Harmonic Maps and the Topology of Stable Hypersurfaces and Manifolds with Non-negative Ricci Curvature

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Harmonic Maps and the Topology of Stable Hypersurfaces and Manifolds with Non-negative Ricci Curvature

RICHARD SCHOEN and SHING TUNG YAU*

This paper may be viewed as a continuation of our previous paper [8] where we discuss certain function theoretic properties of complete Riemannian manifolds and their geometric applications. One useful theorem there is that when f is a non-negative subharmonic function defined on a complete Riemannian manifold M, then f does not belong to $L^p(M)$ for all $p>1$.

In this paper we shall make use of this fact. However, another important tool, the concept of harmonic map, will also be used.

Harmonic maps are critical points of the energy functional defined on the space of maps between two Riemannian manifolds. In [1], Eells and Sampson proved that if N is a compact manifold with non-positive curvature, then any continuous map from a compact manifold into N is homotopic to a harmonic map. Later Hamilton [4] generalized this to the case where the domain manifold is a compact manifold with boundary.

While these theorems have obvious geometric interest, apparently they have rarely been used in the study of geometry of manifolds. In this paper, we make our first attempt in this direction.

To motivate the idea, we make some observations first. It is rather easy to see that every harmonic map from a compact manifold with positive Ricci curvature to a manifold with non-positive curvature is a constant map (cf. [1]). Taking the domain manifold to be the sphere, we conclude immediately that a compact manifold with non-positive curvature is a $K(\pi, 1)$ which is a well-known consequence of a theorem of Cartan-Hadamard. On the other hand, it is also easy to see that the image of every harmonic map from a torus into a compact manifold with negative curvature is a geodesic. Coupling with the above facts, we see that in the fundamental group of a compact manifold of negative curvature, every abelian subgroup is cyclic. This is a theorem of Preissmann.

In order to obtain new results, we turn the above arguments backward and study complete non-compact manifolds with non-negative Ricci curvature.

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As far as we know, there are only a few known topological restrictions to these manifolds. Milnor [6] (subsequently, Wolf [7] and Gromoll-Cheeger [3]) showed that every finitely generated subgroup of the fundamental group of these manifolds has polynomial growth with order less than the dimension of the manifold. Then Gromoll-Meyer [5] showed that every complete non-compact manifold with positive Ricci curvature has at most one end. Using this last fact, we also showed in [8] that for complete manifolds with positive Ricci curvature, the first cohomology group with compact support is torsion.

One of the main theorems in this paper is to prove that given any compact domain D in a complete manifold with non-negative Ricci curvature such that ∂D is simply connected, then any homomorphism from $\pi_1(D)$ into the fundamental group of a compact manifold of non-positive curvature is trivial. To see that this is not a consequence of the above topological obstructions, let M be any compact simply-connected manifold and let $\bar{M}$ be the complement of a point in $S^1 \times M$. Then clearly $\bar{M}$ has only one end and the growth of $\pi_1(\bar{M})$ is small. However, according to our theorem, $\bar{M}$ does not admit complete metric with non-negative Ricci curvature.

The second main theorem of this paper is to apply the same idea to study the topology of complete non-compact stable minimal hypersurfaces in a complete non-negatively curved manifold. We find the same topological obstruction to this class of manifolds. Even if we assume the ambient manifold is the euclidean space, our theorem seems to give the first known topological obstruction to stable hypersurfaces.

Since harmonic maps are not necessarily linear, these theorems may be considered as non-linear vanishing theorems.

1. Existence of a Harmonic Map

Let M be a complete Riemannian manifold and N be a compact manifold with non-positive sectional curvature. Let $f: M \to N$ be a smooth map. Then the energy of f is defined to be the integral $E(f) = \int_M tr_M(f^* ds_N^2)$ where the integrand is the trace of the pull back of the metric tensor of N taken with respect to the metric tensor of M.

In this section, we shall show that when $E(f) < \infty$, we can find a harmonic map $h: M \to N$ such that h is homotopic to f on each compact set of M and $E(h) < \infty$.

Let M_i be a sequence of compact manifolds (with boundary) such that $M = \bigcup_i M_i$. Then according to Hamilton [4] we can find harmonic maps $h_i: M_i \to N$ which are homotopic to $f \mid M_i$. Furthermore, $E(h_i) \leq E(f \mid M_i)$. It remains therefore to prove that there is a subsequence of $\{h_i\}$ which converges uniformly on compact subsets of M.

Let p be an arbitrary point in M. Then we shall find a neighborhood U of p and a subsequence of $\{h_i\}$ which converges uniformly on U.

The first step is to produce a neighborhood of p such that the energy densities $e(h_i) = tr_M(h_i^* \, ds_N^2)$ of h_i are uniformly bounded on this neighborhood. Thus, let U_1 be a compact neighborhood of p and c be the infinimum of the Ricci curvature of U_1. A straightforward computation then shows that on U_1, $\Delta e(h_i) - ce(h_i) \geq 0$.

For our purpose, $c = 0$ is the most interesting case. Hence we give a proof for this special case first.

Let (r, θ) be the geodesic polar coordinate at the point p where θ is a point in the unit sphere Σ. Let $\sqrt{g} r^{n-1} \, dr \, d\theta$ be the volume element associated to this coordinate system. Then Stoke's theorem and $\Delta e(h_i) \geq 0$ shows that

$$\int_{\theta \in \Sigma} \frac{\partial e(h_i)}{\partial r} (\rho, \theta) \sqrt{g} \, d\theta \geq 0 \tag{1.1}$$

when ρ is less than the injectivity radius at p.

Therefore,

$$\frac{\partial}{\partial \rho} \left[\int_{\theta \in \Sigma} e(h_i)(\rho, \theta) \sqrt{g} \, d\theta \right] + \sup_{\theta \in \Sigma} \left| \frac{\partial}{\partial r} \log \sqrt{g}(\rho, \theta) \right| \int_{\theta \in \Sigma} e(h_i)(\rho, \theta) \sqrt{g} \, d\theta \geq 0 \tag{1.2}$$

Integrating this inequality, we obtain

$$e(h_i)(p) \leq \left[\int_{\theta \in \Sigma} e(h_i)(R, \theta) \sqrt{g} \, d\theta \right] \exp \left[R \sup_{\substack{\theta \in \Sigma \\ r \leq R}} \left| \frac{\partial}{\partial r} \log \sqrt{g}(r, \theta) \right| \right] \tag{1.3}$$

where R is less than the injectivity radius at p.

By the mean value theorem, we see that for each R within the injectivity radius at p, there is a number $\bar{R}$ such that

$$\int_{\theta \in \Sigma} e(h_i)(\bar{R}, \theta) \sqrt{g} \, d\theta \leq R^{-n} \int_{R \leq r \leq 2R} e(h_i) \tag{1.4}$$

Therefore, when R is within the injectivity radius at p and M_i covers the ball of radius R,

$$e(h_i)(p) \leq R^{-n} \exp \left[R \sup_{\substack{\theta \in \Sigma \\ r \leq R}} \left| \frac{\partial}{\partial r} \log \sqrt{g} \right| \right] E[h_i \mid M_i] \tag{1.5}$$

In case $c<0$, one can use arguments of [1]. In fact, let R be a small number so that for all point q with distance not greater than nR away from p, the injectivity radius at q is greater than R where $n = \dim M$. Let ϕ be a function defined on the real line so that $\phi(x)=0$ for $|x| \le R^{-n+2}$ and $\phi(x)=x$ for $|x| \ge (R/2)^{-n+2}$. Then for $n \ge 3$, define the function $F(q_1, q_2) = \phi(d(q_1, q_2)^{-n+2})$ on $M \times M$ where $d(q_1, q_2)$ is the distance between q_1 and q_2. (For $n=2$, we replace $d(q_1, q_2)^{-n+2}$ by $\log(q_1, q_2)$.)

As in [1] p. 142, there is a constant A such that

$$e(h_i)(p) \le A \int_{d(p,q)\le R} (F(p,q)+1)e(h_i)(q) \tag{1.6}$$

Iterating this, we obtain

$$e(h_i)(p) \le A^k \int_{d(p,q)\le kR} F_k(p,q)e(h_i)(q) \tag{1.7}$$

where the F_k are defined inductively by $F_1 = F+1$ and

$$F_k(q_1, q_2) = \int_M F_{k-1}(q_1, x)(F(x, q_2)+1) \tag{1.8}$$

When $k > n/2$, $F_k(p, q)$ is bounded for $d(p, q) \le nR$. It follows from this and (1.7) that $e(h_i)(p)$ can be bounded by $E(h_i \mid M_i)$.

Having bounded the energy density, it is then standard (cf. [1]) to estimate the higher derivatives of h_i and hence prove that a subsequence of h_i converges uniformly in a neighborhood of p. By using the diagonal process, we have then proved the claim at the beginning of this section.

2. The Fundamental Group of Complete Manifolds with Non-negative Ricci Curvature

Let M be a complete manifold with non-negative Ricci curvature and N be a compact manifold with non-positive sectional curvature. Let f be a harmonic map from M to N with finite energy. Then we claim that f is a constant map.

By formula (16), p. 123 of [1], we have

$$\Delta e(f) = |\beta(f)|^2 + Q(f^*) \tag{2.1}$$

where $Q(f^*) \ge 0$ under our curvature assumption.

By definition of $\beta(f)$, one can check easily

$$2e(f)\,|\beta(f)|^2 \geq |\nabla e(f)|^2 \tag{2.2}$$

The inequalities (2.1) and (2.2) together show that $\sqrt{e(f)}$ is subharmonic on M.

In [8], we have shown that every non-negative L^2-integrable subharmonic function on a complete Riemannian manifold must be a constant. Applying this to $e(f)$, we conclude that $e(f)$ is a constant.

On the other hand, we have also shown in [8] that when M is a complete non-compact manifold with non-negative Ricci curvature, then the volume of M is infinite (also known to E. Calabi [9]). This forces the constant $e(f)$ to be zero and f to be a constant map. Therefore, we have proved the following

THEOREM 1. *Let M be a complete manifold with non-negative Ricci curvature and N be a compact manifold with non-positive curvature. Let f be any smooth map from M to N with finite energy. Then f is homotopic to constant on each compact set.*

As an application of this theorem, we note the following

COROLLARY. *Let M be a complete manifold with non-negative Ricci curvature. Let D be a compact domain in M with smooth simply connected boundary. Then there is no non-trivial homomorphism from $\pi_1(D)$ into the fundamental group of a compact manifold with non-positive curvature.*

Proof. Suppose $h:\pi_1(D) \to \pi_1(N)$ is a homomorphism where N is a compact manifold with non-positive curvature. Then as N is a $K(\pi, 1)$, there is a smooth map $f:D \to N$ such that $f_* = h$. This map is homotopic to a constant map on ∂D because ∂D is simply connected. Therefore, we can extend f to be a smooth map $\tilde{f}:M \to N$ such that outside a compact set, $\tilde{f}$ is a constant map. Clearly, $\tilde{f}$ has finite energy and we can apply the theorem to see that f is homotopic to a constant map and that h is trivial.

3. Fundamental Group of Stable Immersions

Throughout this section let M^n be a complete non-compact manifold immersed in a manifold $\bar{M}^{n+1}$ having non-negative sectional curvatures. Also, suppose the immersion is stable (i.e., minimal and non-negative second variation of area with respect to compactly supported deformations). Let $e_1, \ldots, e_n, e_{n+1}$ be a local orthonormal frame on $\bar{M}$ such that $e_1, \ldots, e_n$ form an orthonormal frame for M in a neighborhood of a point $x_0 \in M$. Let $\omega_1, \ldots, \omega_{n+1}$ be the dual coframe. Let ω_{ij} be the connection 1-forms of $\bar{M}$ and $\omega_{n+1\,i} = \sum_{j=1}^n h_{ij}\omega_j$ when restricted to M defines the symmetric second fundamental form of M. Since the immersion is minimal, we have $\sum_{i=1}^n h_{ii} = 0$, and the stability inequality is

$$\int_M \left(\sum_{i=1}^n K_{n+1,i,n+1,i} + \sum_{i,j=1}^n h_{ij}^2\right)\varphi^2\, dV_M \leq \int_M |\nabla\varphi|^2\, dV_M$$

where φ is any Lipschitz function with compact support on M, K_{ijkl} is the curvature tensor of $\bar{M}$, and dV_M is the volume element of M (see [10])

LEMMA 1. *M has infinite volume.*

Proof. Let $B_R(x_0)$ be the geodesic ball of radius R in M centered at x_0. Choose φ to be a Lipschitz function with the properties

$$\varphi = \begin{cases} 1 & \text{inside} \quad B_R(x_0) \\ 0 & \text{outside} \quad B_{3R}(x_0) \end{cases} \quad \text{and} \quad |\nabla\varphi| \leq 1/R.$$

The stability inequality implies

$$\int_{B_R(x_0)} \sum_{i,j=1}^n h_{ij}^2\, dV_M \leq \frac{\mathrm{Vol}\,(M)}{R^2}.$$

If $\mathrm{Vol}\,(M) < \infty$, by letting $R \to \infty$ we find that M is totally geodesic. Thus, the sectional curvatures of M are non-negative and by a theorem of [8] we must have $\mathrm{Vol}\,(M) = \infty$, a contradiction.

Let N^k be a compact manifold with non-positive sectional curvature. Let $f: M \to N$ be a harmonic map. Let $x_0 \in M$ and $\bar{e}_1, \ldots, \bar{e}_k$ be an orthonormal frame in a neighborhood of $f(x_0) \in N$. Let $\theta_1, \ldots, \theta_k$ be the dual coframe and $\theta_{\alpha\beta}\, 1 \leq \alpha, \beta \leq k$ the connection forms. Define $f_{\alpha i}\, 1 \leq \alpha \leq k$, $1 \leq i \leq n$ by $f^*\theta_\alpha = \sum_{i=1}^n f_{\alpha i}\omega_i$. Then $e(f) = \sum_{i=1}^n \sum_{\alpha=1}^k f_{\alpha i}^2$. Define $f_{\alpha ij}$ by $df_{\alpha i} + \sum_{\beta=1}^k f_{\beta i} f^*\theta_{\beta\alpha} + \sum_{j=1}^n f_{\alpha j}\omega_{ji} = \sum_{j=1}^n f_{\alpha ij}\omega_j$. Then f is harmonic means $\sum_{i=1}^n f_{\alpha ii} = 0$ for $1 \leq \alpha \leq k$.

LEMMA 2. *If $E(f < \infty$, then f is constant.*

Proof. The stability inequality implies

$$\int_M \left(\sum_{i,j=1}^n h_{ij}^2\right)\varphi^2 \leq \int_M |\nabla\varphi|^2.$$

Replacing φ by $\sqrt{e(f)}\,\varphi$ we obtain

$$\int_M \Big(\sum_{i,j} h_{ij}^2\Big) e(f)\varphi^2 \leq \int_M e(f)\,|\nabla\varphi|^2 + 2\int_M \sqrt{e(f)}\,\varphi\nabla\sqrt{e(f)}\cdot\nabla\varphi$$

$$+\int_M \varphi^2\,|\nabla\sqrt{e(f)}|^2$$

$$=\int_M e(f)\,|\nabla\varphi|^2 - \frac{1}{2}\int_M \varphi^2\,\Delta e(f) + \int_M \varphi^2\,|\nabla\sqrt{e(f)}|^2$$

By the Gauss curvature equation and formula (2.1) using the fact that the sectional curvatures of $\bar{M}$ are non-negative and those of N are non-positive we obtain

$$\Delta e(f) \geq 2\sum_{\alpha,i,j} f_{\alpha ij}^2 - 2\sum_{\alpha,i}\Big(\sum_j h_{ij} f_{\alpha j}\Big)^2$$

$$\geq 2\sum_{\alpha,i,j} f_{\alpha ij}^2 - 2\Big(\sum_{i,j} h_{ij}^2\Big)e(f).$$

Putting this into the above inequality and rearranging, we obtain

$$\int_M \varphi^2\big(\sum f_{\alpha ij}^2 - |\nabla\sqrt{e}|^2\big) \leq \int_M e|\nabla\varphi|^2.$$

Now

$$|\nabla\sqrt{e}|^2 = \frac{\sum_j\big(\sum_{i,\alpha} f_{\alpha i} f_{\alpha ij}\big)^2}{e}$$

and hence

$$\sum f_{\alpha ij}^2 - |\nabla\sqrt{e}|^2 = \frac{1}{2e}\sum_{j,i,k,\alpha,\beta}(f_{\alpha i} f_{\beta kj} - f_{\beta k} f_{\alpha ij})^2$$

$$\geq \frac{1}{2e}\sum_{i,j,\alpha}(f_{\alpha i} f_{\alpha jj} - f_{\alpha j} f_{\alpha ij})^2$$

where we have thrown out terms where $k \neq j$ or $\alpha \neq \beta$. Using the Schwarz inequality we obtain

$$\sum f_{\alpha ij}^2 - |\nabla\sqrt{e}|^2 \geq \frac{1}{2nke}\sum_i \Big(\sum_{j,\alpha} f_{\alpha i}f_{\alpha jj} - \sum_{j,\alpha} f_{\alpha j}f_{\alpha ij}\Big)^2$$

$$= \frac{1}{2nke}\sum_i \Big(\sum_{j,\alpha} f_{\alpha j}f_{\alpha ji}\Big)^2 = \frac{1}{2nk}|\nabla\sqrt{e}|^2$$

where we have used $\sum_j f_{\alpha jj} = 0$ and $f_{\alpha ij} = f_{\alpha ji}$. Therefore, we have $\int \varphi^2 |\nabla\sqrt{e}|^2 \leq 2nk \int e\, |\nabla\varphi|^2$. Choosing φ as in lemma 1 we obtain

$$\int_{B_R(x_0)} |\nabla\sqrt{e}|^2 \leq \frac{2nk}{R^2} E(f).$$

Letting $R \to \infty$ we see that $e \equiv$ constant on M. In light of lemma 1 and the fact that $E(f) < \infty$, we see that $e(f) \equiv 0$. Thus f is constant.

Combining the existence theorem of section 1 with lemma 2 we have the following result.

THEOREM 2. *Let M be a complete non-compact stably immersed hypersurface in a manifold of non-negative curvature and N be a compact manifold with non-positive curvature. Let $f: M \to N$ be a smooth map with $E(f) < \infty$. Then f is homotopic to constant on each compact set.*

As in section 2 we have the following corollary.

COROLLARY. *Let M be as in theorem 2. Let D be a compact domain in M with smooth simply connected boundary. Then there is no non-trivial homomorphism from $\pi_1(D)$ into the fundamental group of a compact manifold with non-positive curvature.*

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