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On Meromorphic Solutions of First-Order Differential Equations

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1. Introduction

In the first part of this paper, we consider first-order differential equations of the form,

$$\sum_{n=0}^{N} R_n(z, w)(w')^n = 0, \tag{1}$$

where for each n, $R_n(z, w) = \sum_{k=0}^{B(n)} a_{kn}(z)P_{kn}(w)$, the $a_{kn}(z)$ are analytic functions in a neighborhood of ∞, having no essential singularity at ∞, and the functions $P_{kn}(w)$ are all defined and analytic on some common open set in the plane. Such equations were treated by A. A. Gol'dberg in [1], and he showed [1; Th 4] that any solution, meromorphic in a neighborhood of ∞, is of finite order of growth, and he obtained estimates (which depend on the equation) for the growth. The main technique used by Gol'dberg in proving this result is 0. Frostman's generalization of the Ahlfors-Shimizu formula (see [3; p. 180] or [10; p. 42]). In the first part of our paper (§§ 3, 4 below), we present an alternate proof of Gol'dberg's result which seems to be more elementary and more transparent than Gol'dberg's proof. In our proof, we determine disks around the sufficiently large a-points of the solution (for most values of a) on which the solution is univalent. From this we obtain estimates for the growth of the counting functions for the a-points, and hence an estimate on the growth of the solution by Nevanlinna's Second Fundamental Theorem.

The second part of the paper deals with algebraic differential equations (i.e. equations of the form $F(z, y, y', \ldots, y^{(n)}) = 0$, where F is a polynomial in all its variables.) In [5], Pólya proved that an entire transcendental function of order zero cannot be a solution of a first-order algebraic differential equation. This result was generalized by Valiron [8], who showed that in the first-order case, the order of an entire transcendental solution must be a positive rational number (and it is now known (see Strelitz [7; p. 70]) that the order must be at least $\frac{1}{2}$.). In addition, Valiron (see [9; pp. 223–225]) found a third-order algebraic differential equation with a transcendental solution of order zero. The remaining case of second-order algebraic differential equations was settled by Zimogljad [11], who showed that these equations cannot possess transcendental entire solutions of order zero. At the present time, the situation concerning transcendental meromorphic solutions of order zero is less clear. In the third-order equation, $F(z, y, y', y'', y''') = 0$, constructed by Valiron in [9], it is easily seen that F is homogeneous as a polynomial in y, y', y'' and y''', and hence the logarithmic derivative of Valiron's solution does provide an example of a transcendental meromorphic solution of order zero of a second-order algebraic differential equation. However, to the authors' knowledge, it was not known whether transcendental meromorphic functions of order zero can satisfy first-order algebraic differential equations. (The result of Gol'dberg [1, Th. 4] hints at the possibility that such solutions may exist, but none had been constructed.) In the second part of our paper (see § 5 below), we construct an example of a transcendental meromorphic function $f(z)$ of order zero which satisfies a first-order algebraic differential equation. The characteristic of our function $f(z)$ satisfies $T(r, f) = O((\log r)^2)$ as $r \to +\infty$ (which is the estimate suggested by Gol'dberg's result), and from the construction of $f(z)$, it is easy to see that $T(r, f) \neq o((\log r)^2)$ as $r \to +\infty$. We conjecture that first-order algebraic differential equations cannot possess transcendental meromorphic solutions whose characteristic is $o((\log r)^2)$ as $r \to +\infty$.

2. Notation

If f is a meromorphic function and λ is a complex number or ∞, we will use the standard notation for the Nevanlinna functions, $T(r, f)$, $N(r, \lambda, f)$ and $m(r, \lambda, f)$, (see [2; p. 6] or [4; pp. 6, 12]). We will also use the notation $n(r, \lambda, f)$ to denote the number of roots of $f(z) = \lambda$ (counting multiplicity) in $|z| \leq r$.

3. THEOREM. *Given the equation,*

$$\sum_{n=0}^{N} R_n(z, w)(w')^n = 0, \tag{2}$$

where each $R_n(z, w)$ is a polynomial in z whose coefficients are meromorphic functions of w, say

$$R_n(z, w) = \sum_{j=0}^{A(n)} z^j P_{jn}(w), \tag{3}$$

where $P_{jn}(w)$ is a meromorphic function of w. Let J be the set of integers n, $0 \leq n \leq N$, for which $R_n(z, w)$ is not identically zero, and we may assume that 0 and N belong to J, and if n belongs to J, then $P_{A(n),n}(w)$ is not identically zero. Assume also that $N \geq 1$ (see § 4(b)), and set

$$L = \max\{((A(j) - A(N))/(N-j)) : j \in J - \{N\}\}. \tag{4}$$

Let $w = f(z)$ be a meromorphic function on the plane which satisfies equation (2) *at every point of analyticity. Then, as $r \to +\infty$,*

(a) $T(r, f) = O(\log r)$ if $L < -1$,

(b) $T(r, f) = 0((\log r)^2)$ if $L = -1$,

(c) $T(r, f) = O(r^{2L+2})$ if $L > -1$.

Proof. Let E be the set of all complex numbers w_0 with the property that if $P_{jn}(w)$ is not identically zero, then $P_{jn}(w)$ is analytic and nonzero at w_0. We now prove a sequence of four lemmas from which the theorem will immediately follow.

LEMMA A. *Let w_0 belong to E. Then there exist real numbers, $b > 0$, $r_1 > 1$, $K_2 > K_1 > 0$ and $a_1 < a_2 < \cdots < a_q \leq L$, such that if $1 \leq j < s \leq q$, then*

$$2K_2 |z|^{a_j} < K_1 |z|^{a_s} \quad \text{if} \quad |z| > r_1, \tag{5}$$

and in addition, if z is a complex number satisfying $|z| > r_1$ and $|f(z) - w_0| \leq b$, then there is a unique element j in the set $\{1, 2, \ldots, q\}$ such that,

$$K_1 |z|^{a_j} \leq |f'(z)| \leq K_2 |z|^{a_j}. \tag{6}$$

Proof. If w_0 belongs to E, then there exist real numbers $b > 0$ and $d_2 > d_1 > 0$ such that on $|w - w_0| \leq b$, we have,

$$d_1 \leq |P_{jn}(w)| \leq d_2, \tag{7}$$

for all P_{jn} which are not identically zero. It now easily follows from (3) that there exists $r_0 > 1$, such that if n belongs to J, then for any z satisfying $|z| > r_0$ and $|f(z) - w_0| \leq b$, we have,

$$c_1 |z|^{A(n)} \leq |R_n(z, f(z))| \leq c_2 |z|^{A(n)}, \tag{8}$$

where $c_1 = d_1/2$ and $c_2 = 2d_2$. For convenience, denote $R_n(z, f(z))$ by $B_n(z)$. Now if z satisfies $|z| > r_0$ and $|f(z) - w_0| \le b$, let k be the largest element of J for which,

$$|B_k(z)(f'(z))^k| = \max\{|B_n(z)(f'(z))^n| : n \in J\}. \tag{9}$$

Then there must exist an element m in $J - \{k\}$ such that,

$$|B_m(z)(f'(z))^m| \ge N^{-1}|B_k(z)(f'(z))^k|, \tag{10}$$

or else equation (2) would clearly be violated at z. If m is the smallest element of $J - \{k\}$ with property (10), we will say that the pair (k, m) is the *index* for z. Denote by α_{km} the number $(A(k) - A(m))/(m - k)$. In addition, let K_2 denote the maximum of all numbers $(Nc_2/c_1)^{1/(j-n)}$, and K_1 the minimum of all numbers $(c_1/Nc_2)^{1/(j-n)}$, where j and n belong to J and $j > n$. (Clearly K_1 and K_2 are independent of z.) From (8), (9) and (10), it easily follows that if z satisfies $|z| > r_0$ and $|f(z) - w_0| \le b$, and if (k, m) is the index for z, then

$$K_1 |z|^{\alpha_{km}} \le |f'(z)| \le K_2 |z|^{\alpha_{km}}. \tag{11}$$

Let F be the set of distinct numbers of the form α_{km} for which there exists a complex number z satisfying $|z| > r_0$ and $|f(z) - w_0| \le b$ having (k, m) for its index. Let F_1 be the subset of F consisting of those elements of F which are larger than L. Let r_2 be so large that $r_2 > r_0$ and

$$r_2^{\alpha - L} > K_2/K_1 \quad \text{for all} \quad \alpha \text{ in } F_1. \tag{12}$$

We now claim that if z satisfies $|z| > r_2$ and $|f(z) - w_0| \le b$, and if (k, m) is the index for z, then

$$\alpha_{km} \le L. \tag{13}$$

If $k = N$, then (13) is clear, so we may assume that $k < N$. If (k, m) is the index for z, then by (9), we have

$$|B_k(z)(f'(z))^k| \ge |B_N(z)(f'(z))^N|. \tag{14}$$

In view of (8) and (4), it follows that $|f'(z)| \le K_2 |z|^L$. But since (k, m) is the index for z, (11) holds, so that $|z|^{\alpha_{km} - L} \le K_2/K_1$. Hence by (12), we see that α_{km} cannot belong to F_1 which proves (13). Now if $a_1 < a_2 < \cdots < a_q$ are the elements of $F - F_1$, and if r_1 is chosen so large that $r_1 > r_2$ and (5) holds for $|z| > r_1$, then the proof of the lemma is complete.

DEFINITION. Let w_0 belong to E, and let b, r_1, K_1, K_2 and $a_1, \ldots, a_q$ be as in Lemma A. If z is a complex number satisfying $|z|>r_1$ and $|f(z)-w_0|\leq b$, then the unique a_j for which (6) holds will be said to be *associated with* z.

LEMMA B. *Let w_0 belong to E with b, r_1, K_1, K_2 and $a_1, \ldots, a_q$ as in Lemma A. Assume that some a_k is less than -1. Then there exists $R_0>r_1$ such that if there is at least one point z_0 satisfying $|z_0|>R_0$ and $f(z_0)=w_0$, whose associated a_j is less than -1, then $f(z)$ is a rational function.*

Proof. Let a_m be the largest a_k less than -1 say $a_m=-1-\eta$, where $\eta>0$. Choose R_0 so large that $R_0>r_1$ and,

$$R_0^{-\eta}\leq \min\{b\eta/4K_2,\, b/8\pi K_2\}. \tag{15}$$

Now let z_0 satisfy $|z_0|>R_0$, $f(z_0)=w_0$, and have associated $a_j<-1$. Then $a_j=-1-\sigma$, where

$$\eta\leq\sigma. \tag{16}$$

If $z_0=|z_0|\, e^{i\varphi}$, we now assert that,

$$|f'(re^{i\varphi})|\leq K_2 r^{a_j} \quad \text{for all} \quad r\geq|z_0|. \tag{17}$$

If (17) fails to hold, then clearly we can find ϵ satisfying $0<\epsilon<K_2$, and a point $z_2=|z_2|\, e^{i\varphi}$, with $|z_2|>|z_0|$, such that,

$$|f'(z_2)|=(K_2+\epsilon)\,|z_2|^{a_j}, \tag{18}$$

while,

$$|f'(re^{i\varphi})|\leq(K_2+\epsilon)r^{a_j} \quad \text{for} \quad |z_0|\leq r\leq|z_2|. \tag{19}$$

Hence from (19),

$$|f(z_2)-f(z_0)|\leq(K_2+\epsilon)\,|z_0|^{-\sigma}/\sigma, \tag{20}$$

so in view of (15) and (16),

$$|f(z_2)-w_0|\leq b/2. \tag{21}$$

By Lemma A, let a_s be associated with z_2. Then by (6) and (18), we have,

$$K_1 |z_2|^{a_s} \leq (K_2 + \epsilon) |z_2|^{a_j} \leq K_2 |z_2|^{a_s}. \tag{22}$$

From the second inequality, it follows easily that $a_s \leq a_j$ is impossible, so $a_j < a_s$. But then the first inequality contradicts (5), thus proving (17). From (17) (together with (15) and (16)), it follows that

$$|f(re^{i\varphi}) - w_0| \leq b/2 \quad \text{for all} \quad r \geq |z_0|. \tag{23}$$

We now assert that if $r > R_0$, then

$$|f'(z)| \leq K_2 |z|^{a_j} \quad \text{on} \quad |z| = r. \tag{24}$$

By (17), we know that (24) holds at $z_1 = re^{i\varphi}$. Hence if (24) failed to hold at some point on $|z| = r$, then we can find ϵ, with $0 < \epsilon < K_2$, and a point $z_2 = re^{i\psi}$, with $\varphi < \psi < \varphi + 2\pi$, such that,

$$|f'(z_2)| = (K_2 + \epsilon) |z_2|^{a_j}, \tag{25}$$

while for $\varphi \leq \theta \leq \psi$,

$$|f'(re^{i\theta})| \leq (K_2 + \epsilon) r^{a_j}. \tag{26}$$

Hence, $|f(z_2) - f(z_1)| \leq 4\pi K_2 r^{a_j+1}$, which with (15), (16) and (23), yields $|f(z_2) - w_0| \leq b$. Thus if a_s is associated with z_2 (by Lemma A), then using (25), we again obtain (22), which as before is impossible. This proves (24), and it easily follows that f is rational.

LEMMA C. *Let w_0 belong to E, and let b, r_1, K_1, K_2 and $a_1, \ldots, a_q$ be as in Lemma A. Assume f is transcendental. Then there is a constant δ_1, with $0 < \delta_1 < \frac{1}{2}$, with the property that if z_0 satisfies $|z_0| > 2r_1$ and $f(z_0) = w_0$, and if the a_j associated with z_0 is at least -1, then f is univalent on the disk $|z - z_0| \leq \delta_1 |z_0|^{-L}$.*

Proof. Let $\lambda = \max\{|a_k| : k = 1, \ldots, q\}$. Let δ be a positive real number such that,

$$\delta < \min\{\tfrac{1}{4}, (b/K_2)2^{-\lambda}\}, \tag{27}$$

and set,

$$\delta_1 = \delta(K_1/K_2)2^{-(2\lambda+2)}. \tag{28}$$

Let z_0 satisfy $|z_0|>2r_1$, $f(z_0)=w_0$, and let a_j be associated with z_0 and satisfy $a_j \geq -1$. Since f is transcendental, the set

$$H=\{z:|z|\geq 3r_1/2, \qquad |f(z)-w_0|=b\}, \tag{29}$$

is not empty. Let z_1 be a point in H such that $|z_0-z_1|=\min\{|z_0-z|:z\in H\}$.

We now assert that,

$$|z_1-z_0|\geq \delta\,|z_0|^{-a_j}. \tag{30}$$

To prove (30), we assume the contrary, so that

$$|z_1-z_0|\leq \delta\,|z_0|^{-a_j}\leq \delta\,|z_0|. \tag{31}$$

Let D be the disk $|z-z_0|\leq|z_1-z_0|$. In view of our assumption (31), it follows that for all z in D, we have $|z|\geq(\frac{3}{2})r_1$ (by (27)) and $|f(z)-w_0|\leq b$ (or else the definition of z_1 would be violated). Hence by Lemma A, each z in D has some a_k associated with it. Using (5) and (6), it follows by an argument very similar to that used to prove (17), that for every z in D, the a_k associated with z is the original a_j associated with z_0, so that (6) holds on D. Now from (27) and (31), we have, $|z_0|/2\leq|z|\leq 2\,|z_0|$ on D, so it follows from (6) that $|f'(z)|\leq K_2 2^{|a_j|}\,|z_0|^{a_j}$. Since z_1 belongs to H, we thus have $b\leq K_2 2^{\lambda}\,|z_0|^{a_j}\,|z_1-z_0|$, which in view of (31) and (27) is impossible. This proves (30).

Let D_1 be the disk $|z-z_0|\leq \delta\,|z_0|^{-a_j}$. Then using (30) and (27), it follows that for all z in D_1, we have $|z|\geq 3r_1/2$ and $|f(z)-w_0|\leq b$ (or else the definition of z_1 would be violated). As before, the a_k associated with each z in D_1 must be the original a_j associated with z_0, so that,

$$K_1\,|z|^{a_j}\leq|f'(z)|\leq K_2\,|z|^{a_j} \quad \text{on} \quad D_1. \tag{32}$$

Since $|z_0|/2\leq|z|\leq 2\,|z_0|$ on D_1, we thus have,

$$K_1 2^{-|a_j|}\,|z_0|^{a_j}\leq|f'(z)|\leq K_2 2^{|a_j|}\,|z_0|^{a_j}, \quad \text{on} \quad D_1. \tag{33}$$

Now let D_2 be the disk, $|z-z_0|\leq(\delta/2)\,|z_0|^{-a_j}$. If z belongs to D_2, then by Cauchy's formula for derivatives (using (33) and the circle of radius $(\delta/2)\,|z_0|^{-a_j}$ around z), we obtain,

$$|f''(z)|\leq K_2 2^{|a_j|+1}\,|z_0|^{2a_j}/\delta \quad \text{on} \quad D_2. \tag{34}$$

Now let D_3 be the disk $|z-z_0|\leq \delta_1 |z_0|^{-a_j}$, where δ_1 is as in (28). Then D_3 is contained in D_2, and by (34) and (33), it easily follows that on D_3,

$$|f'(z)-f'(z_0)|\leq |f'(z_0)|/2. \tag{35}$$

Writing $f'(z)=f'(z_0)+(f'(z)-f'(z_0))$, it now easily follows from (35) (and (33)) that if ζ and σ are distinct points in D_3, then $\int_\sigma^\zeta f'(z)\,dz$ (where the contour is the line segment joining σ to ζ) cannot be zero, and hence f is univalent on D_3. Since $a_j\leq L$, D_3 contains the disk $|z-z_0|\leq \delta_1 |z_0|^{-L}$ and hence the result is proved.

LEMMA D. *Let L and δ_1 be real numbers with $L\geq -1$ and $0<\delta_1<1$. Let $\{z_k\}$ be a sequence of complex numbers such that each disk $|z-z_k|\leq \delta_1 |z_k|^{-L}$ contains no other z_m. Then as $R\to +\infty$, the number of points z_k in the annulus $1\leq |z|\leq R$ is $O(R^{2L+2})$ if $L>-1$, and is $O(\log R)$ if $L=-1$.*

Proof. Let n be a nonnegative integer and let $A_n=\{k: 2^n\leq |z_k|\leq 2^{n+1}\}$. Let $\delta_2=2^{-|L|}\delta_1$. Then it is easy to see that,

$$\delta_1 |z_k|^{-L}\geq \delta_2 2^{-nL} \quad \text{for} \quad k \quad \text{in} \quad A_n. \tag{36}$$

For k in A_n, let $w_k=z_k 2^{-n}$, so $1\leq |w_k|\leq 2$. In view of the hypothesis and (36), it follows that for k in A_n, the disk $|w-w_k|\leq r_n$, where $r_n=\delta_2 2^{-nL} 2^{-n}$, contains no other w_m for m in A_n. Thus the disks $|w-w_k|\leq r_n/3$, for k in A_n. are all disjoint and all lie in $|w|\leq 3$. By an area argument, it follows that if $\sigma(n)$ is the cardinal number of A_n, then,

$$\sigma(n)\leq (81/\delta_2^2)4^{n(L+1)}. \tag{37}$$

Now if $R>2$ is given, let m be such that $2^m\leq R\leq 2^{m+1}$. Then if $\nu(R)$ is the number of z_k in $1\leq |z|\leq R$, clearly,

$$\nu(R)\leq \sum_{n=0}^{m} \sigma(n). \tag{38}$$

From (37) and (38), Lemma D immediately follows.

By Nevanlinna's Second Fundamental Theorem [4; p. 69], it follows that if w_1, w_2 and w_3 are distinct complex numbers, then as $r\to +\infty$,

$$T(r,f)=O\left(\sum_{j=1}^{3} N(2r,w_j,f)+\log r\right), \tag{39}$$

and hence it is now clear that the theorem of § 3 follows immediately from Lemmas A–D.

4. Remarks

(a) It is easy to see that the proof of the theorem of § 3 is valid when the hypotheses are relaxed as follows. In equation (2), for each n,

$$R_n(z, w) = \sum_{k=0}^{B(n)} a_{kn}(z) P_{kn}(w), \tag{40}$$

where the $a_{kn}(z)$ are analytic functions in a neighborhood of ∞, having no essential singularity at ∞, the functions $P_{kn}(w)$ are all defined and analytic on some common open set in the plane, and $f(z)$ is a solution meromorphic in a neighborhood of ∞, say $|z| \geq R_0$. (In the definition of the characteristic $T(r, f)$ of such a function (see [10; p. 49]), only the a-points lying in $R_0 \leq |z| \leq r$ are considered in defining $N(r, a, f)$, and the Second Fundamental Theorem still holds for such functions (see [10; p. 50]).) Let J be as in the statement of the theorem, and for n in J, rearrange terms in $R_n(z, w)$ so that $R_n(z, w)$ has the form,

$$R_n(z, w) = z^{A(n)} g_n(w) + \sum_{k=0}^{B(n)} b_{kn}(z) P_{kn}(w), \tag{41}$$

where $g_n(w)$ is not identically zero, and the highest power of z in the Laurent expansion for each $b_{kn}(z)$ at ∞ is less than $A(n)$. Then with L as defined in (4), the conclusions (a), (b), (c) of the theorem hold. (In this formulation, the theorem is now fully equivalent to Gol'dberg's result [1; Theorem 4].) The proof in this formulation is easily seen to be identical to the proof we gave in § 3, with three minor changes. First, the set E would consist of all complex numbers w_0 with the property that each $g_n(w)$ (for n in J) is analytic and nonzero at w_0, and if $P_{jn}(w)$ is not identically zero, then $P_{jn}(w)$ is analytic and nonzero at w_0. Secondly, the conclusion of Lemma B would be $T(r, f) = O(\log r)$ as $r \to +\infty$, which follows easily from (24) since then $f(z)$ has a finite limit at ∞. Finally, in the hypothesis of Lemma C, we would assume that $T(r, f) \neq O(\log r)$ as $r \to +\infty$, instead of assuming f is transcendental.

(b) If $N = 0$ in the theorem of § 3, then f must be a rational function. This is easily seen as follows. If $A(0) = 0$, clearly f must be a constant, so we may assume $A(0) > 0$. In this case, letting E be as in the proof, we see that if w_0 belongs to E, then there are positive constants b, d_1 and d_2 such that on $|w - w_0| \leq b$, we have

(7). It easily follows that if $|z|$ is sufficiently large, then $|f(z)-w_0|>b$ and hence f must be rational.

5. EXAMPLE. In this section, we construct an example of a transcendental meromorphic function on the plane of order zero, which satisfies a first-order algebraic differential equation.

Let $P(z)$ denote the Weierstrass Pe-function having primitive periods 1 and $2\pi i$ (see [6; p. 368]). If z is a nonzero complex number, and α_1 and α_2 are two values of $\log z$, then clearly $P(\alpha_1)=P(\alpha_2)$. Thus, $w(z)=P(\log z)$ is single-valued on the punctured plane. Now if ζ is a complex number, then clearly there is at least one complex number z such that $z+z^{-1}=\zeta$, and if $z_j+z_j^{-1}=\zeta$ for $j=1,2$, then either $z_1=z_2$ or $z_1=z_2^{-1}$. In either case, $w(z_1)=w(z_2)$ since $P(z)$ is an even function. Thus, the function $u(\zeta)$, defined by,

$$u(\zeta)=w(z) \quad \text{where} \quad z+z^{-1}=\zeta, \tag{42}$$

is single-valued on the plane.

We assert that $u(\zeta)$ is meromorphic on the plane. First, if $\zeta_0\neq\pm2$, then there is an analytic function $h(\zeta)$ around ζ_0 such that $h(\zeta)+(h(\zeta))^{-1}\equiv\zeta$. Since $h(\zeta_0)\neq0$, there is an analytic branch $L(z)$ of $\log z$ on a neighborhood of $h(\zeta_0)$, so that $L(h(\zeta))$ is analytic on a neighborhood of ζ_0. Thus $u(\zeta)=P(L(h(\zeta)))$ is meromorphic on a neighborhood of ζ_0. Now suppose $\zeta_0=2$. By the above argument, $u(\zeta)$ is meromorphic on $0<|\zeta-\zeta_0|<4$. Let $\{\zeta_n\}$ be a sequence converging to ζ_0 such that $\zeta_n\neq\zeta_0$ for $n=1,2,\ldots$, and let z_n be such that $z_n+z_n^{-1}=\zeta_n$. Then clearly $\{z_n\}\to1$ and $z_n\neq1$ for each n. Let $L(z)$ be an analytic branch of $\log z$ on $|z-1|<1$ such that $L(1)=0$. Then for all sufficiently large n, $0<|L(z_n)|<1$, and hence $u(\zeta_n)$ is finite. This shows that $\zeta_0=2$ is an isolated singularity of $u(\zeta)$, and since $P(0)=\infty$, we see that $\zeta_0=2$ is a pole of $u(\zeta)$. A similar argument shows that $\zeta_0=-2$ is a removable singularity of $u(\zeta)$, and hence $u(\zeta)$ is meromorphic on the plane.

From the differential equation for $P(z)$ (see [6; 372]), it easily follows that $u(\zeta)$ satisfies the first-order algebraic differential equation,

$$(\zeta^2-4)(u')^2=4u^3-g_2u-g_3, \tag{43}$$

where g_2 and g_3 are certain constants. Let $z_1=\frac{1}{2}$, $z_2=\pi i$, $z_3=(1+2\pi i)/2$, and $e_j=P(z_j)$ for $j=1$, 2, 3. It follows (see [6; pp. 366, 371]) that e_1, e_2 and e_3 are distinct and that for each j, the set of points where $P(z)=e_j$ is

$$\{z_j+m+2\pi in : m, n=0,\pm1,\pm2,\ldots\}. \tag{44}$$

From this it easily follows that the set of points where $u(\zeta)=e_j$ is

$$\{\exp(z_j+m)+\exp(-(z_j+m)) : m=0, \pm 1, \ldots\}. \tag{45}$$

If $j=1, 2, 3$, then under the change of variable $v=(u-e_j)^{-1}$, it follows (using [6; p. 373. equation (5.9)]) that equation (43) is transformed into an equation of the form,

$$(\zeta^2-4)(v')^2=av^3+bv^2+cv+d, \tag{46}$$

where a, b, c, d are constants and $a\neq 0$. It easily follows that no pole of v can have multiplicity more than 2. Thus the multiplicity of each root of $u(\zeta)=e_j$ is at most 2. It then follows from (45) that for $j=1, 2, 3$, we have $n(r, e_j, u)=O(\log r)$ as $r\to+\infty$, and hence by Nevanlinna's Second Fundamental Theorem, we have $T(r, u)=O((\log r)^2)$ as $r\to+\infty$. Thus $u(\zeta)$ is a transcendental meromorphic solution of equation (43), whose order of growth is zero. (From (45), it easily follows that $T(r, u)\neq o((\log r)^2)$ as $r\to+\infty$.)

BIBLIOGRAPHY

[1] GOL'DBERG, A. A., *On single-valued solutions of first-order differential equations*, Ukrain. Mat. Z., *8* (1956), 254–261, (Russian).

[2] HAYMAN, W. K., *Meromorphic Functions*, Oxford Math. Monographs, Clarendon Press, Oxford, 1964.

[3] NEVANLINNA, R., *Eindeutige analytische Funktionen*, 2nd ed., Springer-Verlag, Berlin, 1953.

[4] NEVANLINNA, R., *Le théorème de Picard-Borel et la théorie des fonctions méromorphes*, Gauthier-Villars, Paris, 1929.

[5] POLYA, G., *Zur Untersuchung der Grössenordnung ganzer Functionen, die einer Differentialgleichung genügen*, Acta Math. *42* (1920), 309–316.

[6] SAKS, S. and ZYGMUND, A., *Analytic Functions*, Monografie Mat., Tom 28 (Engl. transl.), Warsaw, 1952).

[7] STRELITZ, SH., *Asymptotic Properties of Analytic Solutions of Differential Equations*, Vilnius, 1972, (Russian).

[8] VALIRON, G., *Sur les fonctions entières vérifiant une class d'équations différentielles*, Bull. Soc. Math. France, *51* (1923), 33–45.

[9] ——, *Fonctions Analytiques*, Presses Universitaires de France, Paris, 1954.

[10] WITTICH, H., *Neuere Untersuchungen über eindeutige analytische Functionen, Ergebnisse der Mathematik*, Heft 8, Springer-Verlag, Berlin, 1955.

[11] ZIMOGLJAD, V. V., *On the order of growth of transcendental entire solutions of algebraic differential equations of second order*, Mat. Sb. *85* (127), 1971, 286–302 (Russian). (Engl. transl., Math. USSR Sb. *14* (1971), 281–296).

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