Zeitschrift: Commentarii Mathematici Helvetici

Band: 49 (1974)

Artikel: On the genus of finite CW-H-spaces

Autor: Zabrodsky, A.

DOI: https://doi.org/10.5169/seals-37978

On the genus of finite CW–H-spaces

A. ZABRODSKY

The study of the Grothendieck groups of certain stable categories (initiated in [1] and [2] and pursued in [3]) led to the study of two phenomenas: That of non cancellation [4] and that of the genus of a space [7]. Their inter-relations were observed and their study led to the discovery of new finite CW-H-spaces. Our main reference in this study of the genus of an H-space should be [7] but many of the facts studied here are closely related to those studied in [1], [4] and [6]. Here we use the notion of genus with regard to p-equivalence in the sense of Serre ([5]) rather than that of localization which are equivalent for H-spaces (see [7]).

0. Notations, Definitions and Summary of Results

As usual we denote by PA and QA the modules of primitives and indecomposables in a Hopf algebra A.

We denote by $\mathbf{P}$ the set of all primes. All spaces considered are of the homotopy type of simply connected CW complexes of finite type, all graded modules, accordingly, are finitely generated in each dimension.

Let $p\in\mathbf{P}$. A homomorphism $\varphi: G\to G'$ between two finitely generated abelian groups is said to be a p-epimorphism (p-monomorphism) if coker φ (ker φ) is a finite group with order prime to p. If φ is both p-epimorphism and p-monomorphism then it is said to be p-isomorphism. If $\mathbf{P}_1\subset\mathbf{P}$ then φ is said to be a $\mathbf{P}_1$-epimorphism ($\mathbf{P}_1$-monomorphism or $\mathbf{P}_1$-isomorphism) if it is a p-epimorphism (p-monomorphism or p-isomorphism) for every $p\in\mathbf{P}_1$.

A map $\varphi: Y\to X$ is said to be a $\mathbf{P}_1$ equivalence iff $\pi_*(\varphi)$ (equivalenty $H_*(\varphi, Z)$, $H^*(\varphi, Z)$) is a $\mathbf{P}_1$ isomorphism. By the genus of the space X we mean the set $G(X)$ of homotopy classes of spaces Y which are p-equivalent to X for every $p\in\mathbf{P}$.

If X is a CW complex the homotopy (or Postnikov) approximation of X in dim$\leqslant m$ is a pair $(Ht_m(X), \tau_m)$ where $\tau_m: X\to Ht_m(X)$ is such that $\pi_k(\tau_m)$ is an isomorphism for $k\leqslant m$ and $\pi_k(Ht_m(X))=0$ for $k>m$. Similarly, the homology (or Moore) approximation of X is a pair $(Hl_m(X), \tau'_m)$ where $\tau'_m: Hl_m(X)\to X$ yields an isomorphism $H_k(Hl_m(X), Z)\to H_k(X, Z)$ for $k\leqslant m$ and $H_k(Hl_m(X), Z)=0$ for $k>m$.

Though we work in the category of CW complexes and continuous maps by a commutative diagram we mean commutative up to homotopy.

A commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f_2} & Y_2 \\ \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h_1} \\ Y_1 & \xrightarrow{f_1} & X_0 \end{array} \tag{0.1}$$

is said to be a fiber square induced by f_1, h_1 if up to homotopy one has

$$X = \{y_1, y_2, \varphi \in Y_1 \times Y_2 \times PX_0 \mid f_1(y_1) = \varphi(0), h_1(y_2) = \varphi(1)\}.$$

$$h_2 = p_1 \mid X, \quad f_2 = p_2 \mid X$$

where

$$p_i: Y_1 \times Y_2 \times PX_0 \to Y_i \quad \text{are the projections.}$$

(0.1) is said to be a fiber square in dim$\leqslant m$ iff it becomes a fiber square after applying Ht_m to the entire diagram. If X is a finite dimensional H-space then $H^*(X, Q) = \Lambda(\tilde{x}_{n_1}, \tilde{x}_{n_2}, \ldots, \tilde{x}_{n_r})$, $\tilde{x}_{n_i} \in H^{n_i}(X, Q)$ $n_i \leqslant n_{i+1}$. X is then said to be of type $(n_1, n_2, \ldots, n_r)$. Given a vector $\bar{n} = (n_1, n_2, \ldots, n_r)$ of natural numbers we write $K(Z, \bar{n}) = \prod_{i=1}^r K(Z, n_i)$, $S^{\bar{n}} = \prod_{i=1}^r S^{n_i}$.

In this study we consider $G(X)$ for a finite dimensional H-space. Using exactly the same (or dual) methods one can obtain similar (or dual) results for H-spaces with finitely many non-vanishing homotopy groups (or for finite dimensional co-H-spaces).

To state results of this study concerning the structure of $G(X)$ one needs few notations: If A and B are matrices (over any ring R) denote by $A * B$ the matrix

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

Let X be an H-space of type $(n_1, n_2, \ldots, n_r)$. Hom $\mathring{}_Z(QH^*(X, Z)/\text{torsion}, QH^*(X, Z)/\text{torsion})$ can be identified with the set of all matrices (over Z) of the form $A_{s_1} * A_{s_2} * \ldots * A_{s_l}$, A_{s_i} being an $s_i \times s_i$ matrix, $s_1, s_2, \ldots s_l$ uniquely determined by $n_1, n_2, \ldots n_r$. Denote all such matrices by $\mathscr{M}(s_1, s_2, \ldots, s_l; Z)$. Given an integer t let $\mathscr{A} = \mathscr{A}_t$ be the set of all matrices in $\mathscr{M}(s_1, s_2, \ldots, s_l; Z)$ which are invertible modZ_t. Then one has:

Theorem (essentially 2.4*)*. There exists an integer t $(= \bar{t}_k$ in 2.4) and a correspondence $\xi: \mathscr{A} = \mathscr{A}_t \to G(X)$ described as follows: Fix a map $h_0: X \to K(Z, \bar{n})$ so that $QH^*(h_0, Z)/\text{torsion}$ is an isomorphism. Given $A \in \mathscr{A}$ let $f_0 = f_0(A): K(Z, \bar{n}) \to K(Z, \bar{n})$

be given by $H^*(f_0, Z)\, \iota_{n_i} = \sum_j a_{ij}\, \iota_{n_j}$ $A = (a_{ij})$. If

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{f_1} & X \\ \downarrow{\scriptstyle \tilde{h}_1} & & \downarrow{\scriptstyle h_0} \\ K(Z, \bar{n}) & \xrightarrow{f_0} & K(Z, \bar{n}) \end{array}$$

is a fiber square then

(a) $[Hl_k(\tilde{Y})] = [Y] = \xi(A) \in G(X)$ $\quad (k = \dim X)$

(b) $QH^*(h_1, Z)$/torsion is an isomorphism where $h_1: Y \to K(Z, \bar{n})$ is induced by $\tilde{h}: h_1 = \tilde{h}_1 \tau'$.

Corollary (2.6) ξ is onto.

If $\tilde{\alpha}: \mathcal{M}(s_1, s_2, \ldots, s_l; Z) \to \mathcal{M}(s_1, s_2, \ldots, s_l; Z_t)$ is the reduction it induces homomorphism

$$\tilde{\alpha}: GL(s_1, \ldots, s_l; Z) \to GL(s_1, s_2, \ldots, s_l, Z_t)$$

of invertible matrices and $(\operatorname{coker} \tilde{\alpha}) = [(Z_t^*)/\{\pm 1\}]^l$ where Z_t^* are the units in Z_t. Note that $\mathscr{A} = \tilde{\alpha}^{-1}\, GL(s_1, s_2, \ldots, s_l; Z_t)$.

Structure Theorem (2.7) ξ factors through $\operatorname{coker} \tilde{\alpha}$ to obtain a correspondence $\bar{\xi}: [Z_t^*/\{\pm 1\}]^l \to G(X)$ which is onto.

The correspondence is given by

$$\bar{\xi}(d_1, d_2, \ldots, d_l) = \xi(\xi_{d_1} * I_{d_2} * \cdots * I_{d_l})$$

where I_{d_i} is the $s_i \times s_i$ matrix given by

$$I_{d_i} = \begin{pmatrix} I & 0 \\ 0 & d_i \end{pmatrix}$$

One also obtains:

H-*structure theorem* (2.10, *compare with* [6] *and* [7] *lemma* 1.4):

Let X be a finite CW complex. If for every prime p there exists an H-space $X(p)$ and a p-equivalence $f_p: X \to X(p)$ and if $H^*(X(p), Q)$ are all isomorphic as Hopf algebras then X is an H-space.

A Product Theorem (2.11). Let $A, A' \in \mathscr{A}$. Then

$$\xi(A) \times \xi(A') = \xi(I) \times \xi(A \cdot A') = X \times \xi(A \cdot A').$$

Consequently if $A_i, B_i \in \mathscr{A}$ and

$$A_1 \cdot A_2 \cdots \cdot A_m = B_1 \cdot B_2 \cdots \cdot B_m$$

then

$$\xi(A_1)\times\xi(A_2)\cdots\times\xi(A_m)\approx\xi(B_1)\times\xi(B_2)\cdots\times\xi(B_m).$$

Theorem (2.12). If $[Y]\in G(X)$ then

$$Y^{\varphi(t)/2}\approx X^{\varphi(t)/2}$$

where φ is the Euler function. Finally, one obtains

A non cancellation theorem (3.5 *and compare with* [1], [2] *and* [7], p. 83). If $[Y]\in G(X)$ then $Y\times S^{\bar{n}}\approx X\times S^{\bar{n}}$.

1. Some Algebraic Lemmas

The following are well known simple facts concerning finitely generated abelian groups:

1.1. LEMMA: *Let G and G′ be finitely generated abelian groups. If for every prime p there exists a p isomorphism $\varphi_p: G\to G'$ then G and G′ are isomorphic.*

1.2. LEMMA: *Suppose $G\approx G'$. Then $\varphi: G\to G'$ is a p-isomorphism if and only if $\varphi\otimes Z_p$ is an isomorphism.*

1.3. COROLLARY. *Let $\{\varphi_p\,|\,p\in\mathbf{P}_1\}$ be a finite family, $\varphi_p: G\to G'\approx G$ a p-isomorphism. Put $a=\prod_{p\in\mathbf{P}_1}p$. Then $\varphi=\sum_{p\in\mathbf{P}_1}(a\cdot p^{-1})\,\varphi_p$ is a $\mathbf{P}_1$ isomorphism.*

Proof. For every $p_0\in P_1$ $\varphi\otimes Z_{p_0}=a\,p_0^{-1}(\varphi_{p_0}\otimes Z_{p_0})$ is an isomorphism. Hence φ is a $\mathbf{P}_1$ isomorphism.

1.4. COROLLARY. *If $[Y]\in G(X)$ then*

(a) $H^*(Y)\approx H^*(X)$ *and* $\pi(X)\approx\pi(Y)$.

(b) *Let $f: Y\to X$. If $\pi(f)\otimes Z_p$ is an isomorphism for all $p\in\mathbf{P}_1$ then f is a $\mathbf{P}_1$ equivalence.*

Throughout this chapter let X be a finite CW-H-space.

1.5. PROPOSITION. *If $[Y]\in G(X)$ then there exists a partition $\mathbf{P}=\mathbf{P}_1\cup\mathbf{P}_2$ and two maps $f_i: Y\to X$ $i=1,2$ so that f_i is a $\mathbf{P}_i$ equivalence.*

Proof. Choose an arbitrary prime $q\in\mathbf{P}$. Let $f_2: Y\to X$ be a q-equivalence. Then there exists a set $\mathbf{P}_2$ of primes so that $\mathbf{P}_1=\mathbf{P}-\mathbf{P}_2$ is finite and f_2 is a $\mathbf{P}_2$ equivalence. For every $p\in\mathbf{P}_1$ let f_p be a p-equivalence. Let a be the product of all primes in $\mathbf{P}_1$. If $f_1: Y\to X$ is given by $[f_1]=\prod_{p\in\mathbf{P}_1}[f_p]^{a/p}$ (where the product represents a product in the algebraic loop $[Y, X]$ with an arbitrary bracketing) then by 1.3 $\pi_n(f_1) = \sum_{p\in \mathbf{P}_1}(ap^{-1})\,\pi_n(f_p)$ is a $\mathbf{P}_1$ isomorphism and f_1 is a $\mathbf{P}_1$ equivalence.

One notices that $\mathbf{P}_1$ may be replaced by any larger (but finite) set of primes.

1.6. LEMMA. *Given a commutative diagram*

$$\begin{array}{ccc} T & \xrightarrow{h_2} & Y_1 \\ \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle f_1} \\ Y_2 & \xrightarrow{f_2} & X_0 \end{array} \tag{1.6.1}$$

If for $i=1, 2$ h_i *and* f_i *are* $\mathbf{P}_i$ *equivalences in* $\dim\leqslant m$, $\mathbf{P}=\mathbf{P}_1\cup\mathbf{P}_2$ *then* (1.6.1) *is a fiber square in* $\dim\leqslant m$.

Proof. Form the fiber square of f_1 and f_2

$$\begin{array}{ccc} Z' & \xrightarrow{h'_2} & X \\ \downarrow{\scriptstyle h'_1} & & \downarrow{\scriptstyle f_1} \\ Y & \xrightarrow{f_2} & X_0 \end{array}$$

then there exists $\alpha: Z\to Z'$ with $h_i\sim h'_i\alpha$ $i=1, 2$. f_i being $\mathbf{P}_i$ equivalence in $\dim\leqslant m$ implies that h'_i is a $\mathbf{P}_i$ equivalence and as h_i is a $\mathbf{P}_i$ equivalence so is α. Now, α being $\mathbf{P}_1$ and $\mathbf{P}_2$ equivalence in $\dim\leqslant m$ is a homotopy equivalence in that range.

Let R be a ring. If $M=\{M_n \mid 0\leqslant n\leqslant l\}$ is a graded module over R, M_n-free of rank s_n then $\tilde{\mathcal{M}}(R)=\mathrm{Hom}^\circ_R(M, M)$ can be represented by the set of matrices of the form $A_1 * A_2 * \cdots * \mathcal{A}_l$ where A_n is $s_n\times s_n$ matrix over R and $B_1 * B_2$ is the matrix of the form

$$\begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}$$

Denote this set by $\mathcal{M}(s_1, s_2, \ldots, s_l; R)$. Similarly define $GL(s_1, s_2, \ldots, s_l; R)$ and $SL(s_1, s_2, \ldots, s_l; R)$ as the invertible and determinant one matrices in $\mathcal{M}(s_1, s_2, \ldots, s_l; R)$. The product $*g$ in $M(s_1, s_2, \ldots, s_l; R)$ given by

$$(A_1 * A_2 * \cdots * A_l) * g\,(\hat{A}_1 * \hat{A}_2 * \cdots * \hat{A}_l) = (A_1 * \hat{A}_1) * (A_2 * \hat{A}_2) * \cdots * (A_l * \hat{A}_l)$$

corresponds to the natural homeomorphism $\mathrm{Hom}^\circ(M, M)\otimes\mathrm{Hom}^\circ(M, M)\to \mathrm{Hom}^\circ(M\otimes M, M\otimes M)$. Let X be of type $(n_1, n_2, \ldots, n_r)$. Define $s_1, s_2, \ldots, s_l$ by the relation

$$n_{s_1+s_2+\cdots+s_m} < n_{s_1+s_2+\cdots+s_m+1} = n_{s_1+s_2+\cdots+s_m+s_{m+1}} \qquad 0\leqslant m < l \tag{1.7}$$

Put $M_m = QH^n_{s_1+s_2+\cdots+s_m}(X, Z)/\text{torsion}$ and one has

$$\mathrm{Hom}^\circ[QH^*(X, Z)/\text{torsion}, QH^*(X, Z)/\text{torsion}] = \mathcal{M}(s_1, s_2, \ldots, s_l; Z).$$

Let t be an integer. Put

$$\alpha: GL(n, Z) \to GL(n, Z_t)$$

Then one has

1.8. LEMMA. *$GL(n, Z_t)/\mathrm{im}\,\alpha = Z_t^*/\{\pm 1\}$ where Z_t^* are the units in Z_t. The homomorphism $GL(n, Z_t) \to GL(n, Z_t)/\mathrm{im}\,\alpha$ is given by the determinant.*

Let $\tilde{\alpha}: \mathscr{M}(s_1, s_2, \ldots, s_l; Z) \to \mathscr{M}(s_1, \ldots, s_l; Z_t)$ induce

$$\tilde{\alpha}: GL(s_1, s_2, \ldots, s_l; Z) \to GL(s_1, s_2, \ldots, s_l; Z_t)$$

then

$$GL(s_1, s_2, \ldots, s_l; Z_t)/\mathrm{im}\,\tilde{\alpha} \approx [Z_t^*/\{\pm 1\}]^l.$$

2. The Structure of $G(X)$

Throughout this chapter let (X, μ) be an H-space of type $\bar{n} = (n_1, n_2, \ldots, n_r)$. To this one associate the sequence $(s_1, s_2, \ldots, s_l)$ defined by (1.7). Let $\dim X \leqslant k$. Given an integer m, $m \geqslant k$ let $t_m = t_m(G(X))$ be the order of the finite group

$$\sum_{n \leqslant m} (\ker \sigma_n + \operatorname{coker} \sigma_n)$$

where $\sigma_n: \pi_n(X) \to PH_n(X)/\text{torsion}$ is the Hurewicz-Serre homomorphism.

Another interpretation for t_m can be given by the following

2.1. LEMMA. *Let $h_0: X \to K(Z, \bar{n}) = \prod_i K(Z, n_i)$ satisfy: $x_{n_i} = H^*(h_0, Z)\,\iota_{n_i}$, $1 \leqslant i \leqslant r$, represent a basis for $QH^*(X, Z)/\text{torsion}$. Let $F = \text{fiber}\,h_0$. Then* $\text{order}\,\pi_n(F) = \text{order}(\ker \sigma_n + \operatorname{coker} \sigma_{n+1})$. *Hence,* $t_m = \text{order} \sum_{n \leqslant m} \pi_n(F)$.

(Note that $\operatorname{coker} \sigma_{m+1} = PH_{m+1}(X, Z)/\text{torsion} = 0$).

Proof. One has the following commutative diagram

$$\begin{array}{ccc} \pi_n(X) & \xrightarrow{\pi_n(h_0)} & \pi_n(K(Z, \bar{n})) \\ \downarrow{\scriptstyle \sigma_n(X)} & & \approx\downarrow{\scriptstyle \sigma_n(K)} \\ PH_n(X, Z)/\text{torsion} & \xrightarrow[\approx]{h_{0*}} & PH_n(K(Z, \bar{n}), Z)/\text{torsion} \end{array}$$

As $QH^*(h_0, Z)/\text{torsion}$ is an isomorphism so is its dual h_{0*}. Hence, $\ker \sigma_n = \ker \pi_n(h_0)$ and $\operatorname{coker} \sigma_n = \operatorname{coker} \pi_n(h_0)$ and obviously $\text{order}\,\pi_n(F) = \text{order}[\ker \pi_n(h_0) + \operatorname{coker} \pi_{n+1}(h_0)]$.

2.2. PROPOSITION. *Given a set $\{x_{n_1}, x_{n_2}, \ldots, x_{n_r}\}$, $x_{n_i} \in H^{n_i}(X, Z)/\text{torsion}$ which* *reduces to a basis for* $QH^*(X, Z)$/torsion *and given maps* $g: Y \to \tilde{Y}, f: Y \to X$, *(*$Y$ *and* $\tilde{Y}$ *arbitrary* CW *complexes of* $\dim \leqslant m$*) so that* $\mathrm{im}[\tilde{H}(g, Z)/\mathrm{torsion}]$ *is an ideal in* $\tilde{H}(Y, Z)$/torsion. *For any set* $\{z_{n_1}, \ldots, z_{n_r}\}$, $z_{n_i} \in H^{n_i}(\tilde{Y}, Z)$/torsion *there exist maps* $\tilde{f}_i: \tilde{Y} \to X, i = 1, 2, \ldots, r$ *with* torsion $\pi_n(\tilde{Y}) \subset \ker \pi_n(\tilde{f}_i)$ *for* $n \leqslant m$ *so that*

$$[H^*(\hat{f}, Z)/\mathrm{torsion} - H^*(f, Z)/\mathrm{torsion}]\, x_{n_i} = t_m [(H^*(g, Z)/\mathrm{torsion})\, z_{n_i}] \quad (2.2.1)$$

where $\hat{f}$ *is given by*

$$[\hat{f}] = [\tilde{f}_r \circ g]([\tilde{f}_{r-1} \circ g] \cdots ([\tilde{f}_2 \circ g]([\tilde{f}_1 \circ g] \cdot [f]) \cdots))$$

(the products taken in $[Y, X]$*).*

Proof. Let $X_m = Ht_m(X)$. As $\dim Y \leqslant m$, $\dim \tilde{Y} \leqslant m$ and $m \geqslant k$ one can replace all maps in $[Y, X]$ and $[\tilde{Y}, X]$ by the corresponding maps in $[Y, X_m]$ and $[\tilde{Y}, X_m]$. Consider x_{n_i} being in $H^{n_i}(X_m, Z)$/torsion and it suffices to prove relation (2.2.1) for X_m instead of X. By 2.1 $t_m = \mathrm{order}\, \pi_*(F_m)$ (F_m the fiber of $X_m \to K(Z, \bar{n})$). Hence, $t_m[1]$ ($[1] \in [K[Z, \bar{n}), K(Z, \bar{n})]$) lifts to $h_1: K(Z, \bar{n}) \to X$, $[h_0 \circ h_1] = t_m[1]$. We construct $\tilde{f}_i$ inductively. Suppose $\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_{i-1}$ where constructed so that if

$$[\hat{f}_{i-1}] = [\tilde{f}_{i-1} \circ g]([\tilde{f}_{i-2} \circ g](\cdots([\tilde{f}_1 \circ g] \cdot [f])))$$

then

$$[H^*(\hat{f}_{i-1}, Z)/\mathrm{torsion} - H^*(f, Z)/\mathrm{torsion}]\, x_{n_j} = \begin{cases} t_m(H^*(g, Z)/\mathrm{torsion})\, z_{n_j} & j < i \\ t_m(H^*(g, Z)/\mathrm{torsion})\, \tilde{z}_{n_j} & j \geqslant i \end{cases}$$

(there is no problem in starting the induction by starting with $\tilde{f}_0: \tilde{Y} \to X$ being the constant map and $\tilde{z}_{n_j} = 0$). Let $\tilde{f}_i$ be the composition

$$\tilde{Y} \xrightarrow{z_{n_i} - \tilde{z}_{n_i}} K(Z, n_i) \xrightarrow{\mathrm{inj}} K(Z, \bar{n}) \xrightarrow{h_1} X.$$

Then torsion $\pi(\tilde{Y}) \subset \ker \pi(\tilde{f}_i)$. Let $(\ )^*$ denote $H^*(\ , Z)$/torsion. If $[\hat{f}_i] = [\tilde{f}_i \circ g]\, [\hat{f}_{i-j}]$ then

$$\hat{f}_i^* x_{n_j} = \Delta^*(\tilde{f}_i \circ g \times \hat{f}_{i-1})^* \mu^* x_{n_j} = \sum_{s=0}^{s_0} (\tilde{f}_i \circ g)^* x'_{n_j, s} \cdot \hat{f}_{i-1}^* x''_{n_j, s}$$

where $x'_{n_j, s}$ and $x''_{n_j, s}$ are given by $\mu^* x_{n_j} = \sum_{s=0}^{s_0} x'_{n_j, s} \otimes x''_{n_j, s}$ ($x'_{n_j, 0} = x_{n_j} = x''_{n_j, s_0}$, $x''_{n_j, 0} = 1 = x'_{n_j, s_0}$ and for $0 < s < s_0$, $0 < \dim x'_{n_j, s} < n_j$, $0 < \dim x''_{n_j, s} < n_j$).

As $H^n(\tilde{f}_i \circ g, Z)$/torsion$=0$ for $n < n_i$ and $\tilde{f}_i^* x_{ni} = \tilde{f}_i^* h_0^* \iota_{ni} = t_m(z_{n_i} - \tilde{z}_{n_i})$ one has $\hat{f}_i^* x_{n_j} = \hat{f}_{i-1}^* x_{n_j}$ if $j < i$,

$$\hat{f}_i^* x_n = \hat{f}_{i-1}^* x_{n_i} + t_m g^*(z_{n_i} - \tilde{z}_{n_i})$$

and hence for $j \leqslant i$

$$(\hat{f}_i^* - f^*)\, x_{n_j} = (\hat{f}_i^* - \hat{f}_{i-1}^*)\, x_{n_j} + (\hat{f}_{i-1}^* - f^*)\, x_{n_j} = t_m g^* z_n$$

For $j > i$ (as $\tilde{f}_i^* x_{n_j} = 0$ for $j \neq i$) one has

$$\alpha_j = (\hat{f}_i^* - \hat{f}_{i-1}^*)\, x_{n_j} = \sum_{s=1}^{s_0-1} \hat{f}_{i-1}^* x''_{n_j,s} (\tilde{f}_i \circ g)^*\, x'_{n_j,s}$$

and as $\operatorname{im} \overline{(\tilde{f}_i \circ g)^*} \subset t_m \overline{(\operatorname{im} g^*)}$ and as $\overline{\operatorname{im} g^*}$ is an ideal $\alpha_j \in t_m \overline{\operatorname{im} g^*}$ and hence for $i > j$ $(\hat{f}_i^* - f^*)\, x_{n_j} = (\hat{f}_i^* - \hat{f}_{i-1}^*)\, x_n j + t_m g^* \tilde{z}_{n_j} = t_m g^* \tilde{z}'_{n_j}$ and 2.2 follows.

Let $\bar{t}_m$ be the smallest integer divisible by t_m and by every torsion prime in $H^*(X, Z)$. Let $\mathbf{P}_m = \{p \mid p \mid \bar{t}_m\}$ and $\tilde{\mathbf{P}}_m = \mathbf{P} - \mathbf{P}_m$. Applying 2.2 one obtains:

2.3. PROPOSITION. *Let $[Y] \in G(X)$. Given a $\mathbf{P}_k$ equivalence $f: Y \to X$, a set of integral classes $x_{n_1}, x_{n_2}, \ldots, x_{n_r}$ representing a basis for $QH^*(X, Z)$/*torsion *and a set $\tilde{z}_{n_1}, \tilde{z}_{n_2}, \ldots, \tilde{z}_{n_r}$, $\tilde{z}_{n_i} \in H^{n_i}(Y, Z)$, there exists a $\mathbf{P}_k$ equivalence $\hat{f}: Y \to X$ with*

$$[H^*(\hat{f}, Z) - H^*(f, Z)]\, x_{n_j} = \bar{t}_k \tilde{z}_{n_j} + \theta_j$$

where θ_j is a torsion element.

Proof. Apply 2.2 for $Y = \tilde{Y}$, $g = 1$, $z_{n_j} = \bar{t}_k / t_k\, \tilde{z}_{n_j}$. Then one has only to show that $\tilde{f} = \hat{f}$ is a $\mathbf{P}_k$ equivalence. (Note that if $H^*(X, Z)$ is a torsion free $t_k = \bar{t}_k$ and then $H^*(\hat{f}, Z) \otimes Z_{t_k} = H^*(f, Z) \otimes Z_{t_k}$ implies that $\tilde{f}$ is a $\mathbf{P}_k$ equivalence). By 2.2 torsion $\pi(\tilde{f}_i) = 0$ and hence torsion $\pi(\hat{f}) =$ torsion $\pi(f)$.

Consider the following diagram:

$$\begin{array}{ccc}
\pi(Y)/\text{torsion} & \xrightarrow[\pi(f)/\text{torsion}]{\pi(\hat{f})/\text{torsion}} & \pi(X)/\text{torsion} \\
\Big\downarrow \tilde{\sigma}(Y) & & \Big\downarrow \tilde{\sigma}(X) \\
PH_*(Y)/\text{torsion} & \xrightarrow[f_*]{\hat{f}_* = \hat{f}_*} & PH_*(X)/\text{torsion}
\end{array}$$

($\tilde{\sigma}$ – the Hurewicz-Serre monomorphism).

Now all groups are isomorphic and $\det[\tilde{\sigma}(Y)] = \det[\tilde{\sigma}(X)]$

$$H^*(\hat{f}, Z)/\text{torsion} \otimes Z_{\bar{t}_k} = H^*(f, Z)/\text{torsion} \otimes Z_{\bar{t}_k}$$

It follows that $H^*(\hat{f}, Z)$ and its dual $\hat{f}_*$ are $\mathbf{P}_k$ isomorphisms. Now if G is free abelain a homomorphism $\alpha: G \to G$ is a $\mathbf{P}_k$ isomorphism if and only if $(\det \alpha, \bar{t}_k) = 1$. Now,

$$(\det \hat{f}_*)\, (\det \tilde{\sigma}(Y)) = (\det \tilde{\sigma}(X))\, (\det \pi(\hat{f})/\text{torsion})$$

implies $(\det \hat{f}_*) = \det(\pi(\hat{f})/\text{torsion})$ and hence $\pi(\hat{f})$/torsion is a $\mathbf{P}_k$ isomorphism.

As torsion $\pi(\hat{f})$ = torsion $\pi(f)$ and $\pi(\hat{f})$/torsion are $\mathbf{P}_k$ isomorphisms so is $\pi(\hat{f})$ and 2.3 follows. One can construct elements in $G(X)$ as follows:

2.4. PROPOSITION. *Let $h_0: X \to K(Z, \bar{n})$ realize a basis for $QH^*(X, Z)$/*torsion *i.e.: $x_{n_i} = H^*(h_0, Z)\,\iota_{n_i}$ reduces to a basis for $QH^*(X, Z)$/*torsion. *Given any matrix $A = (a_{ij}) \in \mathcal{M}(s_1, s_2, \ldots, s_l; Z)$ with $(\det A, \bar{t}_k) = 1$ let $f_0: K(Z, \bar{n}) \to K(Z, \bar{n})$ be given by $H^*(f_0, Z)\,\iota_{n_i} = \sum_j a_{ij}\,\iota_{n_j}$. Let $Y = Y(A) = Hl_k(\tilde{Y})$ where*

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\tilde{f}_1} & X \\ \downarrow{\scriptstyle \tilde{h}_1} & & \downarrow{\scriptstyle h_0} \\ K(Z, \bar{n}) & \xrightarrow{f_0} & K(Z, \bar{n}) \end{array}$$

is a fiber square induced by h_0, f_0. Then:

(a) $\xi(A) = [Y] \in G(X)$

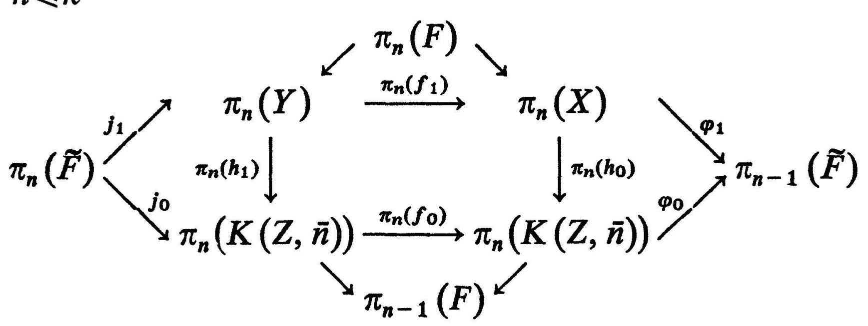
(b) *The set $\{y_{n_i} = H^{n_i}(h_1, Z)\,\iota_{n_i}\}$ reduces to a basis for $QH^*(Y, Z)$/*torsion, *where* $h_1: Y \to K(Z, \bar{n})$ is the composition

$$H_{l_k}(\tilde{Y}) = Y \xrightarrow{\tau'_k} \tilde{Y} \xrightarrow{\tilde{h}_1} K(Z, \bar{n}).$$

Proof. By 2.1 h_0 is a $\tilde{\mathbf{P}}_k = \mathbf{P} - \mathbf{P}_k$ equivalence in dim$\leqslant k$ hence so is $\tilde{h}_1$. It follows that in dim$\leqslant k$ $Y \approx \tilde{Y} \approx_{\tilde{\mathbf{P}}_k} K(Z, \bar{n}) \approx_{\tilde{\mathbf{P}}_k} X$. Consequently, $Y \approx_{\tilde{\mathbf{P}}_k} X$ in dim$\leqslant k$ but both being of dim$\leqslant k$ it follows that $Y \approx_{\tilde{\mathbf{P}}_k} X$. Now, $(\det A, \bar{t}_k) = 1$ implies that $A = \pi(f_0)$ is a $\mathbf{P}_k$ isomorphism and hence f_0 (and consequently $\tilde{f}_1$) are $\mathbf{P}_k$ equivalences. Again in dim $\leqslant k$ $Y \approx \tilde{Y}$ and $Y \approx_{\tilde{\mathbf{P}}_k} X$ and (a) follows. To prove (b) one notices first that in dim$\leqslant k$ one can consider

$$\begin{array}{ccc} Y & \xrightarrow{f_1} & X \\ \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_0} \\ K(Z, \bar{n}) & \xrightarrow{f_0} & K(Z, \bar{n}) \end{array}$$

as a fiber square. Further, if $[Y] \in G(X)$ then coker$\,\sigma_n(Y) \approx$ coker$\,\sigma_n(X)$ and ker$\,\sigma_n(Y) = $ ker$\,\sigma_n(X)$ (as all these groups are finite they involve only a finite set $\mathbf{P}_1$ of primes and by the procedure of 1.5 there exists a $\mathbf{P}_1$ equivalence $Y \to X$ which will yield the desired isomorphisms). One obtains the following commutative diagrams for $n \leqslant k$



where $\tilde{F}=\operatorname{fiber} f_0=\operatorname{fiber} f_1\approx\prod_i K(Z_{\lambda_i}, n_i-1)$ where the matrix $\operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_r)$ is equivalent to A. As $j_0=0$ and as the order of $\pi_n(\tilde{F})$ is prime to that of $\pi_n(F)$ $\pi_n(j_1)=0$ and $\pi_n(f_1)$ is a monomorphism. As φ_1 factors through a free group $\varphi_1(\operatorname{tor}\pi_n(X))=0$ and one has a commutative diagram with exact rows

$$\begin{array}{ccccc}
0\to\pi_n(Y)/\text{torsion} & \xrightarrow{f_{1\#}} & \pi_n(X)/\text{torsion} & \xrightarrow{\tilde{\varphi}_1} & \pi_{n-1}(\tilde{F})\to 0\\
\downarrow{\scriptstyle \sigma_n(Y)} & & \downarrow{\scriptstyle \sigma_n(X)} & & \downarrow{\scriptstyle \sigma'}\\
0\to PH_n(Y)/\text{torsion} & \xrightarrow{f_{1*}} & PH_n(X)/\text{torsion} & \xrightarrow{\varphi} & \operatorname{coker} f_1^*\\
\downarrow{\scriptstyle h_{1*}} & & \downarrow{\scriptstyle h_{0*}} & & \downarrow{\scriptstyle h'}\\
0\to PH_n(K(Z,\bar{n}),Z)/\text{torsion} & \xrightarrow{f_{0*}} & PH_n(K(Z,\bar{n}),Z)/\text{torsion} & \xrightarrow{\tilde{\varphi}_0} & \pi_{n-1}(\tilde{F})\\
{\scriptstyle \sigma_n(K)}\uparrow\approx & & \approx\uparrow{\scriptstyle \sigma_n(K)} & \nearrow{\scriptstyle \varphi_0} & \\
0\to\pi_n(K(Z,\bar{n})) & \xrightarrow{\pi_n(f0)} & \pi_n(K(Z,\bar{n})) & &
\end{array}$$

$\tilde{\sigma}_n(Y)$ and $\tilde{\sigma}_n(X)$ are monomorphisms.

$\operatorname{Coker}\tilde{\sigma}_n(Y)\approx\operatorname{coker}\sigma_n(Y)\approx\operatorname{coker}\sigma_n(X)\approx\operatorname{coker}\tilde{\sigma}_n(X)$ are $\mathbf{P}_k$ torsion groups and as f_1 is a $\mathbf{P}_k$ equivalence $f_{1\#}$ and f_{1*} are $\mathbf{P}_k$ isomorphisms and they induce a $\mathbf{P}_k$ isomorphism $\operatorname{coker}\tilde{\sigma}_n(Y)\to\operatorname{coker}\tilde{\sigma}_n(X)$ and hence an isomorphism. It follows that the left upper square in the last diagram is a push out diagram and σ' and consequently h' are isomorphisms. As h_{0*} is an isomorphism so is h_{1*} and so is its dual $QH^*(h_1, Z)/$torsion.

Again let $\tilde{\alpha}:\mathcal{M}(s_1, s_2, \ldots, s_l; Z)\to\mathcal{M}(s_1, s_2, \ldots, s; Z_{\bar{t}_k})$ and let $\mathcal{A}= =\tilde{\alpha}^{-1}\times GL(s_1, s_2, \ldots, s_l; Z_{\bar{t}_k})$. 2.4(a) defines a correspondence

$$\xi:\mathcal{A}\to G(X). \quad \xi(I)=[X].$$

2.5. PROPOSITION. *Let* $[Y]\in G(X)$. *Let* $f: Y\to X$ *be a* $\mathbf{P}_k$ *equivalence and let* $A\in\mathcal{A}$ *represent* $QH^*(f, Z)$/torsion. *Then*

$$[Y]=\xi(A).$$

Proof. Suppose A represents $QH^*(f, Z)$/torsion with respect to bases represented by $x_{n_1}, \ldots, x_{n_r}$, $x_{n_i}\in H^{n_i}(X, Z)$ and $y'_{n_1}, \ldots, y'_{n_r}$, $y'_{n_i}\in H^{n_i}(Y, Z)$. We shall replace f by a $\mathbf{P}_k$ equivalence $\hat{f}$, $QH^*(\hat{f}, Z)/\text{torsion}=QH^*(f, Z)$/torsion and $y'_{n_1}, \ldots, y'_{n_2}$ by $y_{n_1}, \ldots, y_{n_r}$ representing the same basis as $y'_{n_1}, \ldots, y'_{n_r}$ so that if $h_0: X\to K(Z, \bar{n})$, $h_1: Y\to K(Z, \bar{n})$, $f_0: K(Z, \bar{n})\to K(Z, \bar{n})$ are given by

$$H^*(h_0, Z)\,\iota_{n_i}=x_{n_i}, \qquad H^*(h_1, Z)\,\iota_{n_i}=y_{n_i}$$
$$H^*(f_0, Z)\,\iota_{n_i}=\sum_j a_{ij}\cdot\iota_{n_j} \qquad A=(a_{ij}),$$

then

$$h_0\circ\hat{f}\sim f_0\circ h_1.$$

In the diagram

$$\begin{array}{ccc} Y & \xrightarrow{\hat{f}} & X \\ \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_0} \\ K(Z, \bar{n}) & \xrightarrow{f_0} & K(Z, \bar{n}) \end{array}$$

f_0 and $\hat{f}$ are $\tilde{\mathbf{P}}_k$ equivalences and as $QH(h_\iota, Z)$/torsion $i=0$, 1 are isomorphisms by 2.1 they are $\mathbf{P}_k$ equivalences in dim$\leqslant k$. Hence, by 1.6 the last diagram represents a fiber square in dim$\leqslant k$ and hence $[Y]=\xi(A)$. Equivalently one has to construct $\hat{f}$ and find y_{n_i} so that

$$H^*(\hat{f}, Z)\, x_{n_i} = \sum a_{ij} y_{n_j}.$$

First note that without a loss of generality one may assume A to be diagonal for otherwise $D=E_1AE_2$ where E_i are invertible and D-diagonal, $D=\text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_r)$. Then if E_2^{-1} applied on $y_{n_1}, \ldots, y_{n_r}$ yields the basis $\bar{y}_{n_1}, \bar{y}_{n_2}, \ldots, \bar{y}_{n_r}$ and E_1 applied on $x_{n_1}, \ldots, x_{n_r}$ yields a basis $\bar{x}_{n_1}, \ldots, \bar{x}_{n_r}$ then

$$H^*(f, Z)(\bar{x}_{n_1}, \ldots, \bar{x}_{n_r}) = E_1AE_2E_2^{-1}(y_{n_1}, y_{n_2}, \ldots, y_{n_r}) = D(\bar{y}_{n_1}, \ldots, \bar{y}_{n_r})$$

So $H^*(f, Z)\, x_{n_i}=\sum_j a_{ij}\, y_{n_j}$ if and only if $H^*(f, Z)\, \bar{x}_{n_i}=\lambda_i\bar{y}_{n_i}$. So assume $A=D=\text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_r)$. Hence, it is given that

$$H^*(f, Z)\, x_{n_i} = \lambda_i y'_{n_i} + d_i + \theta_i,$$

d_i-decomposable and θ_i a torsion element. Now $(\det A, \bar{t}_k)=1$ implies $(\lambda_i, \bar{t}_k)=1$. Let $1+a_i\cdot \bar{t}_k=b\cdot\lambda_i$. Now apply 2.3 for $\bar{z}_{n_j}=a_j\cdot d_j$. $\hat{f}$ then will satisfy

$$H^*(\hat{f}, Z)\, x_{n_i} = H^*(f, Z)\, x_{n_i} + \bar{t}_k a_i d_i + \hat{\theta}_i = \lambda_i y'_{n_i} + (1+\bar{t}_k a_i)\, d_i + \hat{\theta}_i + \theta_i.$$

As (order $(\theta_i+\hat{\theta}_i), \lambda_i)=1$ $\theta_i+\hat{\theta}_i=\lambda_i\cdot\theta'_i$ and $H^*(f_1, Z)\, x_{n_i}=\lambda_i(y'_{n_i}+b_i\cdot d_i+\theta'_i)$. $y_{n_i}=y'_{n_i}+b_i d_i+\theta'_i$ are the desired classes.

2.6. COROLLARY. *$\xi: A\to G(X)$ of 2.4(a) is onto.*

Proof. If $[Y]\in G(X)$ then by 1.5 there exists a $\mathbf{P}_k$ equivalence $f: Y\to X$ and by 2.5 $[Y]=\xi(A)$ for some $A\in\mathscr{A}$.

It is quite obvious from the definition of ξ in 2.4 that if $B\in GL(s_1, s_2, \ldots, s_l; Z)$ then $\xi(BA)=\xi(A)$. A simple application of 2.3 and 2.5 shows also that for any $C\in\mathscr{M}(s_1, s_2, \ldots, s_l; Z)$ $\xi(A+\bar{t}_kC)=\xi(A)$. Hence if,

$$\bar{\alpha}: GL(s_1, s_2, \ldots, s_l; Z) \to GL(s_1, s_2, \ldots, s_l; Z_{\bar{t}_k})$$

then ξ factors through coker$\,\bar{\alpha}=(Z^*_{\bar{t}_k}/\{\pm 1\})^l$ (see 1.8). Hence, one obtains the following

2.7. A STRUCTURE THEOREM for $G(X)$. *There exists a correspondence* $\xi:(Z^*_{t_k}/\{\pm 1\})^l \to G(X)$ *which is onto.* ξ *is given by*

$$\xi(d_1, d_2, \ldots, d_l) = \xi(I_{d_1} * I_{d_2} * \cdots * I_{d_l}),$$

$d_i \in Z^*_{t_k}$, where I_{d_i} is the $s_i \times s_i$ matrix given by

$$I(d_i, s_i) = I_{d_i} = \begin{pmatrix} I & 0 \\ 0 & d_i \end{pmatrix}$$

By (2.4) it follows that $\tilde{Y} \approx \text{fiber}(\tilde{g}: X \to K_\lambda)$ for some map $\tilde{g}$ where $K_\lambda = \prod_{i=1}^r K(Z_{\lambda_i}, n_i)$ and the λ_i's are defined by the fact that A is equivalent to $\text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_r)$. Moreover, if $\varrho_{\lambda_i}: H^*(\ , Z) \to H^*(\ , Z_{\lambda_i})$ is the reduction then $H^*(\tilde{g}, Z_{\lambda_i})\, \bar{\imath}_{n_i} = \varrho_{\lambda_i} x_{n_i}$ where $\bar{\imath}_{n_i} \in H_{n_i}(K(Z_{\lambda_i}, n_i), Z_{\lambda i})$ is the fundamental class and $x_{n_1}, x_{n_2}, \ldots, x_{n_r}$ are integral classes representing a basis for $QH^*(X, Z)$/torsion. Note that given $[Y] \in G(X)$ by 1.5 there exists a $\mathbf{P}_{2k}$ equivalence $f: Y \to X$ and by 2.5 $[Y] = \xi(A)$, $(\det A, \bar{\imath}_{2k}) = 1$.

2.8. PROPOSITION. *Let* $m = 2k$. *Given integers* $\lambda_1, \lambda_2, \ldots, \lambda_r$ *prime to* $\bar{\imath}_m$ *and elements* $x_{n_1}, x_{n_2}, \ldots, x_{n_r} \in H^*(X, Z)$ *representing a basis for* $QH^*(X, Z)$/torsion, *there exists an* H-*structure* μ_0 *for* X *with respect to which* $\varrho_{\lambda_i} x_{n_i}$ *are primitive.*

Proof. As all λ_i are prime to all torsion primes of $H^*(X, Z)$ $H^*(X, Z_{\lambda_i}) \approx \approx H^*(X, Z) \otimes Z_{\lambda_i}$. Replacing each λ_i by the product $\lambda = \lambda_1 \lambda_2 \cdot \cdots \cdot \lambda_r$ suffices to prove the theorem for $\lambda_1 = \lambda_2 = \cdots = \lambda_r = \lambda$.

Denote $(\ )^* = H^*(\ , Z)$/torsion. Let $\tilde{x}_{n_i}$ be the image of x_{n_i} in $H^*(X, Z)$/torsion. Given any H-structure μ of X one has

$$\mu^* \tilde{x}_{n_i} = \tilde{x}_{n_i} \otimes 1 + 1 \otimes \tilde{x}_{n_i} + \sum_{s=1}^{a} \tilde{x}'_{n_i, s} \otimes \tilde{x}''_{n_i, s}.$$

Put $\omega_i = \bar{\mu}^* \tilde{x}_{n_i} = \mu^* \tilde{x}_{n_i} - \tilde{x}_{n_i} \otimes 1 - 1 \otimes \tilde{x}_{n_i} = \sum_{s=1}^a x'_{n,s} \otimes x''_{n,s}$ then $\omega_i = \Lambda^* \tilde{\omega}_i$ where $\tilde{\omega}_i \in H^*(X \wedge X, Z)$/torsion and $\Lambda: X \times X \to X \wedge X$ is the reduction. As $\dim X \times X \leqslant m$, $\dim X \wedge X \leqslant m$ one can apply 2.2 for

$$Y = X \times X, \quad \tilde{Y} = X \wedge X, \quad f = \mu, \quad g = \Lambda.$$

(and note that $\overline{\text{im}\Lambda^*}$ is an ideal) and $z_{n_i} = k\tilde{\omega}_i$ where k is an integer satisfying $1 + \bar{\imath}_m k = b\lambda$. If $\bar{f} = \mu_0$ then, as each $\bar{f}_i$ of 2.2 factors through $X \wedge X$, $\bar{f}_i \mid X \vee X = *$ and $\mu_0 \mid X \vee X = \mu \mid X \vee X$. Hence μ_0 is an H-structure. Now,

$$\begin{aligned} \mu_0^* \tilde{x}_{n_i} &= \mu_0^* \tilde{x}_{n_i} + \bar{\imath}_m k \omega_i = \tilde{x}_{n_i} \otimes 1 + 1 \otimes \tilde{x}_{n_i} + \omega_i + t_m k \omega_i \\ &= \tilde{x}_{n_i} \otimes 1 + 1 \otimes \tilde{x}_{n_i} + \lambda b \omega_i. \end{aligned}$$

As $\varrho_\lambda x_{n_i} = \tilde{\varrho}_\lambda \tilde{x}_{n_i}$ ($\tilde{\varrho}_\lambda: H^*(X, Z)/\text{torsion} \to H^*(X, Z_\lambda)$)

$$H^*(\mu_0, Z_\lambda)\,\varrho_\lambda x_{n_i} = H^*(\mu_0, Z_\lambda)\,\tilde{\varrho}_\lambda \tilde{x}_{n_i}$$
$$= \tilde{\varrho}_\lambda \tilde{x}_{n_i} \otimes 1 + 1 \otimes \tilde{\varrho}_\lambda \tilde{x}_{n_i} = \varrho_\lambda x_{n_i} \otimes 1 + 1 \otimes \varrho_\lambda x_{n_i}$$

and $\varrho_\lambda x_{n_i}$ are primitive.

2.9. COROLLARY. *If* $[Y] \in G(X)$. *Then* Y *is an* H-*space and there exists a* P_{2k} *equivalence* $f_1: Y \to X$ *which is an* H-*map with respect to some* H-*structure of* X.

Proof. By the remarks preceding 2.8 one may replace k by $m = 2k$ in 2.7 and then $\tilde{f}_1: \tilde{Y} \to X$ is the fiber of a map $\tilde{g}: X \to K_\lambda$ $K_\lambda = \prod_i K(\lambda_i, n_i)$ $(\lambda_i, \tilde{t}_{2k}) = 1$. By 2.8 choosing the H-structure of X properly $\tilde{g}$ is an H-map and so $\tilde{Y}$ and $\tilde{f}_1$ become an H-space and an H-map. As $Hl_{2k}(\tilde{Y}) = Hl_k(\tilde{Y}) = Y$ Y and $\tau': Y \to \tilde{Y}$ admit H-structure and so does $f_1 = \tilde{f}_1 \circ \tau'$.

2.10. COROLLARY. (See [6] and [7]). *Let* X *be a finite* CW *complex. If for every prime* p *there exist an* H-*space* $X(p)$ *and a* p-*equivalence* $f_q: X \to X(p)$ *and if all* $H^*(X(p), Q)$, $p \in P$, *are isomorphic as Hopf algebras then* X *admits an* H-*structure.*

Proof. Let $H^*(X, Q) \approx H^*(X(p), Q) \approx A(_{n_1}, x_{n_2}, \ldots, x_{n_r})$. As $X \approx (S^{\bar{n}} = S^{n_1} \times S^{n_2} \times \times \cdots \times S^{n_r}$ for almost all primes one may assume that the number of different spaces $X(p)$ and maps f_q is finite. Now, for every $p, q \in \mathbf{P}$ $H^*(X(p), Q) \approx H^*(X(q), Q)$. Hence $K(Z, \bar{n}) = \prod_{i=1}^r K(Z, n_i)$ admits an H-structure and there exist Q equivalences $h_p: X(p) \to K(Z, \bar{n})$ which are H-maps. Moreover, using the procedure of [8] one may replace $X(p)$ by $X'(p)$ so that h_p decomposes into two H-maps – $X(p) \overset{h'_p}{\to} X'(p) \overset{h''_p}{\to} K(Z, \bar{n})$ where h'_p is a p-equivalence and h''_p is a $\mathbf{P} - \{p\}$ equivalence. The pullback $X' = \prod_{h''_p} X'(p)$ is then an H-map, and $X' \approx_p X'(p) \approx_p X$. Hence $[Y] \in G(X')$ and apply 2.9.

Let $A, \hat{A} \in \mathscr{A} \subset M(s_1, s_2, \ldots, s_l; Z)$. Then

$$\bar{\alpha}A \equiv \bar{\alpha}(I(d_1, s_1) * I(d_2, s_2) * \cdots * I(d_l, s_l)) \bmod \operatorname{im} \bar{\alpha}.$$
$$\bar{\alpha}\hat{A} \equiv \bar{\alpha}(I(\hat{d}_1, s_1) * I(\hat{d}_2, s_2) * \cdots * I(\hat{d}_l, s_l)) \bmod \operatorname{im} \bar{\alpha}$$

(If $A = A_1 * A_2 * \cdots * A_l$ then $d_i = \det A_i \in Z_{t_k}^*$ and similarly for $\hat{A}_i$ and $\hat{d}_i$). It can be easily seen that

$$\xi(A) \times \xi(\hat{A}) = \xi(A *_g \hat{A}) \in G(X \times X)$$

As $I(d_i, s_i) * I(\hat{d}_i, s_i) \equiv I * I(d_i \cdot \hat{d}_i, s_i) = I(d_i \cdot \hat{d}_i, 2s_i) \bmod \operatorname{im} \bar{\alpha}$ one has $\bar{\alpha}(A *_g A \equiv \equiv \bar{\alpha}(I *_g A\hat{A}) \bmod \operatorname{im} \bar{\alpha}$ and therefore

$$\xi(A) \times \xi(\hat{A}) = \xi(A *_g \hat{A}) = \xi(I *_g A\hat{A}) = \xi(I) \times \xi(A \cdot \hat{A}).$$

We thus proved

2.11. A PRODUCT THEOREM. *Let* $A, \hat{A} \in \mathscr{A}$. *Then* $\xi(A) \times \xi(\hat{A}) = X \times \xi(A \cdot \hat{A})$ *and consequently if* $A_i, B_i \in \mathscr{A}$ *and if* $A_1 \cdot A_2 \cdot \cdots \cdot A_n$ *and* $B_1 \cdot B_2 \cdot \cdots \cdot B_m$ *have the same image in* $GL(s_1, s_2, \ldots, s_l : Z_{\bar{\imath}_k})/\mathrm{im}\,\bar{\alpha}$ *then*

$$\xi(A_1) \times \xi(A_2) \times \cdots \times \xi(A_m) = \xi(B_1) \times \xi(B_2) \times \cdots \times \xi(B_m).$$

If φ is the Euler function then as $I(d_i, s_i)^{\varphi(\bar{\imath}_k)/2} \equiv I \bmod \mathrm{im}\,\bar{\alpha}$ one obtains (see [7], p. 82):

2.12. COROLLARY. *For every* $[Y] \in G(X)$ $Y^{\varphi(\bar{\imath}_k)/2} \approx X^{\varphi(\bar{\imath}_k)/2}$.
Similarly one obtains

2.13. COROLLARY. *Let* Y *be a finite* CW H-*space if* $QH^n(Y, Z)$/torsion$\neq 0$ *whenever* $QH^n(X, Z)$/torsion$\neq 0$ *then* $G(X \times Y) = \{X\} \times G(Y)$.

3. Genus and non-Cancellation

3.1. DEFINITION. (See [5]). Let K be a finite CW complex, $\mathbf{P}_1$ – a set of primes. K is said to be $\mathbf{P}_1$ universal if for every prime $q \notin \mathbf{P}_1$ there exists a $\mathbf{P}_1$ equivalence $f_q : K \to K$ so that $\bar{H}^*(f_q, Z_q) = 0$.

A space Y is said to be an H_0 space if $H^*(Y, Q)$ is a free (commutative and associative) algebra. If $H^*(Y, Q) = A(x_{n_1}, x_{n_2}, \ldots, x_{n_r})$ then Y is said to be of type $n_1, n_2, \ldots, n_r$. The following is a generalization of [7] Lemma 1.5:

3.2. PROPOSITION. *Let* $\mathbf{P} = \mathbf{P}_1 \cup \mathbf{P}_2$. *Let* X, Y_1, Y_2 *be finite* CW *complexes* $f : X \to Y_1 \times Y_2$. *If* X *is an* H_0-*space,* $Y_i - \mathbf{P}_i$ *universal* $i = 1, 2$ *and if* $p_i \circ f$ *is a* $\mathbf{P}_i$ *equivalence where* $p_i : Y_1 \times Y_2 \to Y_i$, $i = 1, 2$, *is the projection then* f *admits a homotopy left inverse.*

Proof. We may assume that $\mathbf{P}_1 \cap \mathbf{P}_2 = \emptyset$. Let $h_0 : X \to X_0 = K(Z, \bar{n})$ be a rational equivalence. Put $m = \max(n_r, \dim Y_1 \times Y_2)$. Factor $\hat{h}_0 : Ht_m(X) \to X_0$ by maps $h_{i,i-1} : X_i \to X_{i-1}$, $0 < i \leqslant s$ $X_s = Ht_m(X)$, $h_{i,i-1}$ being the fiber of a map $k_i : X_{i-1} \to K(Z_{p_i}, m_i)$ $p_i \in \mathbf{P}$. Let $\alpha_i : Y_1 \times Y_2 \to Y_1 \times Y_2$ be given as follows: If $p_i \notin \mathbf{P}_1$ let $g_i' : Y_1 \to Y_1$ be a $\mathbf{P}_1$ equivalence with $\bar{H}^*(g_i', Z_{pi}) = 0$. If $p_i \notin \mathbf{P}_2$ let $g_i'' : Y_2 \to Y_2$ be defined similarly. Let

$$\alpha_i = \begin{cases} g_i' \times 1 & \text{if } p_i \notin \mathbf{P}_1 \\ 1 \times g_i'' & \text{if } p_i \notin \mathbf{P}_2 \end{cases}$$

suffices to find a left inverse for $\hat{f} = \alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_s \circ f$ and one has $p_i \circ \hat{f}$ is a $\mathbf{P}_i$ equivalence. Let $y_{n_i} = H^*(h_0, Z)\, \iota_{n_i}$, $(\iota_{n_i}, \ldots, \iota_{n_r}) \in H^*(K(Z, \bar{n}), Z)$ the "fundamental vector". As $p_1 \circ f$ is a $\mathbf{P}_1$ equivalence $\lambda_1 y_{n_i} \in \mathrm{im}\, H^*(p_1 \circ \hat{f}, Z)$ λ_1 prime to $\mathbf{P}_1$. Similarly

$\lambda_2 y_{n_i} \in \operatorname{im} H^*(p_2 \circ f, Z)$. As $(\lambda_1, \lambda_2)=1$ $\lambda_1 y_{n_i}$, $\lambda_2 y_{n_i} \in \operatorname{im} H^*(\hat{f}, Z)$ implies $y_{n_i} \in \operatorname{im} H^*(\hat{f}, Z)$. Hence, h_0 factors through $Y_1 \times Y_2$, $h_0 = r_0 \circ \hat{f}$, $r_0: Y_1 \times Y_2 \to$ $\to K(Z, \bar{n}) = X_0$. Suppose one obtains inductively the following commutative diagram:

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y_1 \times Y_2 \xrightarrow{\alpha_s} \cdots Y_1 \times Y_2 & \xrightarrow{\alpha_n} & Y_1 \times Y_2 & & \\ \tau_m \downarrow & & & & \downarrow r_{n-1} & & \\ Ht_m(X) = X_s & \xrightarrow{h_{s\,s-1}} & X_{s-1} \longrightarrow \cdots X_n & \xrightarrow{h_{n,n-1}} & X_{n-1} & \xrightarrow{k_n} & K(Z_{p_n}, m_n) \end{array}$$

Without loss of generality suppose $p_n \notin \mathbf{P}_1$. Then $H^*(\alpha_n, Z_{p_i}) = H^*(g_n', Z_{p_i}) \otimes 1 = \varepsilon \otimes 1$ (where $\varepsilon: H^* \to Z_p \to H^*$ is induced by augmentation). Hence, one obtains

$$\begin{array}{ccccc} Y_1 \times Y_2 & & \xrightarrow{p_2} & & Y_2 \\ \downarrow \alpha_n & & & & \downarrow \bar{r} \\ Y_1 \times Y_2 & \xrightarrow{r_{n-1}} & X_{n-1} & \xrightarrow{k_n} & K(Z_{p_n}, m_n) \end{array}$$

$p_2 \circ \alpha_{n+1} \circ \cdots \circ \alpha_s \circ f$ is a $\mathbf{P}_2$-equivalence, and as $k_n \circ h_{n,n-1} \sim *$, $* \sim k_n \circ r_{n-1} \circ \alpha_n \circ \alpha_{n+1} \circ \cdots \circ$ $\circ \cdots \circ \alpha_s \circ f \sim \bar{r} \circ p_2 \circ \alpha_{n+1} \circ \cdots \circ \alpha_s \circ f$. It follows that $\bar{r} \sim *$, $k_n \circ r_{n-1} \circ \alpha_n \sim *$ and $r_n': Y_1 \times Y_2 \to$ $\to X_n$ exists so that $h_{n,n-1} \circ r_n' \sim r_{n-1} \circ \alpha_n$. Comparing $\varphi_1 = r_n' \circ \alpha_{n+1} \circ \cdots \circ \alpha_s \circ f$ and $\varphi_2 = h_{n+1,n} \circ h_{n+2,n+1} \circ \cdots \circ h_{s,s-1} \circ \tau_m$ there exists $\omega: X \to K(Z_{p_n}, m_n - 1)$ so that $\omega * \varphi_1 \sim$ $\sim \varphi_2$ (where $*$ denotes the action of $[X, K(Z_{p_n}, m_n - 1)]$ on $[X, X_n]$ induced by the principal fibration $K(Z_{p_n}, m_n - 1) \to X_n \to X_{n-1}$). But $\alpha_{n+1} \circ \alpha_{n+2} \circ \cdots \circ \alpha_s \circ f$ is a $\mathbf{P}_2$ equimorphism, hence, ω factors as $\bar{\omega} \circ \alpha_{n+1} \circ \cdots \circ \alpha_s \circ f$, $\bar{\omega}: Y_1 \times Y_2 \to K(Z_{p_n}, m_n - 1)$.

Replacing r_n' by $\bar{\omega} * r_n' = r_n$ one obtains $r_n: Y_1 \times Y_2 \to X_n$ and

$$\begin{aligned} r_n \circ \alpha_{n+1} \circ \cdots \alpha_s \circ f &= (\bar{\omega} * r_n') \circ \alpha_{n+1} \circ \cdots \alpha_0 \sim \\ &\sim (\bar{\omega} \circ \alpha_{n+1} \circ \cdots \circ \alpha_s \circ f) * (r_n' \circ \alpha_{n+1} \circ \cdots \circ \alpha_s \circ f) \sim \\ &\sim \omega * \varphi_1 \sim \varphi_2 = h_{n+1,n} \circ h_{n+2,n+1} \circ \cdots h_{s,s-1} \circ \tau_m . \end{aligned}$$

The final step of lifting $r_s: Y_1 \times Y_2 \to X_s = Ht_m(X)$ to $r: Y_1 \times Y_2 \to X$ with $r \circ f \sim 1$ is automatic as $\dim X \leqslant m$, $\dim Y_1 \times Y_2 \leqslant m$ and hence $[X, \tau_m]: [X, X] \xrightarrow{\approx} [X, X_s]$, $[Y_1 \times Y_2, \tau_m]: [Y_1 \times Y_2, X] \xrightarrow{\approx} [Y_1 \times Y_2, X_s]$ and $\tau_m = [X, \tau_m]$ [1].

3.3. LEMMA. *Let*

$$\begin{array}{ccc} X & \xrightarrow{h_2} & Y_1 \\ h_1 \downarrow & & \downarrow f_1 \\ Y_2 & \xrightarrow{f_2} & X_0 \end{array}$$

be a fiber square. If X_0 is an H-*space then* $X \xrightarrow{(h_1 \times h_2) \circ \Delta} Y_1 \times Y_2 \xrightarrow{[f_2][f_1]^{-1}} X_0$ *is a (quasi) fibration.*

Proof. Denote $\mu_{X_0}(x, x')$ by $x \cdot x'$ and if $c: X \to X$ is the homotopy inverse put

$c(x)=x^{-1}$. Now,

$$X=\{y_1, y_2, \varphi \in Y_1 \times Y_2 \times PX_0 \mid f_1(y_1)=\varphi(0), \varphi(1)=f_2(y_2)\}$$

and

$$F=\{y_1, y_2, \varphi \in Y_1 \times Y_2 \times \mathscr{L}X_0 \mid f_2(y_2)\cdot f_1(y_1)^{-1}=\varphi(1)\}$$

is the fiber of $[f]\,[f_1]^{-1}$. Let $\alpha: X \to F$ be given by $\alpha(y_1, y_2, \varphi)=y_1, y_2, a(f_1(y_1))+\varphi\cdot f_1(y_1)^{-1}$ where $a(z)$ is the homotopy connecting the base point with $z\cdot z^{-1}$. The obvious maps $X \to Y_1 \times Y_2$ and $F \to Y_1 \times Y_2$ are (Hurewicz) fibrations with the same fiber ΩX_0 and $\alpha \mid \Omega X_0=1$. Hence, α is a homotopy equivalence.

Combining 1.6, 3.2 and 3.3 one gets

3.4. THEOREM. *Given a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{h_2} & Y_1 \\ \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle f_1} \\ Y_2 & \longrightarrow & X_0 \end{array}$$

Suppose f_i, h_i *are* $\mathbf{P}_i$ *equivalences in* $\dim \leqslant n$, $i=1, 2$, $\mathbf{P}_1 \cup \mathbf{P}_2=\mathbf{P}$. *If* Y_1 *is* $\mathbf{P}_i$ *universal* $i=1, 2$, X *is an* H_0 *space and* X_0 *is an* H-*space then*

$$Y_1 \times Y_2 \approx X \times X_0 \quad \text{in} \quad \dim \leqslant n.$$

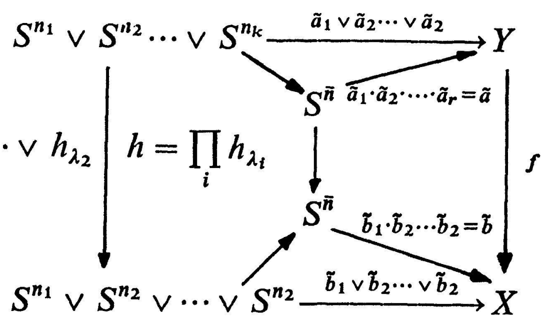
Proof. By 1.6 3.3.1 is a fiber square. By 3.3 one has a fibration

$$X \xrightarrow{(h_1 \times h_2)\Delta} Y_1 \times Y_2 \to X_0$$

where h_i are $\mathbf{P}_i$ equivalences and one can apply 3.2 to find a left inverse for $(h_1 \times h_2)\Delta$ and 3.3 follows.

Let X be again an H-space of type $(n_1, n_2, \dots, n_r)$ and of dimension $\leqslant k$. The set of primes $\mathbf{P}_{2k}$ contains all primes involved in $H_*(X, Z)$, $\pi_n(X)$ $n \leqslant 2k$ and coker$(\pi_*(X)/$torsion$\to PH_*(X, Z)/$torsion). One can easily see that every map $S^{\bar{n}} \to X$ yielding an isomorphism of $\pi(\ )$/torsion is a $\tilde{\mathbf{P}}_k=\mathbf{P}-\mathbf{P}_k$ equivalence. (Hence $\mathbf{P}_k$ contains only regular primes). Let $[Y] \in G(X)$ and let $f: Y \to X$ be a $\mathbf{P}_{2k}$ equivalence which is an H-map (see 2.9). Diagonalizing $\pi(f)$/torsion one obtains bases $a_1, \dots, a_r \in \pi(Y)$/torsion and $b_1, \dots, b_r \in \pi(X)$/torsion so that $(\pi(f)/\text{torsion})\, a_1=\lambda_i b_i$, λ_i prime to $\mathbf{P}_{2k}$. If $\tilde{a}_i \in \pi_{n_i}(Y)$, $\tilde{b}_i \in \pi_{n_i}(X)$ represent a_i and b_i respectively then $\pi(f)\ \tilde{a}_i=\lambda_i \tilde{b}_i+\theta_i$, θ_i – a torsion element. But λ_i is prime to the torsion primes in $\pi_n(X)$ $n \leqslant 2k$ hence to the order of θ_i and $\theta_i=\lambda_i \tilde{\theta}_i$. Replacing $\tilde{b}_i$ by $\tilde{b}_i+\tilde{\theta}_i$ if necessary one may assume $\pi(f)\ \tilde{a}_i=$

$=\lambda_i \tilde{b}_i$. Hence one obtains the following commutative diagram:



As f is an H-map one can add $h=\prod_i h_{\lambda_i}$ to the above diagram to obtain a commutative diagram:

$$\begin{array}{ccc} S^{\bar{n}} & \xrightarrow{\tilde{a}} & Y \\ \downarrow h & & \downarrow f \\ S^{\bar{n}} & \xrightarrow{\tilde{b}} & X \end{array}$$

Now f and h are $\mathbf{P}_k$ equivalences while $\tilde{a}$ and $\tilde{b}$ are $\mathbf{P}_k$-equivalences and one can apply 3.4 to get the following:

3.5. A NON CANCELATION THEOREM. (Compare with [1], [2] and [7], p. 83). *If* $[Y]\in G(X)$ *then* $Y\times S^{\bar{n}} \approx X \times S^{\bar{n}}$.

REFERENCES

[1] Freyd, P., *Stable homotopy*. Proceeding of the conference on categorical algebra (La Jolla, 1965). Springer 1966.
[2] ——, *Stable homotopy II*. AMS, Proceedings of symposia in pure mathematics, Vol. XVII, 1970, pp. 161–183.
[3] Hilton, P. J., *On the Grothendieck group of compact polyhedra*, Fund. Math. *61* (1967), pp. 199–214.
[4] Hilton, P. J. and Roitberg, J., *On principal S^3 bundles over spheres*, Ann. of Math. *90* (1969), pp. 91–107.
[5] Mimura, M. and Toda, H., *On p-equivalences and p-universal spaces*, Comment. Math. Helv. *46* (1971), pp. 87–97.
[6] Mislin, G., *H-spaces* mod*p*. *H-spaces conference*, Neuchâtel 1970, Springer lecture notes 196.
[7] ——, *The Genus of an* H-*space*, Symp. on algebraic topology, Seattle 1971, Springer lecture notes 249, pp. 75–83.
[8] Zadbrodsky, A., *Homotopy associativity and finite* CW *complexes*, Topology *9* (1970), pp. 121–128.

The Hebrew University of Jerusalem, Israel
and Forschungsinstitut für Mathematik ETH Zurich, Switzerland

Received July 16, 1973