Commutators of Diffeomorphisms

Commutators of Diffeomorphisms[1])

by JOHN N. MATHER

In this paper, we will show that certain groups of diffeomorphisms are perfect, i.e., equal to their own commutator subgroups. Epstein has shown [2] that for quite general groups of homeomorphisms, the commutator subgroup is simple. In particular, his result shows that for the groups of diffeomorphisms which we consider, the commutator subgroup is simple. Combining his results with our result, we see that the groups we consider are simple. In §7, we obtain a result concerning the connectivity of Haefliger's classifying space for foliations as a corollary of our proof and a result of Thurston [4].

We say an isotopy H_t of a space M *has compact support* if there is a compact set K in M such that $H_t(x)=x$ for all $x\in M-K$ and all t.

Let M be a smooth manifold. We define $\mathrm{Diff}(M, r)$ to be the group of C^r diffeomorphisms of M which are isotopic to the identity through compactly supported C^r isotopies. Thus, any element of $\mathrm{Diff}(M, r)$ has compact support. Our main result is the following.

THEOREM 1. *If* $\infty > r \geqslant n+2$, *then* $\mathrm{Diff}(M, r)$ *is perfect, where* $n=\dim M$.

This is slighly different from a result announced in [3]. We will also prove the result announced there.

The case $r=\infty$ of this theorem has previously been proved. For $M=T^n$, it is an easy consequence of a theorem due to J. Moser. The generalization to arbitrary manifolds is due to Thurston [4].

As mentioned above, Theorem 1, and Epstein's result imply:

COROLLARY. *If* $\infty > r \geqslant n+2$, *then* $\mathrm{Diff}(M, r)$ *is simple.*

Actually, Theorem 1 is an immediate consequence of the special case when $M=\mathbf{R}^n$. For, any member h of $\mathrm{Diff}(M, r)$ can be decomposed as a product:

$$h=g_1 \cdots g_q,$$

where for each g_i there is an open subset U_i of M, diffeomorphic to $\mathbf{R}^n$, such that g_i is the identity outside U_i, and $g_i \mid U_i$ is in $\mathrm{Diff}(U_i, r)$. Then the special case of the theorem implies each g_i is a product of commutators, and it follows that h is a product of commutators.

Theorem 1 is a consequence of Theorem 2, stated in §2.

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The question of simplicity of groups of homeomorphisms has generated a fair amount of interest and number of papers. See Epstein [2] for older references. See Thurston [4] for more recent references on simplicity of groups of homeomorphisms. See also Thurston [5] for an interesting result concerning volume preserving diffeomorphisms.

§1. Moduli of Continuity

The results which we will actually prove are refinements of the results stated in the previous section. In fact, we will need to prove these refinements to prove the results stated there. To state the refinements we need the notion of a modulus of continuity.

DEFINITION. A *modulus of continuity* is a continuous strictly increasing real-valued function α defined on an interval $[0, \varepsilon]$, where $\varepsilon > 0$, such that $\alpha(0) = 0$, and $\alpha(tx) \leqslant t\alpha(x)$ for any $x \in [0, \varepsilon]$ and $t \geqslant 1$, such that $tx \in [0, \varepsilon]$.

DEFINITION. A mapping $f: X \to Y$ of metric spaces is *α-continuous* if there exists C, ε' with $C > 0, \varepsilon > \varepsilon' > 0$ such that for any $x, y \in X$ satisfying $\text{dist.}(x, y) \leqslant \varepsilon'$, we have

$$\text{dist.}(f(x), f(y)) \leqslant C\alpha(\text{dist.}(x, y)).$$

We say f is *locally α-continuous* if each point $x \in X$ has a neighborhood U such that $f \mid U$ is α-continuous.

For example, if $\alpha(x) = x$, then f is α-continuous if and only if it is Lipschitz. If $\alpha(x) = x^\beta$, where $0 < \beta \leqslant 1$, then f is α-continuous if and only if it is Hölder continuous with Hölder constant β.

SUMS. Let X be a metric space and Y a normed vector space. If $f, g: X \to Y$ are α-continuous, then so is their sum.

COMPOSITION. Consider mappings of metric spaces $X \to Y \to Z$. If f is α-continuous and g is Lipschitz, or f is Lipschitz and g is α-continuous, then $g \circ f$ is α-continuous. However, in general the composition of α-continuous mappings is not α-continuous.

PRODUCTS. Consider a bilinear mapping $B: Y \times Z \to W$ of normed vector spaces. If $f: X \to Y$ and $g: X \to Z$ are mappings, we define their "product" $fg: X \to W$ by $fg(x) = B(f(x), g(x))$. If f and g are α-continuous, and their images are bounded, then fg is α-continuous. This is because if Y_0 and Z_0 are bounded sets in Y and Z, respectively, then $B \mid Y_0 \times Z_0$ is Lipschitz, so we can apply our remarks about composition to $fg = B \circ (f, g)$.

§2. Mappings of Class $C^{r,\alpha}$

We will say a mapping from an open set in $\mathbf{R}^n$ to $\mathbf{R}^p$ is of class $C^{r,\alpha}$ if it is of class C^r and its rth derivative is locally α-continuous. When $r=\infty$, $C^{r,\alpha}$ will mean C^∞. We can extend this notion to mappings between smooth manifolds in the usual way. The class $C^{r,\alpha}$ has a number of properties which will be useful in what follows.

SUMS and PRODUCTS. If we form sums and products of mappings as in the previous section, we find that sums and products of $C^{r,\alpha}$ mappings are $C^{r,\alpha}$.

COMPOSITION. If $r \geqslant 1$, any composition of $C^{r,\alpha}$ mappings is $C^{r,\alpha}$. The proof is slightly different in the cases $r=1$ and $r>1$.

In the case $r=1$, we use the formula $D(f \circ g)(x) = Df(g(x)) \cdot Dg(x)$, or

$$D(f \circ g) = (Df \circ g) \cdot Dg. \tag{1}$$

Then $Df \circ g$ is the composition of a locally α-continuous mapping and a C^1 mapping, and is therefore locally α-continuous. Therefore $D(f \circ g)$ is the "product" of locally α-continuous mappings, so it is itself locally α-continuous.

In the case $r>1$, we use the formula for the rth derivative of a composition:

$$D^r(f \circ g) = (D^r f \circ g) \cdot (Dg)^r + (Df \circ g) \cdot D^r g + \text{other terms}, \tag{2}$$

where each of the other terms has the form

$$C(D^i f \circ g) \cdot (D^{j_1} g \times \cdots \times D^{j_i} g) \tag{3}$$

with C an integer, $1<i<r$, $1 \leqslant j_l$, and $j_1 + \cdots + j_i = r$.

The sum of the "other terms" is C^1. It follows that $D^r(f \circ g)$ is the sum of locally α-continuous mappings, and is therefore itself locally α-continuous.

INVERSES. If $r \geqslant 1$, any C^1 inverse of a $C^{r,\alpha}$ mapping is $C^{r,\alpha}$. If E and F are Banach spaces, we let $\mathrm{Iso}(E, F)$ denote the space of isomorphisms of E into F. We let

$$\mathrm{Inv}: \mathrm{Iso}(E, F) \to \mathrm{Iso}(F, E)$$

be defined by $\mathrm{Inv}(\varphi) = \varphi^{-1}$. Then Inv is C^∞ and for a C^1 invertible mapping f, we have $D(f^{-1})(x) = (Df(f^{-1}(x))^{-1}$ or

$$D(f^{-1}) = \mathrm{Inv} \circ Df \circ f^{-1}. \tag{3}$$

It is well known that if f is C^r and has a C^1 inverse, then its inverse is C^r. (In fact, this is an easy consequence of the above formula, and induction on r.) Now if f is $C^{r,\alpha}$ and has a C^1 inverse we know that f^{-1} is C^r, that Df is $C^{r-1,\alpha}$, and that Inv is C^∞, so we get from the above formula that $D(f^{-1})$ is $C^{r-1,\alpha}$. Hence f^{-1} is $C^{r,\alpha}$.

DEFINITION. If M is a smooth manifold, we let $\mathrm{Diff}(M, r, \alpha)$ be the group of $C^{r,\alpha}$ diffeomorphisms of M which are isotopic to the identity through compactly supported $C^{r,\alpha}$ isotopies.

The following is our refinement of Theorem 1.

THEOREM 2. *If* $\infty > r \geqslant n+2$, *and* α *is a modulus of continuity, or* $r=n+1$ *and* $\alpha(x)=x^\beta$ *for some* $0<\beta\leqslant 1$, *then* $\mathrm{Diff}(M, r, \alpha)$ *is perfect, where* $n=\dim M$.

Note that Theorem 1 is an immediate consequence, since

$$\mathrm{Diff}(M, r) = \bigcup_\alpha \mathrm{Diff}(M, r, \alpha),$$

where the union is taken over all moduli of continuity.

The case $\alpha(x)=x$ of this theorem is theorem of our announcement [3].

From Epstein's theorem [2], we get:

COROLLARY. *If* $\infty > r \geqslant n+2$, *and* α *is a modulus of continuity, or* $r=n+1$, *and* $\alpha(x)=x^\beta$ *for some* $0<\beta\leqslant 1$, *then* $\mathrm{Diff}(M, r, \alpha)$ *is simple.*

Of course, just as it is enough to prove Theorem 1 for the case $M=\mathbf{R}^n$, it is enough to prove Theorem 2 for the case $M=\mathbf{R}^n$, to obtain the general result. We reduce the proof in §3 to the construction of certain mappings $\Psi_{i,A}$. The mappings are constructed in subsequent sections.

§3. Strategy of the Proof

In this section, we discuss the strategy of the proof of Theorem 2 in the case $M=\mathbf{R}^n$ and $r<\infty$. The idea is to construct lots of conjugate elements. Roughly speaking, if we can find enough pairs of conjugate elements in a group, then it is perfect. If u is an element in a group, we will denote its image in the commutator quotient group by $[u]$. If u and v are conjugate, then $[u]=[v]$.

Let $A>1$. Let

$$D_{i,A} = \{x \in \mathbf{R}^n : -2 \leqslant x_j \leqslant 2,\ 1 \leqslant j \leqslant i \quad \text{and} \quad -2A \leqslant x_j \leqslant 2A,\ i<j\leqslant n\}.$$

Thus each $D_{i,A}$ is a rectilinear parallelpiped and $D_{n,A} \subset D_{n-1,A} \subset \cdots \subset D_{0,A}$. The rectilinear parallelpipeds $D_{i-1,A}$ and $D_{i,A}$ have the same width in all coordinates but the ith coordinate. We note $D_{n,A}$ is independent of A. We will write $D_n = D_{n,A}$.

The main technical step in the proof is the construction of certain mappings $\Psi_{i,A}$ of function spaces, and the proof of a number of properties of the $\Psi_{i,A}$. If X is a subset of topological space, we let $\operatorname{int} X$ denote the interior of X. The domain of $\Psi_{i,A}$ is a C^1 neighborhood of the identity in the space of C^1 diffeomorphisms of $\mathbf{R}^n$ with support in $\operatorname{int} D_{i-1,A}$. The range of $\Psi_{i,A}$ is the set of C^1 diffeomorphisms of $\mathbf{R}^n$ with support in $\operatorname{int} D_{i,A}$. Here, we list the properties we will show $\Psi_{i,A}$ to have.

3.1. *Properties of* $\Psi_{i,A}$

1) $\Psi_{i,A}(\mathrm{id}) = \mathrm{id}$, where id denotes the identity mapping of $\mathbf{R}^n$.

2) If u is $C^{r,\alpha}$, then so is $\Psi_{i,A}(u)$. (Hence, if u is C^r, then so is $\Psi_{i,A}(u)$.)

3) The restriction of $\Psi_{i,A}$ to the set of C^r diffeomorphisms in its domain is continuous with respect to the C^r topologies on its domain and range.

4) If u is in the domain of $\Psi_{i,A}$ then u is isotopic to the identity through an isotopy with support in $\operatorname{int} D_{i-1,A}$ and $\Psi_{i,A}(u)$ is isotopic to the identity through an isotopy with support in $\operatorname{int} D_{i,A}$.

Notice that if we have constructed a mapping $\Psi_{i,A}$ satisfying (1)–(3), then by replacing the domain of $\Psi_{i,A}$ by a possibly smaller neighborhood of id, we can arrange for (4) to hold. Notice also that (2) and (4) imply that if u is in the domain of $\Psi_{i,A}$ and u is $C^{r,\alpha}$, then $u, \Psi_{i,A}(u) \in \mathrm{Diff}(\mathbf{R}^n, r, \alpha)$.

5) If u is in the domain of $\Psi_{i,A}$ and u is $C^{r,\alpha}$, then

$$[u] = [\Psi_{i,A}(u)]$$

in the commutator quotient group of $\mathrm{Diff}(\mathbf{R}^n, r, \alpha)$.

The last property of $\Psi_{i,A}$ that we need is expressed in terms of pseudo-norms on the space of $C^{r,\alpha}$ mappings. If u is a $C^{r,\alpha}$ mapping from an open set U in $\mathbf{R}^n$ into $\mathbf{R}^p$, and the domain of α is $[0, \varepsilon]$, we let

$$\|u\|_{r,\alpha} = \sup\left\{\frac{\|D^r u(x) - D^r u(y)\|}{\alpha(\|x-y\|)} : x, y \in U \text{ and } \|x-y\| \leqslant \varepsilon\right\}.$$

Note that if α' is the restriction of α to a possibly smaller interval, then $\| \ \|_{r,\alpha} \geqslant \| \ \|_{r,\alpha'} \geqslant \lambda \| \ \|_{r,\alpha}$ for some $\lambda > 0$, provided U is convex.

If u is a $C^{r,\alpha}$ diffeomorphism of $\mathbf{R}^n$ with compact support, we set

$$\mu_{r,\alpha}(u) = \|u - \mathrm{id}\|_{r,\alpha}.$$

Thus $\mu_{r,\alpha}(u) = \|u\|_{r,\alpha}$ if $r > 1$.

6) There exists $\delta > 0$, $C > 0$ such that

$$\mu_{r,\alpha}(\Psi_{i,A}(u)) \leqslant CA\mu_{r,\alpha}(u),$$

if u is of class $C^{r,\alpha}$, lies in the domain of $\Psi_{i,A}$, and satisfies $\mu_{r,\alpha}(u)<\delta$. Moreover, C is independent of A.

However, δ depends on r, α, n, and A, and C depends on r, α, and n.

In the rest of this section, we finish the proof of Theorem 2, assuming the existence of $\Psi_{i,A}$ satisfying (1)–(6). Consider $f\in \mathrm{Diff}(\mathbf{R}^n, r, \alpha)$ with support in $\mathrm{int}\, D_{n,A}$. We wish to show f is in the commutator subgroup if it is sufficiently close to id.

We let $A:\mathbf{R}^n\to\mathbf{R}^n$ denote the mapping defined by $A(x)=A\cdot x$. For any $u\in \mathrm{Diff}(\mathbf{R}^n, r, \alpha)$ with support in $\mathrm{int}\, D_n$, we try to define

$$\begin{aligned} u_0 &= AfuA^{-1} \\ u_1 &= \Psi_{1,A}(u_0) \\ u_2 &= \Psi_{2,A}(u_1) \\ &\dots \\ u_n &= \Psi_{n,A}(u_{n-1}). \end{aligned}$$

If u and f are sufficiently close to the identity in the C^1 topology, these will actually be defined, by properties (1)–(3) of $\Psi_{i,A}$.

It is easily seen that u_0 is conjugate to fu in $\mathrm{Diff}(\mathbf{R}^n, r, \alpha)$. For, we can choose $\tilde{A}\in \mathrm{Diff}(\mathbf{R}^n, r, \alpha)$ such that $\tilde{A}\mid D_n=A$, and then we have

$$u_0=\tilde{A}fu\tilde{A}^{-1}.$$

Thus, $[u_0]=[fu]$ in the commutator quotient group of $\mathrm{Diff}(\mathbf{R}^n, r, \alpha)$. Thus

$$[fu]=[u_n]. \tag{$*$}$$

LEMMA. *Suppose* $n+2\leqslant r<\infty$, *or* $r=n+1$ *and* $\alpha(x)=x^\beta$ *for some* $0<\beta\leqslant 1$. *There exists* A_0 *such that if* $A\geqslant A_0$, *then for* $\varepsilon>0$ *sufficiently small,* $\mu_{r,\alpha}(u)\leqslant\varepsilon$ *and* $\mu_{r,\alpha}(f)\leqslant\varepsilon$ *imply* $\mu_{r,\alpha}(u_n)\leqslant\varepsilon$.

We will prove this lemma below. Assuming it, we can give a very quick proof of Theorem 2.

Proof of Theorem 2. Let B_ε denote the set of $u\in \mathrm{Diff}(\mathbf{R}^n, r, \alpha)$ with support in $\mathrm{int}\, D_n$ such that $\mu_{r,\alpha}(u)\leqslant\varepsilon$. If ε is sufficiently small, the mapping $u\mapsto u-\mathrm{id}$ is a bijection of B_ε onto the set B'_ε of $C^{r,\alpha}$ mappings $v:\mathbf{R}^n\to\mathbf{R}^n$ with support in $\mathrm{int}\, D_n$ and $\mu_{r,\alpha}(v)\leqslant\varepsilon$. Moreover the mapping $u\mapsto u-\mathrm{id}$ is a homeomorphism with respect to the C^r topologies. The set B'_ε is compact with respect to the C^r topology, by Ascoli's theorem, and is convex. Therefore by the Schauder-Tychonoff theorem [1, V. 10.5], it has the fixed point property, for mappings which are continuous with respect to the C^r topology.

Since B_ε is homeomorphic to B'_ε, it also has the fixed point property for mappings which are continuous with respect to the C^r topology.

Now choose A sufficiently large and ε sufficiently small so that the lemma holds. Let $f \in B_\varepsilon$, and for any $u \in B_\varepsilon$, let $u_0, \ldots, u_n$ be defined as above. We will write $\theta_f(u)$ for u_n. This defines a mapping θ_f of B_ε into itself by the lemma. By property (3) of $\Psi_{i,A}$, the mapping θ_f is continuous, with respect to the C^r topology. Therefore θ_f has a fixed point.

Let u be a fixed point of θ_f. Then, by (*), we have

$$[fu] = [u]$$

or

$$[f] = 1.$$

In other words, f is in the commutator subgroup of $\mathrm{Diff}(\mathbf{R}^n, r, \alpha)$.

Since f is an arbitrary element of B_ε, we have shown that every element of B_ε is in the commutator subgroup. But, $\mathrm{Diff}(\mathbf{R}^n, r, \alpha)$ is generated by conjugates of elements of B_ε, so we obtain the desired result. Q.E.D.

Proof of the Lemma. We start by proving an estimate which we will use later. Let K be a compact subset of $\mathbf{R}^n$. Let $r \geqslant 1$ and let α be a modulus of continuity. Then there exist $\delta > 0$ (small) and $C > 0$ (large) such that the following holds. Let f, g be $C^{r,\alpha}$ diffeomorphisms of $\mathbf{R}^n$ whose rth derivatives vanish outside K, and suppose $\mu_{r,\alpha}(f), \mu_{r,\alpha}(g) < \delta$. Then

$$\mu_{r,\alpha}(gf) \leqslant \mu_{r,\alpha}(g) + \mu_{r,\alpha}(f) + C\mu_{r,\alpha}(g)\,\mu_{r,\alpha}(f). \tag{!}$$

In the case $r = 1$, this estimate is an easy consequence of formula (1) in §2. In the case $r > 1$, it is an easy consequence of formula (2) in §2.

From this estimate, we have that if $\varepsilon > 0$ is sufficiently small, and the hypotheses $\mu_{r,\alpha}(u) \leqslant \varepsilon$ and $\mu_{r,\alpha}(f) \leqslant \varepsilon$ of the lemma are satisfied, then

$$\mu_{r,\alpha}(fu) \leqslant 3\varepsilon.$$

From the definition of u_0, we have

$$\mu_{r,\alpha}(u_0) \leqslant A^{1-r}\mu_{r,\alpha}(fu)$$

and

$$\mu_{r,\alpha}(u_0) = A^{1-r-\beta}\mu_{r,\alpha}(fu) \quad \text{if} \quad \alpha(x) = x^\beta.$$

Thus

$$\mu_{r,\alpha}(u_0) \leqslant 3A^{1-r}\varepsilon$$
$$\mu_{r,\alpha}(u_0) \leqslant 3A^{1-r-\beta}\varepsilon \quad \text{if} \quad \alpha(x) = x^\beta.$$

From condition (6) on the mappings $\Psi_{i,A}$, and the definition of $u_1, \dots, u_n$, it follows that if $\varepsilon > 0$ is sufficiently small, then

$$\mu_{r,\alpha}(u_n) \leqslant 3C^n A^{1-r+n}\varepsilon,$$
$$\mu_{r,\alpha}(u_n) \leqslant 3C^n A^{1-r-\beta+n}\varepsilon, \quad \text{if} \quad \alpha(x) = x^\beta,$$

where C is the number appearing in (6). Under the hypothesis of the lemma, the exponent of A is negative, so by taking A sufficiently large, we can arrange that

$$3C^n A^{1-r+n} < 1$$

(in the case $r \geqslant n+2$) or

$$3C^n A^{1-r-\beta+n} < 1$$

(in the case $r = n+1$ and $\alpha(x) = x^\beta$). In either case, we have $\mu_{r,\alpha}(u_n) \leqslant \varepsilon$. Q.E.D.

§4. A Criterion of Conjugacy

By the previous section, in order to prove Theorem 2, it is enough to construct the mappings $\Psi_{i,A}$ having properties (1)–(6). Here we want to focus on property (5). The diffeomorphisms in the domain of $\Psi_{i,A}$ have support in $\operatorname{int} D_{i-1,A}$ and the diffeomorphisms in the range of $\Psi_{i,A}$ have support in $\operatorname{int} D_{i,A}$. Thus, to construct $\Psi_{i,A}$, we need at least to solve the following problem: given u with support in $\operatorname{int} D_{i-1,A}$, find v with support in $\operatorname{int} D_{i,A}$ such that $[u] = [v]$.

Our method is to construct an auxiliary diffeomorphism τ_i, and then for given u with support in $\operatorname{int} D_{i-1,A}$ to construct v with support in $\operatorname{int} D_{i,A}$ such that $\tau_i u$ is conjugate to $\tau_i v$. In this section, we consider a preliminary question: given u and v satisfying suitable hypotheses, find sufficient conditions for $\tau_i u$ and $\tau_i v$ to be conjugate in the group $\operatorname{Diff}(\mathbf{R}^n, r, \alpha)$.

CONSTRUCTION of τ_i. If X is a vector field which generates a one-parameter group $\{\varphi_t\}$, we let $\exp X = \varphi_1$. We call $\exp X$ the *exponential* of X.

Let ϱ_1 be a C^∞ non-negative function on $\mathbf{R}$ whose support is a finite interval. Suppose $\varrho_1 = 1$ on $[-2A, 2A]$. Let $\varrho(x) = \varrho_1(x_1) \cdots \varrho_1(x_n)$ for any $x = (x_1, \dots, x_n) \in \mathbf{R}^n$. We let

$$\tau_i = \exp(\varrho \partial_i),$$

where ∂_i denotes the unit vector field on $\mathbf{R}^n$ in the direction of the ith coordinate.

We will also need another auxiliary mapping φ_i.

CONSTRUCTION of φ_i. We let φ_i be the mapping which is uniquely characterized by the following three conditions.

a) $\operatorname{dom}\varphi_i=\{x\in\mathbf{R}^n:|x_j|\leqslant 2A \quad \text{for} \quad i\neq j\}$.
b) $\varphi_{i*}(\partial_i)=\varrho\partial_i$
c) $\varphi_i \mid D_{0,A}=\text{id}$.
Note that b and c are compatible, since $\varrho=1$ on $D_{0,A}$.

Clearly φ_i maps each line parallel to the x_i axis into itself. Moreover $\operatorname{im}\varphi_i$ is a bounded subset of $\mathbf{R}^n$, open in $\operatorname{dom}\varphi_i$. In Figure 1 we have indicated schematically the domain and image of φ_1 in $\mathbf{R}^2$, as well as $D_{0,A}$.

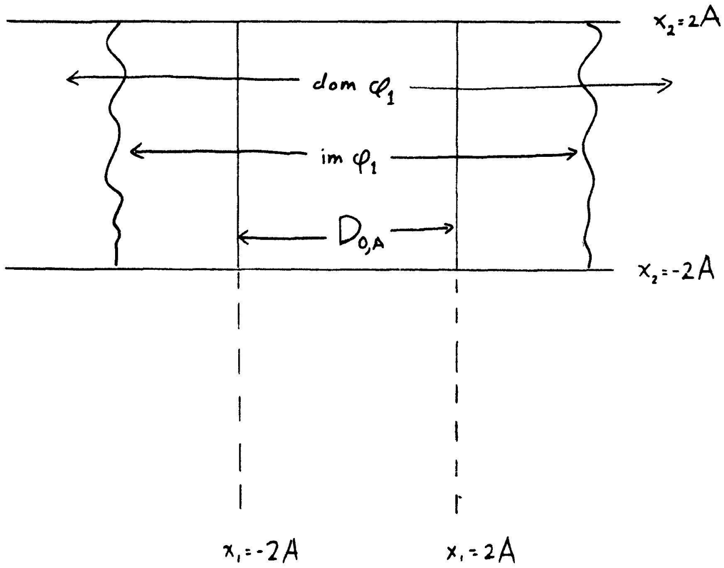

Fig. 1.

DEFINITION of T_i. Let T_i denote the unit translation in the ith coordinate:

$$T_i(x_1,\ldots,x_n)=(x_1,\ldots,x_i+1,\ldots,x_n).$$

Since $T_i=\exp(\partial_i)$, $\tau_i=\exp(\varrho\partial_i)$ it follows from the definition of φ_i that

$$\tau_i\varphi_i=\varphi_i T_i.$$

DEFINITION of $\mathscr{C}_i$. Let $\mathscr{C}_i$ denote the set of $(x_1,\ldots,\theta_i,\ldots,x_n)$, where each x_j is a real number and θ_i is a real number mod. 1. Let $\pi:\mathbf{R}^n\to\mathscr{C}_i$ denote the projection.

DEFINITION of Γ_u^i. Let u be a diffeomorphism of $\mathbf{R}^n$, close to the identity, with support in D_0. We define $\Gamma_u^i=\Gamma_u:\mathscr{C}_i\to\mathscr{C}_i$ as follows. Let $\theta\in\mathscr{C}_i$ and let $x\in\mathbf{R}^n$ be such that $\pi(x)=\theta$, and $x_i<-2A$. If for some $j\neq i$, we have $|x_j|>2A$, we let $\Gamma_u(\theta)=\theta$. Otherwise, we choose a positive integer N so large that $(T_iu)^N(x)_i>2A$, and let

$$\Gamma_u(\theta)=\pi(T_iu)^N(x).$$

It is possible to choose such an N if u is sufficiently close to the identity. Then Γ_u is a $C^{r,\alpha}$ diffeomorphism of $\mathscr{C}_i$, if u is of class $C^{r,\alpha}$, and sufficiently close to id.

DEFINITION of $\mathscr{G}$. Let $\mathscr{G}=\mathscr{G}^{r,\alpha}$ denote the group of $C^{r,\alpha}$ diffeomorphisms h of $\mathscr{C}_i$ such that

$$h(x_1,\ldots,\theta_i,\ldots,x_n)_j=h_j(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n),\quad j\neq i$$
$$h(x_1,\ldots,\theta_i,\ldots,x_n)_i=\theta_i+\theta(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n),$$

where $h_j:\mathbf{R}^{n-1}\to\mathbf{R}$ and $\theta:\mathbf{R}^{n-1}\to\mathbf{R}/\mathbf{Z}$.

LEMMA. *Suppose u, v are $C^{r,\alpha}$ diffeomorphisms of $\mathbf{R}^n$ which have support in* $\operatorname{int}D_{0,A}$. *If u and v are C^1 close to the identity and $\Gamma_v\Gamma_u^{-1}\in\mathscr{G}$, then $\tau_i u$ and $\tau_i v$ are conjugate elements of the group* Diff$(\mathbf{R}^n, r, \alpha)$.

Proof. The assumptions that u is C^1 close to id and has support in $\operatorname{int}D_{0,A}$ imply that for any $x\in\mathbf{R}^n$, there exists a positive integer N such that $(T_iu)^{-N}(x)_i<-2A$. For any $x\in\mathbf{R}^n$, we choose N so that this inequality holds, and let

$$\Lambda(x)=(T_iv)^N(T_iu)^{-N}(x).$$

Clearly, since v also has support in $\operatorname{int}D_{0,A}$, this is independent of N. Moreover, Λ is a $C^{r,\alpha}$ diffeomorphism of $\mathbf{R}^n$. Its inverse is given by

$$\Lambda^{-1}(x)=(T_iu)^N(T_iv)^{-N}(x)$$

for large N.

It is easily verified that

$$\Lambda T_iu\Lambda^{-1}=T_iv. \tag{1}$$

Clearly $\Lambda(x)=x$ if $|x_j|\geqslant 2A$ for some $j\neq i$ or $x_i<-2A$. Furthermore,

$$\Gamma_v\Gamma_u^{-1}\pi(x)=\pi\Lambda(x)$$

if $x_i>2A$.

From this and the hypothesis that $\Gamma_v\Gamma_u^{-1}\in\mathscr{G}$, we get:

a) If $j\neq i$, then there exists a mapping $\Lambda_j:\mathbf{R}^{n-1}\to\mathbf{R}$ such that

$$\Lambda_j(x_1,\ldots,\hat{x}_i,\ldots,x_n)=\Lambda(x)_j$$

if $x_i>2A$.

b) There exists a mapping $\beta:\mathbf{R}^{n-1}\to\mathbf{R}$ such that

$$\beta(x_1,\ldots,\hat{x}_i,\ldots,x_n)=\Lambda(x)_i-x_i$$

if $x_i>2A$.

We let

$$\Lambda'(x')=(\Lambda_1(x'),\ldots,\Lambda_{i-1}(x'),\Lambda_{i+1}(x'),\ldots,\Lambda_n(x'))$$

if $x'\in\mathbf{R}^{n-1}$. Clearly Λ' is of class $C^{r,\alpha}$ and equal to the identity outside of the set $\{|x_j|>2A, j\neq i\}$. It is easily seen that if u and v are close to the identity, then Λ' is close to the identity, and there is a $C^{r,\alpha}$ isotopy $\{\Lambda'_t\}$ of Λ' to the identity, with support in the set $\{|x_j|<2A, j\neq i\}$. Thus $\Lambda'_0=\Lambda'$ and $\Lambda'_1=\mathrm{id}$. We may (and do) assume that if we extend Λ'_t by setting $\Lambda'_t=\Lambda'$ for $t\leqslant 0$ and $\Lambda'_t=\mathrm{id}$ for $t\geqslant 1$, then the mapping

$$(x,t)\to\Lambda'_t(x):\mathbf{R}^{n-1}\times\mathbf{R}\to\mathbf{R}^{n-1}$$

is of class $C^{r,\alpha}$.

Let $B>0$ be such that $\varrho=0$ for $x_i>B-1$, where ϱ is the function which appears in the construction of τ_i. We define a mapping λ of $\mathbf{R}^n$ into itself, as follows.

$$\begin{aligned}\lambda(x)&=\varphi_i\Lambda\varphi_i^{-1}(x), \quad x\in\operatorname{im}\varphi_i\\ &=(\Lambda'(x'),\exp(\varrho\beta\partial_i)(x)_i), 2A\leqslant x_i\leqslant B\\ &=(\Lambda'_t(x'),x_i), \quad x_i=B+t, 0\leqslant t\leqslant 1\\ &=x \quad \text{otherwise.}\end{aligned}$$

Here, $x'=(x_1,\ldots,\hat{x}_i,\ldots,x_n)$. In the second equation above $\beta(x)$ is defined to be $\beta(x')$. In the second and third equations above, we have abused notation by writing the ith coordinate last.

We will show that λ is the conjugating diffeomorphism. First, we have to verify that λ is well-defined and of class $C^{r,\alpha}$. Before we begin the verifications, we recall that

$$D_0\subset\operatorname{im}\varphi_i\subset\operatorname{dom}\varphi_i.$$

In Fig. 1, we have indicated the relation between these sets schematically, in the case $n=2$, $i=1$. Recall $\varphi_i=\mathrm{id}$ on D_0.

The only x for which we have given two different definitions of $\lambda(x)$ are those x for which $x\in\operatorname{im}\varphi_i$ and $2A\leqslant x_i\leqslant B$. But

$$\begin{aligned}\Lambda(x)&=(\Lambda'(x'),x_i+\beta(x'))\\ &=(\Lambda'(x'),\exp(\beta\partial_i)(x)_i), \quad x_i\geqslant 2A,\end{aligned}$$

and

$$\varphi_{i*}\partial_i=\varrho\partial_i,$$

by the definition of φ_i and τ_i. Furthermore φ_i maps each line parallel to the x_i-axis into itself, and β is independent of x_i. The equation

$$\varphi_i A \varphi_i^{-1}(x) = (A'(x'), \exp(\varrho\beta\partial_i)(x)_i)$$

follows immediately.

It is clear that λ is $C^{r,\alpha}$ on $\mathrm{im}\,\varphi_i$. On the other hand, $\lambda(x)=x$ if $x_i < -2A$, and it is clear from our definition that $\lambda(x)$ is $C^{r,\alpha}$ on $\{x_i > 2A\} \cap \mathrm{dom}\,\varphi_i$, so λ is $C^{r,\alpha}$ on $\mathrm{dom}\,\varphi_i$. Since $\lambda(x)=x$ outside $\mathrm{dom}\,\varphi_i$ it is $C^{r,\alpha}$ on the complement of $\mathrm{dom}\,\varphi_i$. But $\lambda(x)=x$ also in a neighborhood of any boundary point of $\mathrm{dom}\,\varphi_i$, so it is $C^{r,\alpha}$ there also.

It is clear λ has compact support. In fact

$$\mathrm{supp}\,\lambda \subset \mathrm{dom}\,\varphi_i \cap \{-A \leqslant x_i \leqslant B+1\}.$$

It is also clear from the defining formulas for λ that it is an immersion. Moreover, by taking u and v close enough to the identity (with respect to the C^1 topology) we can arrange for λ to be arbitrarily close to the identity. But, if λ is sufficiently near the identity, then $\lambda \in \mathrm{Diff}(\mathbf{R}^n, r, \alpha)$.

All that remains to show is that λ conjugates $\tau_i u$ and $\tau_i v$. We assert

$$\lambda \tau_i u \lambda^{-1} = \tau_i v. \tag{2}$$

To begin with, $\mathrm{im}\,\varphi_i$ is invariant under λ, τ_i, and u. It is invariant under λ because of the definition of λ, under τ_i because of the definition of φ_i, and under u because $\mathrm{supp}\,u \subset D_0 \subset \mathrm{im}\,\varphi_i$. Moreover,

$$\varphi_i T_i \varphi_i^{-1} = \tau_i$$

by the definition of φ_i, and

$$\varphi_i u \varphi_i^{-1} = u$$

because $\mathrm{supp}\,u \subset D_0$. Hence, conjugating equation (1) by φ_i, we get that (2) holds on $\mathrm{im}\,\varphi_i$.

From our assumption that $\{\varrho > 0\}$ is convex, it follows that $\tau_i = \mathrm{id}$ on $\mathrm{dom}\,\varphi_i - \mathrm{im}\,\varphi_i$. We have $\tau_i u = \tau_i v = \tau_i$ outside $\mathrm{im}\,\varphi_i$, and it is clear that λ commutes with τ_i there, since $\lambda = \mathrm{id}$ outside $\mathrm{dom}\,\varphi_i$ and $\tau_i = \mathrm{id}$ on $\mathrm{dom}\,\varphi_i - \mathrm{im}\,\varphi_i$. This proves (2). Q.E.D.

§5. Construction of the Mappings $\Psi_{i,A}$

We consider again the problem: given u with support in $\mathrm{int}\,D_{i-1,A}$, find v with support in $\mathrm{int}\,D_{i,A}$ such that $\tau_i u$ and $\tau_i v$ are conjugate. In the previous section, we found sufficient conditions for $\tau_i u$ and $\tau_i v$ to be conjugate. In this section, we construct, for u sufficiently close to the identity, a $v = \Psi_{i,A}(u)$ satisfying these conditions.

We will assume in this section that u is a C^1 diffeomorphism of $\mathbf{R}^n$, $\operatorname{supp} u \subset \operatorname{int} D_{i-1}$, and u is C^1 close to the identity.

DEFINITION of h. We let h be the unique diffeomorphism of $\mathscr{C}_i$ onto itself which is the identity on the subset $\{\theta_i = 0\}$ of $\mathscr{C}_i$ and satisfies $h(\Gamma_u^i)^{-1} \in \mathscr{G}$. It is easily seen that if u is sufficiently close to the identity there is one and only one such diffeomorphism h.

DEFINITION of h_0, h_1. Since $\mathscr{C}_i = \mathbf{R}^n/\mathbf{Z}$, the manifold $\mathscr{C}_i$ has the structure of an abelian Lie group, induced by the group structure on $\mathbf{R}^n$. We use "+" for the group operation. We add mappings into $\mathscr{C}_i$ pointwise.

If h is sufficiently close to the identity, we can lift $h - \mathrm{id}$ to a mapping $\gamma: \mathscr{C}_i \to \mathbf{R}^n$, such that $\pi\gamma = h - \mathrm{id}$, and $\|\gamma(\theta)\| < 1$ for all $\theta \in \mathscr{C}_i$. Here $\pi: \mathbf{R}^n \to \mathscr{C}_i$ denotes the projection, as in the previous section.

We let ζ be a C^∞ bump function on the circle $\mathbf{R}/\mathbf{Z}$. We require that ζ be equal to 1 on a neighborhood of 0 mod. 1, equal to 0 in a neighborhood of 1/2 mod. 1 and that $0 \leqslant \zeta \leqslant 1$ everywhere.

We let

$$\zeta\gamma(x_1, \dots, \theta_i, \dots, x_n) = \zeta(\theta_i)\, \gamma(x_1, \dots, \theta_i, \dots, x_n)$$

$$h_0 = \pi \circ (\zeta\gamma) + \mathrm{id}$$

$$h_1 = h h_0^{-1}$$

If h is close to the identity, then so are h_0 and h_1, so they are diffeomorphisms of $\mathscr{C}_i$. Clearly

$$h = h_1 h_0,$$

h_0 is the identity in a neighborhood of the set of $(x_1, \dots, \theta_i, \dots, x_n)$ such that $\theta_i = 1/2$ mod. 1 and h_1 is the identity in a neighborhood of the set of $(x_1, \dots, \theta_i, \dots, x_n)$ such that $\theta_i = 0$ mod. 1.

CONSTRUCTION of v. Let E_- be the set of $(x_1, \dots, x_n)$ in $\mathbf{R}^n$ such that $-3/2 < x_i < -1/2$, and let E_+ be the set of $(x_1, \dots, x_n)$ in $\mathbf{R}^n$ such that $0 < x_i < 1$. We let $v \mid \mathbf{R}^n - E_- - E_+ = \mathrm{id}$. We let $v \mid E_-$ be the unique diffeomorphism of E_- into itself such that $\pi v \mid E_- = h_0 \pi \mid E_-$. We let $v \mid E_+$ be the unique diffeomorphism of E_+ into itself such that $\pi v \mid E_+ = h_1 \pi \mid E_+$.

We set $\Psi_{i,A}(u) = v$.

Of the properties (1)–(6) of $\Psi_{i,A}$ listed in §3, properties (1)–(3) are easy to see. The verification of these properties will be omitted. As noted there, we can arrange for property (4) to hold by replacing the domain of $\Psi_{i,A}$ with a possibly smaller neighborhood of the identity.

To prove property (5), it is enough to notice that if u is of class $C^{r,\alpha}$ then $\tau_i u$ and $\tau_i v$ are conjugate in Diff$(\mathbf{R}^n, r, \alpha)$. But it is easily seen that $\Gamma_v^i = h$. Hence $\Gamma_v^i(\Gamma_u^i)^{-1} \in \mathscr{G}$, and it follows from the lemma in the previous section that $\tau_i u$ and $\tau_i v$ are conjugate.

§6. Property (6)

In this section, we complete the verification that the mappings $\Psi_{i,A}$ have the properties listed in §3. The only property we have not verified is (6).

Before we begin the verification of (6), we introduce some more notation.

The projection mapping $\pi: \mathbf{R}^n \to \mathscr{C}_i$ gives us a preferred system of coordinates in a neighborhood of any point of $\mathscr{C}_i$. The transition mappings between different coordinate systems which we obtain in this way are all translations. It follows that the rth derivative of any C^r mapping of $\mathscr{C}_i$ into itself is defined independently of the choice of preferred coordinate system. The rth derivative of such a mapping v is a mapping $D^r v: \mathscr{C}_i \to SL^r(\mathbf{R}^n, \mathbf{R}^n)$ of $\mathscr{C}_i$ into the space of symmetric r-linear mappings of $\mathbf{R}^n$ into itself. If v vanishes outside of a compact subset of $\mathscr{C}_i$, we define $\|v\|_{r,\alpha}$ by the same formula as we used to define $\|u\|_{r,\alpha}$ in §3, but with U replaced by $\mathscr{C}_i$ and u replaced by v. If v is the identity outside a compact subset of $\mathscr{C}_i$, we define $\mu_{r,\alpha}(v) = \|v - \mathrm{id}\|_{r,\alpha}$.

There are three steps in the proof of (6) in §3. In step 1, we show if u is $C^{r,\alpha}$ and sufficiently near the identity, then

$$\mu_{r,\alpha}(\Gamma_u) \leqslant 8A\mu_{r,\alpha}(u). \tag{1}$$

In step 2, we show that if u is $C^{r,\alpha}$ and sufficiently near the identity, then

$$\mu_{r,\alpha}(h) \leqslant 3\mu_{r,\alpha}(\Gamma_u), \tag{2}$$

where h is the diffeomorphism of $\mathscr{C}_i$ constructed in the previous section.

In step 3, we show that there exists a constant $C_1 > 0$, independent of A, such that if u is sufficiently near the identity, then

$$\mu_{r,\alpha}(\Psi_{i,A}(u)) \leqslant C_1 \mu_{r,\alpha}(h). \tag{3}$$

Then (6) of §3 follows, with $C = 24C_1$.

Step 1. From the estimate (!) in §3, it follows that if N is a positive integer, then there exists $\delta > 0$, such that if u has support in intD_{i-1} and $\mu_{r,\alpha}(u) < \delta$, then

$$\mu_{r,\alpha}((T_i u)^N) \leqslant (N+1)\, \mu_{r,\alpha}(u).$$

Hence

$$\mu_{r,\alpha}(\Gamma_u) \leqslant (N+1)\, \mu_{r,\alpha}(u),$$

where N is large enough so that for any $\theta\in\mathscr{C}_i$ there exists $x\in\mathbf{R}^n$ such that $\pi(x)=\theta$, $x_i<-2A$, and $(T_iu)^N(x)_i>2A$.

We can take N to be any integer $\geqslant 4A+2$, if u is close enough to the identity. In particular, we can suppose $N+1\leqslant 4A+4$. Hence

$$\mu_{r,\alpha}(\Gamma_u)\leqslant(4A+4)\,\mu_{r,\alpha}(u).$$

Since $A\geqslant 1$, the inequality (1) follows.

Step 2. We recall that h is the unique diffeomorphism such that h is the identity on $\{\theta_i=0\}$ and $h\Gamma_u^{-1}\in\mathscr{G}$. Let $g=h\Gamma_u^{-1}$. Then g^{-1} is the unique element of $\mathscr{G}$ which equals Γ_u on $\{\theta_i=0\}$. It follows easily that

$$\mu_{r,\alpha}(g^{-1})\leqslant\mu_{r,\alpha}(\Gamma_u).$$

(We are assuming $\mathbf{R}^n$ is provided with the norm $\|(x_1,\dots,x_n)\|=|x_1|+\cdots+|x_n|$.) On the other hand, for any $\lambda>1$, we have

$$\mu_{r,\alpha}(g)\leqslant\lambda\mu_{r,\alpha}(g^{-1}),$$

if g is in a sufficiently small $C^{r,\alpha}$ neighborhood of the identity, depending on λ. Since $h=g\Gamma_u$, the inequality (2) follows immediately, for u in a sufficiently small $C^{r,\alpha}$ neighborhood of the identity.

Step 3. From the definition of $v=\Psi_i(u)$ in the previous section, it is clear that

$$\mu_{r,\alpha}(v)=\sup\{\mu_{r,\alpha}(h_0),\,\mu_{r,\alpha}(h_1)\}.$$

Moreover, if $\lambda>1$, and u (and therefore also h) is in a sufficiently small $C^{r,\alpha}$ neighborhood of the identity, then

$$\mu_{r,\alpha}(h_1)\leqslant\lambda(\mu_{r,\alpha}(h)+\mu_{r,\alpha}(h_0)),$$

since $h_1=hh_0^{-1}$. Thus, it is enough to estimate $\mu_{r,\alpha}(h_0)$ in terms of $\mu_{r,\alpha}(h)$.

In view of the definition of h_0 and Leibniz's formula for the derivative of a product, it is enough to prove the following assertion. There exists $C>0$ such that

$$\mu_{s,\alpha}(h)\leqslant C\mu_{r,\alpha}(h),\qquad 0\leqslant s\leqslant r$$

if h is a $C^{r,\alpha}$ mapping of $\mathscr{C}_i$ into itself which is the identity outside a compact set, and is also the identity on $\{\theta_i=0\}$. However, this is an easy consequence of decreasing induction on s, the mean value theorem, and the fact that every point of $\mathscr{C}_i$ is of distance $\leqslant 1/2$ to the set $\{\theta_i=0\}$.

This concludes the verification of the properties of $\Psi_{i,A}$ which were listed in §3. Thus, we have completed the proof of Theorem 2.

§7. Application to Haefliger's Classifying Space

Let $F\Gamma_n^r$ denote Haefliger's classifying space for C^r foliations of codimension n with framed normal bundle. What we have done and known results show that $F\Gamma_n^r$ is $(n+1)$-connected for $\infty > r \geqslant n+2$. This was already known by Thurston's work for $r=\infty$. We also get that $F\Gamma_n^{r,\alpha}$ is $(n+1)$-connected, under the hypotheses of Theorem 2.

There are two known results that we need. One is Haefliger's theorem that $F\Gamma_n^r$ is n-connected.

The other is a theorem of Thurston that was announced in [4]. To state this theorem, we need to introduce some more definitions. If G is a topological group, we let G_δ denote G with the discrete topology. We let $\bar{G}$ denote the homotopy theoretic fiber of the identity mapping $G_\delta \to G$. In the usual realization, this is the space of paths emanating from the origin, topologized with the topology generated by the compact-open topology and the discrete topology on end-points. It is a topological group.

We think of $\mathrm{Diff}(\mathbf{R}^n, r)$ as the direct limit of its subgroups consisting of diffeomorphisms having support in open balls, and topologize it with the direct limit topology. According to Thurston [4], there is a mapping

$$B\,\overline{\mathrm{Diff}}(\mathbf{R}^n, r) \to \Omega^n F\Gamma_n^r$$

which induces isomorphism in integer homology. It follows that $F\Gamma_n^r$ is $(n+1)$-connected if and only if

$$H_1(B\,\overline{\mathrm{Diff}}(\mathbf{R}^n, r)) = 0. \tag{1}$$

We assert that what we have done in this paper shows that this homology group vanishes.

In the notation of §3, let u be a fixed point of θ_f. We have shown,

$$u_0 = \tilde{A} f u \tilde{A}^{-1}$$
$$\tau_i u_{i-1} = \lambda_i^{-1} \tau_i u_i \lambda_i, \qquad 1 \leqslant i \leqslant n,$$

where λ_i is a suitable element of $\mathrm{Diff}(\mathbf{R}^n, r)$. Moreover, since u is a fixed point of θ_f, we have $u_n = u$. Here $\tilde{A}$ and τ_i are fixed elements of $\mathrm{Diff}(\mathbf{R}^n, r)$, given in advance. By restricting f to lie in a sufficiently small neighborhood of the identity, we can arrange for the u_i and λ_i to lie in an arbitrarily small neighborhood of the identity. From these facts, it follows easily that (1) holds.

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