to T(\nu_M).$$

This means there is a map $H: S^t \bigcup_\lambda D^{4k+t} \to T(\gamma)$ which is of degree one in dimensions t and $4k+t$. If we choose M to be a spin manifold (this is possible by [2]), we can use the above information to show there is a smooth closed spin manifold N^{4k} with $p_\omega[N] = p_\omega(-\gamma)[M^*]$ for all Pontrjagin numbers p_ω. Indeed, Stong has shown [15] that such a manifold N exists provided a certain integrality condition (Atiyah-Hirzebruch) holds. In our context, the condition is that $\langle ph(\delta) \cdot \hat{A}(-\gamma), [M^*] \rangle$ is in $a_k Z$ for all vector bundles δ on M^*. (Here ph is the Pontrjagin character.) Let $t \equiv 0 \bmod 8$. Then $\hat{A}(-\gamma) = \Phi^{-1}(ph\, U)$, where U is the K-theory Thom class of γ and Φ is the Thom isomorphism in ordinary cohomology. Thus

$$\begin{aligned}\langle ph(\delta) \cdot \hat{A}(-\gamma), [M^*] \rangle &= \langle \Phi(ph(\delta) \cdot \hat{A}(-\gamma)), \Phi[M^*] \rangle \\ &= \langle ph(\delta) \cdot ph(U), \Phi[M^*] \rangle \\ &= \langle ph(\delta \cdot U), \Phi[M^*] \rangle.\end{aligned}$$

Now use the map H. We have

$$\langle ph(\delta \cdot U), \Phi[M^*] \rangle = \langle ph(\xi), d^{4k+t} \rangle,$$

where $\xi = H^*(\delta \cdot U)$ and d^{4k+t} is the generator of $H_{4k+t}(S^t \bigcup_\lambda D^{4k+t})$. By the definition of the e-invariant, $e(\lambda) = 0$ means that $\langle ph(\xi), d^{4k+t} \rangle$ is in $a_k Z$ for any bundle ξ on $S^t \bigcup_\lambda D^{4k+t}$. Thus the integrality condition is satisfied and there is a spin manifold N with $p_\omega[N] = p_\omega(-\gamma)[M^*]$. Let W be the connected sum $M^* \# (-N)$. We may use W to calculate $g(\lambda)$. If γ' is the vector bundle on W, then $p_\omega(-\gamma')[W] = 0$ for all ω. Hence $L_k(-\gamma') = 0$. Thus $g(\lambda) = -\tfrac{1}{8}(\sigma(W)) \bmod 1$. Note that except for the Pontrjagin number p_k, we have $p_\omega[W] = p_\omega(-\gamma')[W] = 0$. Thus W is a spin manifold (smoothable on the complement of a $4k$-disk) with all decomposable Pontrjagin numbers zero. Since the signature of such a manifold is divisible by 8 (see [3] and [17]), Theorem 1 is proved.

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