Finiteness Properties of Duality Groups

ROBERT BIERI and BENO ECKMANN

0. Introduction

0.1. In this paper we show that groups with homological duality (generalizing Poincaré duality, cf. [2]) always satisfy certain finiteness conditions. We emphasize that the definition of a duality group as given in [2] does not involve any a priori finiteness property of the group.

Let G be a *duality group*, of dimension n and with dualizing module C. Here is a list of finiteness properties automatically fulfilled:

(1) G is finitely generated.

(2) The dualizing module C is finitely presented over $\mathbf{Z}G$.

(3) The dualizing module C admits a *finite* projective resolution over $\mathbf{Z}G$; i.e., a resolution which is finitely generated in each dimension and of finite length m. There is always a resolution of length $m \leqslant n+1$, of length $m=n$ if C is $\mathbf{Z}$-free.

(4) The integral cohomology groups $H^k(G; \mathbf{Z})$ are finitely generated.

(5) The integral homology groups $H_k(G; \mathbf{Z})$ are finitely generated.

(6) If G is a *Poincaré duality group* (i.e., if the Abelian group underlying C is $\mathbf{Z}$), then G is of type (FP). – A group is called of type (FP) if the trivial G-module $\mathbf{Z}$ admits a finite projective G-resolution.

0.2. With regard to the proofs of these statements, we make the following preliminary remarks.

(1) has already been established in [2], Theorem 4.6. The proof of (2) is based on a known criterion which we include for completeness; the proof of (3) on a generalization of that criterion. (4) is an easy consequence of (3). The statement (5) follows from (4) via the universal-coefficients theorem and a lemma which seems new and which may also be useful in other contexts: If A is an Abelian group such that $\mathrm{Hom}(A, \mathbf{Z})$ and $\mathrm{Ext}(A, \mathbf{Z})$ are finitely generated, then A is finitely generated.

Statement (6), concerning Poincaré duality, is essentially a corollary of (3). We do not know whether duality groups in the general sense must also be of type (FP). We recall from [2], Section 4, that groups of type (FP) are easier to investigate, with respect to duality, than arbitrary groups.

The fact that Poincaré duality groups are of type (FP) can be established by a second method which does not use the cap-product nor any naturality – just the existence of duality isomorphisms. From this it turns out (Theorem 3.4 below) that a group satisfying Poincaré duality isomorphisms – not supposed to be given by a cap-product with a fundamental class nor even to be natural – is a true Poincaré duality group.

The contents of this paper have been announced in a Comptes Rendus Note [3].

1. Finitely Presented Modules and Finitely Generated Free Resolutions

1.1. Let R be a ring with unit. We recall that a right R-module is said to be *finitely presented* if there is a short exact sequence of modules

$$K \rightarrowtail F \twoheadrightarrow B \tag{1.1}$$

with F being R-free and F and K finitely generated over R. The sequence (1.1) is called a finite (free) presentation of B.

If $\{A_i\}$ is an inverse system of left R-modules, then clearly $\{\mathrm{Tor}_k^R(B, A_i)\}$ is an inverse system of Abelian groups, and one has a unique natural homomorphism

$$\mathrm{Tor}_k^R(B, \varprojlim A_i) \to \varprojlim \mathrm{Tor}_k^R(B, A_i), \qquad k=0, 1, \ldots \tag{1.2}$$

(Similar homomorphisms are available for $\mathrm{Ext}_R^k(B, A_i)$, B a left module). We consider the special case where $\lim A_i$ is the direct product $\prod_I R$ of copies of R (indexed by some index set I). For $k=0$ one has the homomorphism

$$\mu_B : B \otimes_R \prod_I R \to \prod_I B \tag{1.3}$$

given by $\mu_B(b \otimes \prod_{i \in I} r_i) = \prod_{i \in I} br_i$, $b \in B$, $r_i \in R$.

LEMMA 1.1. (i) *μ_B is an epimorphism for every direct product $\prod_I R$ if and only if B is finitely generated.*

(ii) *μ_B is an isomorphism for every direct product $\prod_I R$ if and only if B is finitely presented.*

Proof. (i) We take B itself as index set I and assume that μ_B is an epimorphism. Then there is an element $c \in B \otimes_R \prod_B R$ with $\mu_B(c) = \prod_{b \in B} b$. The element c is of the form

$$c = \sum_{k=1}^{m} (b_k \otimes \prod_{b \in B} r_{bk}),$$

hence $\mu_B(c) = \sum_{k=1}^m \prod_{b \in B} b_k r_{bk} = \prod_{b \in B} b$. It follows that for each $b \in B$ one has $b = \sum_{k=1}^m b_k r_{bk}$; i.e., B is generated by the finite set $b_1, b_2, \ldots, b_m$. In the other direction, (i) is trivial.

(ii) Let B be finitely generated, and $K \rightarrowtail F \twoheadrightarrow B$ a free presentation with F finitely generated. Naturality of (1.3) yields the commutative diagram with exact rows

$$\begin{array}{ccccccc} K\otimes_R\prod R & \to & F\otimes_R\prod R & \to & B\otimes_R\prod R & \to & 0 \\ \downarrow\mu_K & & \downarrow\mu_F & & \downarrow\mu_B & & \\ 0\to \prod K & \to & \prod F & \to & \prod B & \to & 0 \end{array}$$

for an arbitrary direct product $\prod$. It is easy to see that μ_F is an isomorphism. By (i), μ_B is an epimorphism. By the five lemma, μ_B is a monomorphism if and only if μ_K is an epimorphism; i.e., by (i), if K is finitely generated. –

The above proof shows that, if B is a finitely presented module, any exact sequence $K \rightarrowtail F \twoheadrightarrow B$ with F finitely generated free is a finite presentation of B.

1.2. An R-resolution

$$\cdots \to X_k \to X_{k-1} \to \cdots \to X_1 \to X_0 \twoheadrightarrow B$$

(in short $\mathfrak{X} \twoheadrightarrow B$) of the R-module B is said to be finitely generated if the modules X_k are finitely generated for all $k \geqslant 0$. In this section we give necessary and sufficient conditions for a module B to admit a *finitely generated free* resolution.

Let B be finitely presented, and $K_0 \rightarrowtail F_0 \twoheadrightarrow B$ a finite free presentation. In the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0\to \mathrm{Tor}_1^R(B,\prod R) & \to & K_0\otimes_R\prod R & \to & F_0\otimes_R\prod R & \to & B\otimes_R\prod R & \to & 0 \\ \downarrow & & \downarrow\mu_{K0} & & \downarrow\mu_{F0} & & \downarrow\mu_B & & \\ 0 & \to & \prod K_0 & \to & \prod F_0 & \to & \prod B & \to & 0 \end{array}$$

for an arbitrary direct product $\prod$, μ_B and μ_{F_0} are isomorphisms, and μ_{K_0} is an epimorphism. By Lemma 1.1, K_0 is finitely presented if and only if μ_{K_0} is a monomorphism, i.e., if $\mathrm{Tor}_1^R(B, \prod R)=0$ for all $\prod$. If this is the case, we take a finite free presentation $K_1 \rightarrowtail F_1 \twoheadrightarrow K_0$ and apply the same argument: K_1 is finitely presented if and only if $\mathrm{Tor}_1^R(K_0, \prod R)=0$. But $\mathrm{Tor}_1^R(K_0, \prod R) \cong \mathrm{Tor}_2^R(B, \prod R)$, by the exact Tor-sequence. Iterating the argument we get the following criterion.

PROPOSITION 1.2. *The R-module B admits a free resolution* $\mathfrak{F} \twoheadrightarrow B$ *with* F_i *finitely generated for all* $i \leqslant k$ *if and only if B is finitely presented and* $\mathrm{Tor}_i^R(B, \prod R)=0$ *for* $i=1, 2, \ldots, k-1$ *and all direct products* $\prod$.

2. The Dualizing Module

2.1. We recall that a group G is termed *duality group* of dimension $n \geqslant 0$ (cf. [2]) if there is a "dualizing (right) G-module" C and a "fundamental class" $e \in H_n(G; C)$

such that the cap-product with e yields isomorphisms

$$(e\cap -): H^k(G; A) \xrightarrow{\cong} H_{n-k}(G; C\otimes A)$$

for every (left) G-module A and all $k\in\mathbf{Z}$. The tensor product $C\otimes A$ over the integers is understood to be endowed with the diagonal G-module structure. We recall the following facts from [2]; they will be used without further reference.

PROPOSITION 2.1. *For a duality group G of dimension n and with dualizing module C one has*

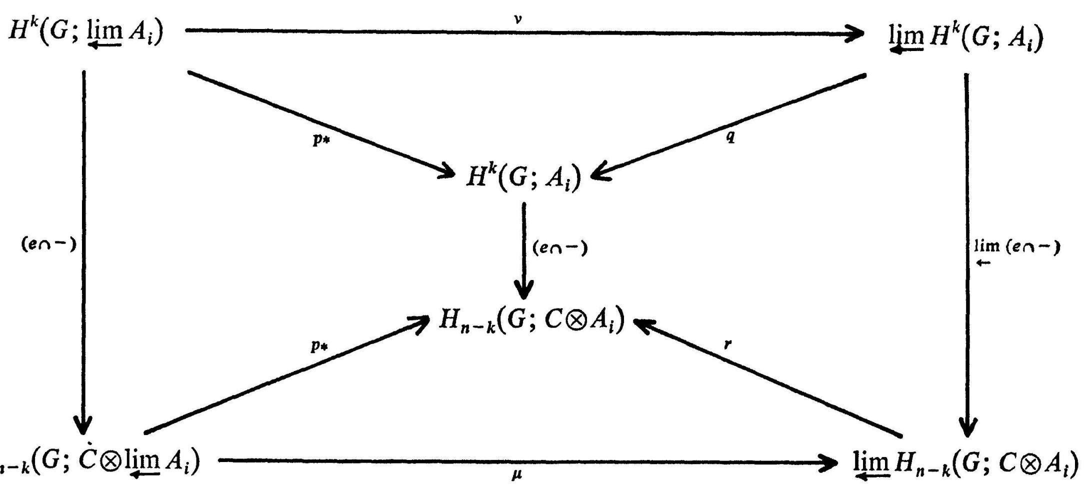
(i) *$C\cong H^n(G; \mathbf{Z}G)$ as right G-modules*

(ii) *C is torsion-free as an Abelian group*

(iii) *n is equal to the cohomology dimension cd$G$ and to the homology dimension hd$G$ of G*

(iv) *$H^k(G; \mathbf{Z}G)=0$ for all $k\neq n$.*

2.2. Let G be an arbitrary group, and C a right G-module, $\{A_i\}$ an inverse system of left G-modules. Clearly $\{H^k(G; A_i)\}$ and $\{H_{n-k}(G; C\otimes A_i)\}$ are inverse systems of Abelian groups. We consider, for integers n and k and an element $e\in H_n(G; C)$, the diagram



where ν and μ are limiting homomorphisms (cf. (1.2)), p, q, r the obvious projections from $\varprojlim$. The left-hand square is commutative by the naturality of the cap-product. The right-hand square and the triangles are commutative by the definition of μ, ν and $\varprojlim(e\cap -)$.

Now the two maps $r\circ\mu\circ(e\cap -)$ and $r\circ\varprojlim(e\cap -)\circ\nu: H^k(G; \varprojlim A_i)\to$

$H_{n-k}(G; C\otimes A_i)$ coincide, for each index i. Therefore the two maps $\mu\circ(e\frown -)$ and $\varprojlim(e\frown -)\circ\nu$ themselves must coincide; i.e., the outer square is commutative. We thus have established the following result.

LEMMA 2.2. *Let G be an arbitrary group, C a right G-module, and* $\{A_i\}$ *an inverse system of left G-modules. For arbitrary integers n, k and elements* $e\in H_n(G; C)$ *the diagram*

$$\begin{array}{ccc} H^k(G; \varprojlim A_i) & \xrightarrow{\nu} & \varprojlim H^k(G; A_i) \\ {\scriptstyle (e\frown -)}\downarrow & & \downarrow{\scriptstyle \varprojlim(e\frown -)} \\ H_{n-k}(G; C\otimes \varprojlim A_i) & \xrightarrow[\mu]{} & \varprojlim H_{n-k}(G; C\otimes A_i) \end{array}$$

is commutative.

2.3. Let, in particular, G be a duality group of dimension n, C its dualizing module and $e\in H_n(G; C)$ a fundamental class for G. Taking for $\varprojlim A_i$ an arbitrary direct product of copies of $\mathbf{Z}G$, Lemma 2.2 yields the commutative diagram

$$\begin{array}{ccc} H^k(G; \prod \mathbf{Z}G) & \xrightarrow{\nu} & \prod H^k(G; \mathbf{Z}G) \\ {\scriptstyle (e\frown -)}\downarrow & & \downarrow{\scriptstyle \Pi(e\frown -)} \\ H_{n-k}(G; C\otimes \prod \mathbf{Z}G) & \xrightarrow[\mu]{} & \prod H_{n-k}(G; C\otimes \mathbf{Z}G) \end{array} \qquad (2.1)$$

for all integers k. The vertical arrows are isomorphisms. Since the direct product is an exact inverse limit, H^k commutes with $\prod$, i.e., ν is an isomorphism. Hence μ is an isomorphism. For $k=n$, the map $\mu: H_0(G; C\otimes\prod \mathbf{Z}G) = C\otimes_G\prod \mathbf{Z}G \to \prod C$ is just the homomorphism μ_C of (1.3). Since it is an isomorphism, it follows from Lemma 1.1 that C is *finitely presented.*

Moreover, for an arbitrary integer j, the group $H_j(G; C\otimes\prod \mathbf{Z}G)$ can be transformed as follows. Since $\prod \mathbf{Z}G$ is torsion-free as an Abelian group, the standard associativity formula for Tor (cf. [4], p. 352) yields

$$H_j(G; C\otimes\prod \mathbf{Z}G) = \mathrm{Tor}_j^G(C\otimes\prod \mathbf{Z}G, \mathbf{Z}) \cong \mathrm{Tor}_j^G(C, \prod \mathbf{Z}G).$$

Since $H^k(G; \mathbf{Z}G)=0$ for $k\neq n$, this implies $\mathrm{Tor}_j^G(C, \prod \mathbf{Z}G)=0$ for $j=n-k\neq 0$. By Proposition 1.2, C being finitely presented, it follows that C admits a finitely generated G-free resolution $\mathfrak{F}\twoheadrightarrow C$. We summarize:

THEOREM 2.3. *Let G be a duality group of dimension n. Its dualizing module* $C=H^n(G; \mathbf{Z}G)$ *admits a finitely generated free resolution over* $\mathbf{Z}G$*. In particular, C is finitely presented over* $\mathbf{Z}G$*.*

2.4. As a corollary of this theorem and of the fact that $cdG=n$ we can obtain information on the length of *projective* resolutions of C over $\mathbf{Z}G$, as follows.

The associativity spectral sequence for Ext (cf. [4], p. 351) yields a spectral sequence

$$H^p(G; \operatorname{Ext}^q(C, A)) \Rightarrow \operatorname{Ext}_G^{p+q}(C, A)$$

for all G-modules A. Since $H^p(G; -)=0$ for $p>n$, we have $\operatorname{Ext}_G^{n+2}(C, A)=0$ for all A. Hence there exists a projective resolution of C of length $\leqslant n+1$. More precisely, the finitely generated free resolution $\mathfrak{F} \twoheadrightarrow C$ above splits in all dimensions $\geqslant n+1$. Hence there exists a *finite* projective resolution of C, of length $\leqslant n+1$. Since $H_n(G; C)\neq 0$, the length cannot be $<n$.

In case the dualizing module is $\mathbf{Z}$-free, we even have $\operatorname{Ext}_G^{n+1}(C, A)=0$ for all A; i.e., C admits a finite projective resolution of length n. We thus have

COROLLARY 2.4. *Let G be a duality group of dimension n. Its dualizing module C admits a finite projective resolution over $\mathbf{Z}G$, of length n or $n+1$; if C is $\mathbf{Z}$-free, of length n.*

2.5. We now prove that all integral (co)homology groups of a duality group are finitely generated.

THEOREM 2.5. *All homology and cohomology groups $H_k(G; \mathbf{Z})$ and $H^k(G; \mathbf{Z})$ of a duality group are finitely generated.*

Proof. The cohomology part is an immediate consequence of Theorem 2.4, since $H^k(G; \mathbf{Z})\cong H_{n-k}(G; C)=\operatorname{Tor}_{n-k}^G(C, \mathbf{Z})$. The homology part of Theorem 2.5 follows from the general fact that, for an arbitrary group G, the cohomology groups $H^k(G; \mathbf{Z})$ are finitely generated for all k if and only if the homology group $H_k(G; \mathbf{Z})$ are.

To prove this general fact, we consider the universal-coefficient exact sequence

$$\operatorname{Ext}(H_{k-1}(G; \mathbf{Z}), \mathbf{Z}) \rightarrowtail H^k(G; \mathbf{Z}) \twoheadrightarrow \operatorname{Hom}(H_k(G; \mathbf{Z}), \mathbf{Z})$$

for all integers k. Obviously, if the $H_k(G; \mathbf{Z})$ are all finitely generated, so are the $H^k(G; \mathbf{Z})$. The converse follows from the lemma below.

LEMMA 2.6. *Let A be an Abelian group. If the groups* $\operatorname{Hom}(A, \mathbf{Z})$ *and* $\operatorname{Ext}(A, \mathbf{Z})$ *are finitely generated, then A itself is finitely generated.*

Proof. If $\operatorname{Hom}(A, \mathbf{Z})\neq 0$, there is an epimorphism $A \twoheadrightarrow \mathbf{Z}$, hence $A\cong A_1\oplus\mathbf{Z}$. The rank of $\operatorname{Hom}(A_1, \mathbf{Z})$ is less than the rank of $\operatorname{Hom}(A, \mathbf{Z})\cong\operatorname{Hom}(A_1, \mathbf{Z})\oplus\mathbf{Z}$. Thus iterating the argument, we find a decomposition $A\cong B\oplus F$, with F free Abelian of finite rank and $\operatorname{Hom}(B, \mathbf{Z})=0$. Then $\operatorname{Ext}(B, \mathbf{Z})\cong\operatorname{Ext}(A, \mathbf{Z})$ is finitely generated.

Let T be the torsion subgroup of B. From the exact sequence (where $\operatorname{Hom}(B, \mathbf{Z})$

$=0$, $\mathrm{Hom}(T, \mathbf{Z})=0$)

$$0 \to \mathrm{Hom}(B/T, \mathbf{Z}) \to \mathrm{Hom}(B, \mathbf{Z}) \to \mathrm{Hom}(T, \mathbf{Z}) \to \mathrm{Ext}(B/T, \mathbf{Z}) \to \mathrm{Ext}(B, \mathbf{Z})$$

we see that $\mathrm{Hom}(B/T, \mathbf{Z})=0$ and $\mathrm{Ext}(B/T, \mathbf{Z})$ is finitely generated. The latter group being divisible, it must be 0. But (cf. [5], p. 182) $\mathrm{Hom}(B/T, \mathbf{Z})=0$ and $\mathrm{Ext}(B/T, \mathbf{Z})=0$ imply $B/T=0$; i.e., $B=T$, $A \cong T \oplus F$.

It remains to show that T is finite. With the natural imbedding $Z \to \mathbf{R}$ we associate the exact sequence

$$0 \to \mathrm{Hom}(T, \mathbf{Z}) \to \mathrm{Hom}(T, \mathbf{R}) \to \mathrm{Hom}(T, \mathbf{R}/\mathbf{Z}) \to \mathrm{Ext}(T, \mathbf{Z}) \to 0,$$

i.e., $\mathrm{Hom}(T, \mathbf{R}/\mathbf{Z}) \cong \mathrm{Ext}(T, \mathbf{Z})$, hence finitely generated. But the group $\mathrm{Hom}(T, \mathbf{R}/\mathbf{Z})$ (the character group of T) can be given its natural compact topology. Being finitely generated, it must be finite. Hence T itself is finite.

3. Groups of Type (FP) and Poincaré Duality

3.1. A duality group is said to be a *Poincaré duality group* (cf. [1]) if its dualizing module C is infinite cyclic as an Abelian group; in this case we write $\tilde{\mathbf{Z}}$ for C (the symbol $\mathbf{Z}$ being reserved for the trivial G-module). A Poincaré duality group is said *orientable* or *non-orientable* according to whether $\tilde{\mathbf{Z}}=\mathbf{Z}$ or $\neq \mathbf{Z}$. By [1], Korollar 2.1.2, a non-orientable Poincaré duality group contains a unique orientable one of index 2.

If G is a Poincaré duality group of dimension n, Corollary 2.5 yields a finite projective resolution (right G-modules)

$$0 \to \tilde{P}_n \to \tilde{P}_{n-1} \to \cdots \to \tilde{P}_0 \twoheadrightarrow \tilde{\mathbf{Z}}.$$

Let sgn denote the homomorphism $G \twoheadrightarrow \mathbf{Z}_2 = \{1, -1\}$ defined by the G-action on $\tilde{\mathbf{Z}}$: for $x \in G$ and $1 \in \tilde{\mathbf{Z}}$, $1 \cdot x = \mathrm{sgn}(x)$. Using sgn, we define left G-modules P_k by taking the underlying Abelian group of P_k to be that of $\tilde{P}_k$, and by putting

$$xp = \mathrm{sgn}(x)\, p \cdot x^{-1}, \qquad x \in G, \qquad p \in P_k,$$

for $k=0, 1, \ldots, n$. The P_k are still finitely generated projective; the same procedure turns $\tilde{\mathbf{Z}}$ into the (left) module $\mathbf{Z}$. We thus get a finite projective resolution over $\mathbf{Z}G$

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \twoheadrightarrow \mathbf{Z}$$

of the trivial G-module $\mathbf{Z}$; i.e., G is of type (FP) according to the terminology explained in the introduction (Section 0). We thus have proved

THEOREM 3.1. *All Poincaré duality groups are of type* (FP).

3.2. Remark. Recall that a *Poincaré complex* X is a CW-complex dominated by a finite CW-complex and whose (co)homology with arbitrary local coefficients satisfies Poincaré duality. The duality isomorphisms are understood to be given by the cap-product with a fundamental class $[X]\in H_n(X; \tilde{\mathbf{Z}})$, for a suitable $\pi_1(X)$-module $\tilde{\mathbf{Z}}$. If a group G admits an Eilenberg-MacLane complex $K(G, 1)$ which is a Poincaré complex[1]), then clearly G is a Poincaré duality group, and moreover *finitely presented.* Conversely, Theorem 3.1 shows that any Poincaré duality group, provided it is finitely presented, admits a $K(G, 1)$ which is a Poincaré complex (since a finitely presented group of type (FP) admits a $K(G, 1)$ which is dominated by a finite CW-complex).

3.3. In the remainder of this section we apply to Poincaré duality groups directly the criterion for finitely generated free resolutions established in Section 1. This will provide, among other things, a second proof of Theorem 3.1 from which different features emerge.

If G is an arbitrary group, and $\mathbf{Z}$ the trivial G-module, we will say that G is *of type* $(\overline{FP})$ if $\mathbf{Z}$ admits a finitely generated free resolution over $\mathbf{Z}G$.

PROPOSITION 3.2. *A group G is of type* $(\overline{FP})$ *if and only if the two conditions hold:*

(i) *G is finitely generated*

(ii) $H_k(G; \prod \mathbf{Z}G)=0$ *for all* $k\geqslant 1$ *and all direct products* $\prod \mathbf{Z}G$.

Moreover, G is of type (FP) *if and only if in addition to* (i) *and* (ii)

(iii) $cdG<\infty$.

Proof. From the short exact augmentation sequence $IG\rightarrowtail \mathbf{Z}G\twoheadrightarrow \mathbf{Z}$ one sees that $\mathbf{Z}$ is finitely presented over $\mathbf{Z}G$ if and only if IG is finitely generated, i.e., if G is finitely generated. Hence Proposition 3.2 follows from Proposition 1.2.

As a minor application, we mention briefly that Proposition 3.2 provides a very simple proof of the following well-known facts.

PROPOSITION 3.3. a) *The class of groups of type* $(\overline{FP})$ *is extension closed, and so is the class of groups of type* (FP).

b) *Let S be a subgroup of finite index in a torsion-free group G. If S is of type* (FP), *so is G.*

Proof. Clearly condition (i) of Proposition 3.2 is extension closed; and by the "maximum principle" for the Lyndon spectral sequence of the extension the same holds for (iii). As for (ii), let $N\rightarrowtail G\twoheadrightarrow Q$ be a short exact sequence of groups, and consider the initial terms $E^{(2)}_{p,q}=H_p(Q; H_q(N; \prod \mathbf{Z}G))$ of the spectral sequence. As N and Q are assumed to admit finitely generated free resolutions of $\mathbf{Z}$, the homology functors commute with all direct products. Thus we get $E^{(2)}_{p,q}=0$ whenever $pq\neq 0$,

[1]) Johnson-Wall [6] use the term "Poincaré duality group" for such groups.

whence (ii). Combining a) with Serre's theorem ([7], Théorème 1), we get the proof of b).

3.4. Let now G be a duality group of dimension n. Conditions (iii) and (i) of Proposition 3.2 are fulfilled, since $cdG=n$ and G is finitely generated by [2], Theorem 4.6. Unfortunately we are not able to check (ii) in the general case. In the Poincaré duality case, however, i.e., if the dualizing module $C=\tilde{\mathbf{Z}}$, we have

$$H^{n-k}(G;\tilde{\mathbf{Z}}\otimes\prod \mathbf{Z}G)\cong H_k(G;\tilde{\mathbf{Z}}\otimes\tilde{\mathbf{Z}}\otimes\prod \mathbf{Z}G).$$

Now $\tilde{\mathbf{Z}}\otimes\tilde{\mathbf{Z}}$ with diagonal action is G-isomorphic to $\mathbf{Z}$, whence

$$H_k(G;\prod \mathbf{Z}G)\cong H^{n-k}(G;\tilde{\mathbf{Z}}\otimes\prod \mathbf{Z}G),$$

which, by Lemma 1.1, is $\cong H^{n-k}(G;\prod(\tilde{\mathbf{Z}}\otimes \mathbf{Z}G))\cong\prod H^{n-k}(G;\tilde{\mathbf{Z}}\otimes \mathbf{Z}G)\cong \prod H_k(G;\mathbf{Z}G)=0$ for $k\geqslant 1$. Hence (ii) is fulfilled, and we have a second proof of Theorem 3.1.

It is worth-while noticing that this present argument does not involve the cap-product $e\cap -$, nor even any naturality properties of the duality isomorphisms – just the assumption that these exist. This provides the following result.

THEOREM 3.4. *Let G be a group with a homomorphism* $\text{sgn}: G\to \mathbf{Z}_2$ *defining the G-module* $\tilde{\mathbf{Z}}$, *n an integer* $\geqslant 0$. *If one has isomorphisms (not assumed to be natural)*

$$H^k(G;A)\cong H_{n-k}(G;\tilde{\mathbf{Z}}\otimes A) \tag{3.1}$$

for all k and all G-modules A, then G is a Poincaré duality group (of dimension n, with dualizing module $\tilde{\mathbf{Z}}$*).*

Proof. As remarked above, G is of type (FP). Since (3.1) implies $H^k(G;\mathbf{Z}G)\cong H_{n-k}(G;\tilde{\mathbf{Z}}\otimes \mathbf{Z}G)\cong H_{n-k}(G;\mathbf{Z}G)=0$ for $k\neq n$, and $H^n(G;\mathbf{Z}G)\cong H_0(G;\tilde{\mathbf{Z}}\otimes\mathbf{Z}G)=\tilde{\mathbf{Z}}$ torsion-free, the assertion follows from [2], Theorem 4.6.

Remark. In [1], Satz 2.6, it was shown that if isomorphisms (3.1) are assumed to exist and to be natural in A, then they are automatically induced by $e\cap -$ for a certain fundamental class $e\in H_n(G;\tilde{\mathbf{Z}})$. Of course, this result and Theorem 4.3 do not imply each other.

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Eidg. Techn. Hochschule,
Zürich, Switzerland

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