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Explicit Quasiconformal Extensions for some Classes of Univalent Functions

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1. Introduction. Notations

Let S be the class of functions analytic and univalent in $\Delta=\{z:|z|<1\}$ for which $f(0)=f'(0)-1=0$.

We say that $f\in S_k$, $0\le k<1$, if $f\in S$ and f has a quasiconformal extension on the whole plane $\mathbb{C}$ with complex dilatation $\mu_f=f_{\bar z}/f_z$ that satisfies $|\mu_f(z)|\le k$ almost everywhere in $\mathbb{C}$. The symbols f_z, $f_{\bar z}$ denote formal derivatives of f.

Let $S^*(\alpha), 0\le\alpha<1$, denote the subclass of S consisting of strongly starlike functions of order α, cf. [1], [5], i.e. of functions f that satisfy:

$$\left|\arg\frac{zf'(z)}{f(z)}\right|\le\alpha\frac{\pi}{2}\quad,\quad z\in\Delta. \tag{1}$$

As shown in [1], $f(\Delta)$ is a Jordan domain for any $f\in S^*(\alpha)$.

In this paper we find an explicit quasiconformal extension for an arbitrary function $f\in S^*(\alpha)$. We show that $S^*(\alpha)\subset S_k$, where $k\le\sin\alpha\pi/2$.

We construct this extension by means of an auxiliary mapping which may be called a reflection with respect to a starlike Jordan curve (Lemma 1). In what follows we call a k-circle a Jordan curve that is a homeomorphic image of the unit circumference under a quasiconformal mapping F of the extended plane $\bar{\mathbb{C}}$ onto itself whose complex dilatation μ_F satisfies $|\mu_F(z)|\le k<1$ a.e.

We obtain explicit quasiconformal extensions for bounded convex functions and for functions with bounded boundary rotation (Theorems 3,4). In particular we show that any convex Jordan curve contained in an annulus $\{w:r\le|w|\le R\}$ is a k-circle with $k\le\sqrt{1-(r/R)^2}$.

Similarly, any strongly starlike curve of order α is a k-circle with $k\le\sin\alpha\pi/2$.

2. Quasiconformal Extension for the Class $S^*(\alpha)$

In this section we shall prove

THEOREM 1. *If* $f \in S^*(\alpha)$, $0 \le \alpha < 1$, *then the mapping* F *defined by the formula*

$$F(z) = \begin{cases} f(z) & \text{for } |z| \le 1 \\ |f(\xi)|^2 \Big/ \overline{f\left(\dfrac{1}{\bar z}\right)}, & \text{for } |z| \ge 1, \end{cases} \tag{2}$$

where ζ *satisfies the conditions:* $|\zeta| = 1$, $\arg f(\zeta) = \arg f(1/\bar z)$, *belongs to the class* S_k *and* $|\mu_F(z)| \le k = \sin \alpha\pi/2$ a.e.

We first prove

LEMMA 1. *Suppose that* G *is a domain bounded by a Jordan curve* Γ *starlike with respect to the origin. Suppose, moreover, that*

$$w = R(\varphi) e^{i\varphi}, \qquad 0 \le \varphi \le 2\pi, \tag{3}$$

is the parametric equation of Γ, *where* $R(\varphi)$ *is absolutely continuous and positive,* $R(0) = R(2\pi)$ *and*

$$|R'(\varphi)|[R^2(\varphi) + R'^2(\varphi)]^{-1/2} \le k < 1 \tag{4}$$

almost everywhere in $[0, 2\pi]$. *Then the mapping*

$$\phi(w) = R^2(\varphi)/\bar w, \qquad \varphi = \arg w, \tag{5}$$

is an antiquasiconformal mapping of G *onto* $\hat{\mathbb{C}} \setminus \bar G$ *whose complex dilatation* $\phi_w/\phi_{\bar w}$ *is bounded by* k *in absolute value.*

Proof. Obviously ϕ is a sense-reversing homeomorphism in G. Moreover, if $w = re^{i\varphi}$, $0 < r < R(\varphi)$, then

$$\Phi(w) = \Phi(re^{i\varphi}) = R^2(\varphi) e^{i\varphi}/r$$

and we have for almost all φ in $[0, 2\pi]$:

$$\mu_\phi = \frac{\phi_w}{\phi_{\bar{w}}} = e^{-2i\varphi} \frac{\phi_r + \phi_\varphi / ir}{\phi_r - \phi_\varphi / ir} = -e^{-2i\varphi} \frac{R'(\varphi)}{R'(\varphi) + iR(\varphi)}$$

so that $|\mu_\phi(w)| \leq k$ almost everywhere in G by (4).

We now prove that ϕ has the ACL-property in $G \backslash \{0\}$. The function $R^2(\varphi)e^{i\varphi}/r$ is absolutely continuous in φ with fixed $r > 0$ because by (4) $R'(\varphi)$ is essentially bounded and also absolutely continuous in r, $r \in [\delta, R(\varphi)]$, for fixed φ, $\delta > 0$. Thus the ACL-property holds in the log w-plane. Since the ACL-property is invariant under composition with conformal mapping, φ has in fact the ACL-property in $G \backslash \{0\}$. This ends the proof.

The condition (4) has a simple geometrical interpretation. Suppose that $R'(\varphi)$ does exist. Then Γ has a tangent intersecting the radius vector at an angle $\psi = \arctan R/R'$ and consequently

$$R'(R^2 + R'^2)^{-1/2} = \cos \psi.$$

Hence (4) means that the angle ψ is bounded away from 0 and π at points where the tangent does exist.

The mapping $\phi(w)$ will be called a reflection with respect to the starshaped curve Γ. It is a sense-reversing homeomorphism for any starshaped Jordan curve Γ. Moreover, if the angle between the radius vector and the tangent of Γ is bounded away from 0 and π a.e., the reflection $\phi(w)$ is an anti-quasiconformal mapping.

Proof of Theorem 1. If $f \in S^*(\alpha)$ with $0 \leq \alpha < 1$ then f has a continuous extension on $\bar{\Delta}$, $f(e^{i\theta})$ is absolutely continuous and $d/d\theta\, f(e^{i\theta}) = ie^{i\theta} f'(e^{i\theta})$ a.e. in $[0, 2\pi]$, cf. [1]. Hence the definition of F in (2) makes sense. Let Γ be the Jordan curve $w = f(e^{i\theta}) = R(\varphi)e^{i\varphi}$, $0 \leq \theta \leq 2\pi$. After differentiation with respect to θ of the identity:

$$\log f(e^{i\theta}) = \log R(\varphi) + i\varphi,$$

we obtain

$$\frac{e^{i\theta} f'(e^{i\theta})}{f(e^{i\theta})} = \left[1 - \frac{iR'(\varphi)}{R(\varphi)}\right] \frac{d\varphi}{d\theta}$$

and hence by (1)

$$\left|\arg\left\{1-i\frac{R'(\varphi)}{R(\varphi)}\right\}\right|\leq\frac{\alpha\pi}{2},$$

or

$$|R'(\varphi)|[R^2(\varphi)+R'^2(\varphi)]^{-1/2}\leq\sin\frac{\alpha\pi}{2}\quad\text{a.e.}$$

This means that Γ satisfies the condition (4) with $k=\sin\alpha\pi/2$ and therefore the reflection with respect to Γ is anti-quasiconformal with complex dilatation bounded by $\sin\alpha\pi/2$. Now, the mapping $F(z)$ for $|z|>1$ is composed of the following mappings: reflection in $|z|=1$, conformal mapping f and a reflection with respect to Γ. Complex dilatation of F has the form

$$\mu_F=\left(\frac{z}{\bar{z}}\right)^2\frac{\overline{f'(1/\bar{z})}}{\overline{f(1/\bar{z})}}\frac{\phi_w}{\phi_{\bar{w}}}.$$

Therefore F is a quasiconformal mapping in $\{z:|z|>1\}$ and $|\mu_F(z)|\leq\sin\alpha\pi/2$ a.e. by Lemma 1. Obviously F as defined by (2), is a homeomorphism of the sphere $\hat{\mathbb{C}}$ onto itself which is conformal in Δ and quasiconformal in $\mathbb{C}\setminus\bar{\Delta}$. Since $\partial\Delta$, $\{\infty\}$, $\{0\}$ are removable sets, cf. [4], F is quasiconformal in $\hat{\mathbb{C}}$.

COROLLARY 1. *If Γ is a Jordan curve starshaped with respect to $w=0$ and intersecting the radius vectors at an angle bounded away from 0 and π by $\beta\pi/2$, $0<\beta\leq 1$, then Γ is a k-circle with $k\leq\cos\beta\pi/2$.*

3. Some Applications of Theorem 1

It is well-known that, if

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n\quad\text{in}\quad\Delta \tag{5}$$

and

$$\sum_{n=2}^{\infty}n\,|a_n|\leq 1 \tag{6}$$

then f is a starlike univalent function. The condition (6) does not imply the possibility of quasiconformal extension of f (e.g. $f(z)=z+\frac{1}{2}z^2$ satisfies (6) and obviously has no quasiconformal extension on $\hat{\mathbb{C}}$).

Consider the class $\tilde{S}(k)$ of functions f of the form (5) that satisfy the condition

$$\sum_{n=2}^{\infty} n\,|a_n|\le k<1. \tag{6'}$$

We prove

LEMMA 2. *If $f\in\tilde{S}(k)$, then $f\in S^*(\alpha)$ with $\alpha=(2/\pi)$ arc sin k.*

Proof. The condition (6′) implies

$$\left|\frac{zf'(z)}{f(z)}-1\right|\le k,$$

because

$$\left|\frac{zf'(z)}{f(z)}-1\right|=\left|\frac{\sum_{n=2}^{\infty}(n-1)a_nz^{n-1}}{1+\sum_{n=2}^{\infty}a_nz^{n-1}}\right|\le\frac{\sum_{n=2}^{\infty}(n-1)\,|a_n|}{1-\sum_{n=2}^{\infty}|a_n|}\le k.$$

Hence f satisfies (1) with $\alpha=(2/\pi)$ arc sin k.

From Lemma 2 and Theorem 1 we immediately obtain

THEOREM 2. *If $f\in\tilde{S}(k)$ then $f\in S_k$.*

Another quasiconformal extension of $f\in\tilde{S}(k)$ can be obtained in a different way, similarly as in [2].

THEOREM 2′. *Let $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ belong to $\tilde{S}(k)$. Then the mapping $G(z)$ defined by the formula*

$$G(z)=\begin{cases} z+\sum_{n=2}^{\infty}a_nz^n & \text{for } |z|\le 1,\\ z+\sum_{n=2}^{\infty}a_n\bar{z}^{-n} & \text{for } |z|\ge 1\end{cases} \tag{7}$$

is a quasiconformal extension of f onto $\hat{\mathbb{C}}$ and $|\mu_G(z)|\le k$.

The mapping G satisfies the following condition:

$$|z_1-z_2|\,(1-k)\le|G(z_1)-G(z_2)|\le|z_1-z_2|\,(1+k) \tag{8}$$

for $z_1, z_2\in\Delta$ and also for $z_1, z_2\in\mathbb{C}\setminus\bar{\Delta}$. It is well known that a function lipschitzian in Δ has a continuous extension on $\bar{\Delta}$ that satisfies (8) also in $\bar{\Delta}$. Hence G as defined by (7) is a sense-preserving homeomorphism in $\hat{\mathbb{C}}$. Its complex dilatation satisfies

$$|\mu_G(z)|=|G_{\bar{z}}/G_z|=\left|\sum_{n=2}^{\infty} na_n z^{-n-1}\right|\le\sum_{n=2}^{\infty} n\,|a_n|\le k$$

in $\mathbb{C}\setminus\bar{\Delta}$. Since $\partial\Delta$ is a removable set, G is a quasiconformal in $\hat{\mathbb{C}}$.

Let $C(B)$ denote the subclass of S consisting of all convex functions for which $|f(z)|\le B$, $z\in\Delta$. Next, let $V_\lambda(B)$ denote the subclass of S consisting of all bounded functions $|f(z)|\le B$ for which $f(\Delta)$ has boundary rotation at most $\lambda\pi$, cf. [3].

Moreover, let d_f denote the radius of the largest open disc centered at the origin which is contained in $f(\Delta)$.

In [1] Brannan and Kirwan have found the following relations between $C(B)$, $V_\lambda(B)$ and $S^*(\alpha)$.

(i) If $f\in C(B)$, then $f\in S^*(\alpha)$ with $\alpha=1-(2/\pi)\arcsin d_f/B$.

(ii) If $f\in V_\lambda(B)$ and $(\lambda-2)\pi<2\arcsin(d_f/B)$, then $f\in S^*(\alpha)$ with $\alpha=\lambda-1-(2/\pi)\arcsin(d_f/B)$.

The above stated relations yield at once as immediate consequences of Theorem 1 the following results.

THEOREM 3. *If $f\in C(B)$, then f has a quasiconformal extension F on the whole plane defined by the formula* (2) *and*

$$|\mu_F(z)|\le\sqrt{1-\left(\frac{d_f}{B}\right)^2}.$$

COROLLARY 2. *If Γ is a convex Jordan curve contained in the annulus $\{w: r\le|w|\le R\}$, then Γ is a k-circle with $k\le\sqrt{1-(r/R)^2}$.*

THEOREM 4. *If $f\in V_\lambda(B)$ and $(\lambda-2)\pi<2\arcsin(d_f/B)$, then the function F defined by* (2) *is a quasiconformal extension of f and*

$$|\mu_F(z)|\le\sin\left[(\lambda-1)\frac{\pi}{2}-\arcsin\frac{d_f}{B}\right].$$

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