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1. Introduction

For convenience we work in the PL category. A *knot* of S^{n-1} in S^{n+1} is a locally flat submanifold of S^{n+1} homeomorphic to S^{n-1}. The closure of the complement of a regular neighbourhood of a knot is its *exterior*. Two knots are *equivalent* if there is a homeomorphism of S^{n+1} taking one to the other. Equivalent knots therefore have homeomorphic exteriors. As far as the converse is concerned, it is known that for $n \geq 3$ there are at most two inequivalent knots with a given exterior [6], [2], [10], [11], [15]. Recently, Cappell and Shaneson have shown that for $n = 4, 5$, inequivalent knots with homeomorphic exteriors do exist [4]. Their examples are certain knots whose exteriors fibre over S^1 with fibre T^n-open disc, where T^n is the n-dimensional torus (compare [3]). Since this approach uses the generalized Poincaré conjecture, however, in the case $n = 3$ it only yields knots in homotopy 4-spheres.

In the present paper, we use twist-spun knots to prove

THEOREM 1.1. *There exist inequivalent knots $K_1, K_2, \ldots, K_1^*, K_2^*, \ldots$ of S^2 in S^4, such that K_i and K_i^* have homeomorphic exteriors* $(i = 1, 2, \ldots)$.

In the course of the proof, we show that removing a regular neighbourhood of a twist-spun knot in S^4 and sewing it back differently always gives S^4 (Theorem 3.1). In particular, this answers a question of Zeeman [18, p. 494, problem 1], and enables us to give some new counterexamples to the 4-dimensional Smith conjecture [5], [9], [14].

2. Notation etc.

B^n, D^n will both denote n-balls, with centre O, S^n the n-sphere, and R^n Euclidean n-space.

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We will use the notation $K=(S^{n+1}, S^{n-1})$ for a knot of S^{n-1} in S^{n+1}. S^{n-1} then has a regular neighbourhood $S^{n-1}\times D^2\subset S^{n+1}$ (with S^{n-1} corresponding to $S^{n-1}\times O$), and X, the exterior of K, is $S^{n+1}-S^{n-1}\times \operatorname{int} D^2$. In this way ∂X is identified with $S^{n-1}\times S^1$.

So let $K=(S^{n+1}, S^{n-1})$ have exterior X, and let (M, S^{n-1}) be a knot of S^{n-1} in an $(n+1)$-manifold M, whose exterior is homeomorphic to X. Then

$$(M, S^{n-1})\cong(S^{n-1}\times D^2\cup_\gamma X, S^{n-1}\times O),$$

where $\gamma: S^{n-1}\times S^1\to S^{n-1}\times S^1$ is some 'gluing' homeomorphism. Clearly (M, S^{n-1}) depends (up to equivalence, i.e. homeomorphism of pairs) only on the pseudo-isotopy class of γ.

Now it is known [6], [2], [10], [11], [15] that for $n\geq 3$, two homeomorphisms of $S^{n-1}\times S^1$ are pseudo-isotopic if and only if they are homotopic. The group of pseudo-isotopy classes is thus isomorphic to $\mathbf{Z}_2\times\mathbf{Z}_2\times\mathbf{Z}_2$, where the first two factors correspond to orientation-reversal of S^{n-1} and S^1 respectively, and the third is generated by τ, defined by

$$\tau(x, \theta)=(\rho(\theta)(x), \theta),$$

where $\rho(\theta)$ is rotation of S^{n-1} about its polar S^{n-3} through the angle θ.

Since generators of the first two factors extend to homeomorphisms of $(S^{n-1}\times D^2, S^{n-1}\times O)$, it follows that (M, S^{n-1}) is equivalent to either K or

$$K^*=(S^{n-1}\times D^2\cup_\tau X, S^{n-1}\times O).$$

Moreover, K and K^* are equivalent if and only if there is a homeomorphism of X which on ∂X restricts to $\varepsilon\tau$, where the pseudo-isotopy class of ε belongs to the first $\mathbf{Z}_2\times\mathbf{Z}_2$.

We write $\Sigma(K)$ for $S^{n-1}\times D^2\cup_\tau X$; it is a homotopy $(n+1)$-sphere.

The following convention will be useful in §3. Given $\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbf{Z})$, we can define a homeomorphism (orientation-preserving) of $S^1\times S^1$ by

$$(\theta, \phi)\mapsto(a\theta+b\phi, c\theta+d\phi).$$

Regard the first S^1 as ∂D^2, and the second as ∂B^2. Then, for example, homeomorphisms corresponding to matrices of the form $\begin{pmatrix} 1 & b\\ 0 & 1\end{pmatrix}$ extend to $D^2\times$

∂B^2 by defining

$$((r, \theta), \phi) \mapsto ((r, \theta + b\phi), \phi),$$

where (r, θ) denotes polar co-ordinates in D^2. Similarly matrices of the form $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$ define homeomorphisms of $\partial D^2 \times B^2$, those of the form $\begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix}$ homeomorphisms $\partial D^2 \times B^2 \to D^2 \times \partial B^2$, and so on. In §3 we shall frequently be dealing with homeomorphisms of $E \times \partial D^2 \times \partial B^2$, $E \times D^2 \times \partial B^2$, etc., where E is some space, and we shall say that such a homeomorphism corresponds to a matrix $A \in SL_2(\mathbf{Z})$ if it is of the form $id \times h_A$, where id denotes the identity on E, and h_A is the appropriate extension (by polar co-ordinates) of the homeomorphism of $\partial D^2 \times \partial B^2$ given by A.

3. Twist-spun Knots

If K is a knot of S^{n-2} in S^n, and k an integer, then *k-twist-spinning* [18] produces a knot $K^{(k)}$ of S^{n-1} in S^{n+1}. In particular, $K^{(0)}$ is just the classical spin of K [1].

Recalling from §2 the definition of $\Sigma(K)$, the main result of this section is

THEOREM 3.1.[1] *Let K be a knot of S^1 in S^3, and let $K^{(k)}$ be the k-twist-spin of K, where k is any integer. Then $\Sigma(K^{(k)}) \cong S^4$.*

Remarks. (1) Zeeman was motivated in [18] by Mazur's candidate for a counterexample to the 4-dimensional Poincaré conjecture [12], and in particular showed that this candidate was S^4 after all. Now there is a choice of tubular neighbourhood involved in Mazur's construction (see [12] and [18, p. 473]), and Zeeman [18, Question 4, p. 493] raises the question of what happens if a different choice is made. Theorem 3.1 shows that the result is still S^4.

(2) Theorem 3.1 will follow from Propositions 3.2 and 3.3, whose proofs are valid in the smooth category as well as the *PL*, and apply explicitly (as this involves no extra effort) to knots K of S^{n-2} in S^n for all n. It therefore follows that in fact $\Sigma(K^{(k)})$ is always diffeomorphic to S^{n+1}.

(3) Theorem 3.1 enables us to sharpen some of the results of [9] (and hence, indirectly, [5]) on the 4-dimensional Smith conjecture. Shifting dimension by one for consistency with the notation in [9], let K be a knot of S^{n-3} in S^{n-1} and $K^{(k)}$

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[1] Added in proof. P. Pao has since given an alternative proof of this result, based on an examination of the natural S^1-action on $\Sigma(K^{(k)})$ (to appear).

its k-twist-spin. Then the argument in [9, §4], with an appropriate modification of [9, §4, Remark (2)], implies that $K^{(k)*}$ admits a strong $\mathbf{Z}_m$-action which embeds in an S^1-action (see [9] for definitions) for all m such that $(m, k)=1$. (This observation, incidentally, gives a proof of [9, Theorem 3] which avoids spinning.) Since we now know that $K^{(k)*}$ is a knot in S^n even when $n=4$, the restriction $n \geq 5$ in [9, Theorem 3] can be replaced by $n \geq 4$. In particular, we have examples of knots in S^4 which are the fixed-point sets of particularly nice transformations of even period (see also [14] and [9, §3]).

(4) It is worth noting that the proof of Theorem 3.1 does not use Zeeman's fibration theorem on twist-spun knots [18, Main Theorem] (although this will be used crucially in §4).

We prove Theorem 3.1 by induction on k:

PROPOSITION 3.2 $\Sigma(K^{(k)}) \cong \Sigma(K^{(k+1)})$.

Proof. Let $K=(S^n S^{n-2})$ be a knot, $n \geq 3$, with tubular neighbourhood $S^{n-2} \times D^2 \subset S^n$. Write S^{n-2} as the union of its two hemispheres $E_+ \cup E_-$, identified along their common boundary $\partial E_{\pm}$. Consider the ball-pair associated with K

$$(B^n, E_+) = (S^n, S^{n-2}) - (\operatorname{int} E_- \times \operatorname{int} D^2, \operatorname{int} E_- \times O).$$

Then

$$\partial(B^n, E_+) = \partial(E_- \times D^2, E_- \times O).$$

Given $k \in \mathbf{Z}$, let

$$f: \partial(E_- \times D^2) \times \partial B^2 \to \partial(E_- \times D^2) \times \partial B^2$$

be the restriction of the homeomorphism of $E_- \times D^2 \times \partial B^2$ corresponding (as in §2) to the matrix $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$. Note that f restricts to the identity on $\partial E_{\pm} \times O \times \partial B^2$. The k-twist-spin of K is then (see [18, p. 485]) defined by

$$K^{(k)} = (S^{n+1}, S^{n-1}) = \partial(B^n, E_+) \times B^2 \cup_f (B^n, E_+) \times \partial B^2.$$

$$S^{n-1} = \partial E_{\pm} \times B^2 \cup E_+ \times \partial B^2$$

has tubular neighbourhood

$$N = \partial E_\pm \times D^2 \times B^2 \cup_{f'} E_+ \times D^2 \times \partial B^2$$

where $f' = f \mid \partial E_\pm \times D^2 \times \partial B^2$. Clearly f' extends to a homeomorphism e of $E_+ \times D^2 \times \partial B^2$, namely that corresponding to $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$. Then $id \cup e$ induces an explicit homeomorphism from

$$S^{n-1} \times D^2 = \partial E_\pm \times D^2 \times B^2 \cup E_+ \times D^2 \times \partial B^2$$

to N.

Let X be the exterior of K, so

$$X = S^n - S^{n-2} \times \operatorname{int} D^2 = B^n - E_+ \times \operatorname{int} D^2,$$

and

$$\partial X = (E_+ \cup E_-) \times \partial D^2.$$

Then the exterior of $K^{(k)}$, which is $S^{n+1} - \operatorname{int} N$, is given by

$$X^{(k)} = E_- \times \partial D^2 \times B^2 \cup_g X \times \partial B^2$$

where $g = f \mid E_- \times \partial D^2 \times \partial B^2$. Also

$$\partial X^{(k)} = \partial N = \partial E_\pm \times \partial D^2 \times B^2 \cup_{g'} E_+ \times \partial D^2 \times \partial B^2$$

where $g' = g \mid \partial E_\pm \times \partial D^2 \times \partial B^2$. The restriction of $id \cup e$ to $\partial E_\pm \times \partial D^2 \times B^2 \cup E_+ \times \partial D^2 \times \partial B^2$ defines the appropriate homeomorphism from $S^{n-1} \times \partial D^2$ to $\partial X^{(k)}$.

Now the homeomorphism τ (see §2) of $S^{n-1} \times \partial D^2 = \partial E_\pm \times \partial D^2 \times B^2 \cup E_+ \times \partial D^2 \times \partial B^2$ may be taken to be $\lambda \cup \nu$, where λ, ν both correspond to the matrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Hence

$$\Sigma(K^{(k)}) = S^{n-1} \times D^2 \cup_\tau X^{(k)}$$

can be expressed as the union of four pieces

$$(A \cup B) \cup_{\lambda \cup \lambda'} (C \cup_g D) \tag{1}$$

where

$$A = \partial E_{\pm} \times D^2 \times B^2,$$
$$B = E_{+} \times D^2 \times \partial B^2,$$
$$C = E_{-} \times \partial D^2 \times B^2,$$
$$D = X \times \partial B^2,$$

and

$$\lambda' = (e \mid E_{+} \times \partial D^2 \times \partial B^2)\nu.$$

We now come to the main step in the proof, which exploits the symmetry of this situation, as follows. Rearranging the decomposition (1), let

$$P = A \cup_{\lambda} C, \quad \text{and} \quad Q = B \cup_{\lambda'} D. \tag{2}$$

First look at Q. Let

$$u : E_{+} \times \partial D^2 \times B^2 \to E_{+} \times D^2 \times \partial B^2$$

be the homeomorphism given by $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$, and let $u' = u \mid E_{+} \times \partial D^2 \times \partial B^2$. Write h for the composition $\lambda' u'$. Then $u^{-1} \cup id$ induces a homeomorphism from Q to

$$E_{+} \times \partial D^2 \times B^2 \cup_h X \times \partial B^2$$

and h corresponds to the matrix $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & k+1 \\ 0 & 1 \end{pmatrix}$ (u was chosen so as to achieve this).

Now consider P. Let

$$v : E_{-} \times \partial D^2 \times B^2 \to E_{-} \times D^2 \times \partial B^2$$
$$w : \partial E_{\pm} \times D^2 \times B^2 \to \partial E_{\pm} \times D^2 \times B^2$$

be the homeomorphisms given by $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ respectively. Let $v' = v \mid \partial E_{\pm} \times \partial D^2 \times B^2$ and $w' = w \mid \partial E_{\pm} \times D^2 \times \partial B^2$. Since $v' \lambda w'$ is just the identity,

$w^{-1} \cup v$ induces a homeomorphism from P to

$$\partial E_{\pm} \times D^2 \times B^2 \cup E_{-} \times D^2 \times \partial B^2.$$

Writing

$$B^* = E_{-} \times D^2 \times \partial B^2$$

and

$$C^* = E_{+} \times \partial D^2 \times B^2$$

(compare B, C above), we have thus established homeomorphisms

$$P \cong A \cup B^*, \quad \text{and} \quad Q \cong C^* \cup_h D. \tag{3}$$

Referring back to (1) and (2), it follows that $\Sigma(K^{(k)})$ can be expressed as

$$(A \cup B^*) \cup_{\mu \cup \mu'} (C^* \cup_h D) \tag{4}$$

where μ, μ' are certain homeomorphisms of $\partial E_{\pm} \times \partial D^2 \times B^2$ and $E_{-} \times \partial D^2 \times \partial B^2$ respectively. Following through the homeomorphisms we have applied to the various pieces of (1), we see that μ corresponds to $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$, and μ' (as it must do for consistency) to the product $\begin{pmatrix} 1 & k+1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$.

We now note that the decomposition (4) differs from (1) only in that E_{+} and E_{-} have been interchanged, λ (corresponding to $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$) has been replaced by μ (corresponding to $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$), and, elsewhere, k has been replaced by $k+1$. In the definition of twist-spinning, however, we could equally well have used the ball-pair

$$(B^n, E_{-}) = (S^n, S^{n-2}) - (\text{int } E_{+} \times \text{int } D^2, \text{int } E_{+} \times O)$$

instead of (B^n, E_{+}), since the two ball-pairs $(E_{\pm} \times D^2, E_{\pm} \times O) \subset (S^n, S^{n-2})$ are ambient isotopic. Hence (4) is also a decomposition of

$$S^{n-1} \times D^2 \cup_{\tau^{-1}} X^{(k+1)}$$

where $X^{(k+1)}$ is the exterior of $K^{(k+1)}$. But τ and τ^{-1} are isotopic (since $\pi_1(SO_n) \cong \mathbf{Z}_2$), and hence

$$\Sigma(K^{(k)}) \cong S^{n-1} \times D^2 \cup_\tau X^{(k+1)} = \Sigma(K^{(k+1)})$$

as required.

To complete the proof of Theorem 3.1, it remains only to show that the induction starts. One way of doing this would be to start at $k=1$, using Zeeman's result [18, Corollary 2] that $K^{(1)}$ is always the unknot. Zeeman's theorem can be avoided, however, by starting at $k=0$. For $K^{(0)}$ is just the spin of K, and therefore by [6, §22] (see also [3]), its exterior $X^{(0)}$ admits a homeomorphism which restricts to τ on $\partial X^{(0)}$. Hence $\Sigma(K^{(0)}) \cong S^4$ (indeed $K^{(0)*}$ is equivalent to $K^{(0)}$). Since a proof can easily be given in terms of our present notation, we include one for completeness.

PROPOSITION 3.3 (Gluck). *$\tau : \partial X^{(0)} \to \partial X^{(0)}$ extends to a homeomorphism of $X^{(0)}$.*

Proof. (Compare [3, Theorem 8].) Using the notation introduced in the proof of Proposition 3.2,

$$X^{(0)} = E_- \times \partial D^2 \times B^2 \cup X \times \partial B^2$$

(identified along $E_- \times \partial D^2 \times \partial B^2$), and τ is given by $\lambda \cup \nu$. Since λ extends in the obvious way to $E_- \times \partial D^2 \times B^2$, it remains only to extend ν compatibly to $X \times \partial B^2$. But projection $S^{n-2} \times \partial D^2 \to \partial D^2$ extends to $p : X \to \partial D^2$ (see [18, Lemma 2]; if $n=3$, then the trivialization of the tubular neighbourhood $S^{n-2} \times D^2$ has to be chosen correctly). The homeomorphism of $X \times \partial B^2$ defined by

$$(x, \phi) \mapsto (x, p(x) + \phi)$$

is then an extension of ν, which agrees with the extension of λ on $E_- \times \partial D^2 \times \partial B^2$.

4. The Main Theorem

Let M be a (closed) n-manifold ($n \geq 3$), $B^n \subset M$ an n-ball with centre O, and let $M_0 = M - \operatorname{int} B^n$. Consider $M \times S^1$, taking $(O, 0)$ as base-point, and identify M,

S^1 with $M\times 0$, $O\times S^1$ respectively. Let ε, τ be as in §2, and write $\bar{\varepsilon}$, $\bar{\tau}$ for the homeomorphisms of $B^n\times S^1$ which naturally extend ε, τ.

In the following proposition, f_* denotes the homomorphism induced on $\pi_1(M\times S^1)=\pi_1(M)\times\pi_1(S^1)$. The basic philosophy behind its proof is the same as that of [4].

PROPOSITION 4.1. *If the universal cover of M is R^n, then there is no map $f: M\times S^1\to M\times S^1$ such that*

(i) $f(B^n\times S^1)\subset B^n\times S^1$ *and* $f(M_0\times S^1)\subset M_0\times S^1$,
(ii) $f\mid B^n\times S^1=\bar{\varepsilon}\bar{\tau}$
(iii) f_* *is an isomorphism,*
(iv) $f_*(\pi_1(M))\subset\pi_1(M)$.

Proof. By (ii), (iii) and (iv), f_* induces isomorphisms $f_*\mid\pi_1(S^1):\pi_1(S^1)\to\pi_1(S^1)$ and $f_*\mid\pi_1(M):\pi_1(M)\to\pi_1(M)$. Hence pulling back, via f, the cover $R^n\times S^1$ of $M\times S^1$ associated with the subgroup $\pi_1(S^1)$, we get a proper lift $\tilde{f}: R^n\times S^1\to R^n\times S^1$ of f, such that, if $B^n_+\subset R^n$ is an n-ball covering B^n, $\tilde{f}\mid B^n_+\times S^1=\bar{\varepsilon}\bar{\tau}$.

Let $p=M\times S^1\to M$ be projection, and let $g: M\times S^1\to M\times S^1$ be $h\times id$, where $h: M\to M$ is defined by $h(x)=pf(x,0)$. Then $\tilde{g}$ similarly lifts to a proper map $\tilde{g}=\tilde{h}\times id: R^n\times S^1\to R^n\times S^1$, such that $\tilde{g}\mid B^n_+\times S^1$ is just $\bar{\varepsilon}$.

By composing f (if necessary) with $id\times$(orientation-reversal of S^1), we may assume that $\bar{\varepsilon}$ involves at most an orientation-reversal of B^n. Then $f_*\mid\pi_1(S^1)$ is the identity, and so $f_*=g_*:\pi_1(M\times S^1)\to\pi_1(M\times S^1)$. Since $M\times S^1$ is a $K(\pi,1)$, a standard obstruction theory argument shows that $f\simeq g\operatorname{rel}(O,0)$. Hence $\tilde{f}\overset{p}{\simeq}\tilde{g}$, where $\overset{p}{\simeq}$ denotes proper homotopy. Note also that (by (i)) $\tilde{f}$ and $\tilde{g}$ take $(R^n-\operatorname{int}B^n_+)\times S^1$ to itself.

Now any proper map $\alpha: R^n\times S^1\to R^n\times S^1$ induces a map $\alpha^c: S^n\times S^1\to S^n\times S^1$ by taking the one-point compactification at each level of the map $R^n\times S^1\to R^n\times S^1$ given by

$$(x,\theta)\mapsto(p'\alpha(x,\theta),\theta),$$

where p' is projection $R^n\times S^1\to R^n$. Moreover $\alpha\overset{p}{\simeq}\beta$ implies $\alpha^c\simeq\beta^c$.

Doing this with $\tilde{f}$ and $\tilde{g}$, we get $\tilde{f}^c$, $\tilde{g}^c: S^n\times S^1\to S^n\times S^1$ such that if we write S^n as the union of its two hemispheres $B^n_+\cup B^n_-$, then $\tilde{f}^c$ and $\tilde{g}^c$ take $B^n_\pm\times S^1$ to $B^n_\pm\times S^1$, $\tilde{f}^c\mid B^n_+\times S^1=\bar{\varepsilon}\bar{\tau}$, and $\tilde{g}^c\mid B^n_+\times S^1=\bar{\varepsilon}$. By coning at each level $B^n_-\times\theta$, it is easy to define homotopies (rel $B^n_+\times S^1$) which show that $\tilde{f}^c\simeq\varepsilon'\tau'$ (where ε', τ' correspond to ε, τ, but one dimension higher), whereas $\tilde{g}^c\simeq\varepsilon'$. This contradicts $\tilde{f}\overset{p}{\simeq}\tilde{g}$, and so completes the proof.

PROPOSITION 4.2. *Let K be a knot of S^{n-2} in S^n, and k an odd integer. If the universal cover of the k-fold branched cyclic cover of K is R^n, then the knots $K^{(k)}$, $K^{(k)*}$ are inequivalent.*

Proof. Recalling §2, the knots $K^{(k)}$, $K^{(k)*}$ will be equivalent if and only if some $\varepsilon\tau: \partial X^{(k)} \to \partial X^{(k)}$ extends to a homeomorphism $\phi: X^{(k)} \to X^{(k)}$. If $p_m: X_m^{(k)} \to X^{(k)}$ is the m-fold cyclic cover of $X^{(k)}$, then any map $X^{(k)} \to X^{(k)}$ lifts to a map $X_m^{(k)} \to X_m^{(k)}$. Hence if $K^{(k)}$ and $K^{(k)*}$ are equivalent, we have a homeomorphism ϕ_m of $X_m^{(k)}$ (the lift of ϕ) such that, identifying $\partial X_m^{(k)}$ with $S^{n-1} \times S^1$ in the obvious way, $\phi_m \mid \partial X_m^{(k)} = \varepsilon\tau^m$.

Let M be the k-fold branched cyclic cover of K, $B^n \subset M$ an equivariant n-ball, $M_0 = M - \operatorname{int} B^n$, and $h: M_0 \to M_0$ the restriction of the canonical covering transformation of M. Then the Main Theorem of [18] implies that $X^{(k)} \cong M_0 \times I/h$, that is, $M_0 \times I$ with $M_0 \times 0$ and $M_0 \times 1$ identified via h. Since $h^k = id$, there is therefore a homeomorphism $\psi: X_k^{(k)} \to M_0 \times S^1$. Taking $m = k$ in the previous paragraph then gives a homeomorphism, $\psi\phi_k\psi^{-1}$, of $M_0 \times S^1$, which restricts to $\psi'(\varepsilon\tau^k)\psi'^{-1}$ on $\partial M_0 \times S^1$, where $\psi' = \psi \mid \partial X_k^{(k)}$. If k is odd, $\psi'(\varepsilon\tau^k)\psi'^{-1}$ is pseudo-isotopic to $\varepsilon\tau$ (see §2). Hence we get a homeomorphism f_0 of $M_0 \times S^1$ such that $f_0 \mid \partial M_0 \times S^1 = \varepsilon\tau$. Extending f_0 to $M \times S^1$ finally gives a map f which clearly satisfies all the hypotheses of Proposition 4.1, except possibly (iv). With the natural choice for ψ, however, $(p_k\psi^{-1})_*$ takes $\pi_1(M_0) \subset \pi_1(M_0 \times S^1)$ isomorphically onto the (fibre) subgroup $\pi_1(M_0) \subset \pi_1(X^{(k)})$, which (since $X^{(k)}$ is a knot exterior) is precisely the commutator subgroup of $\pi_1(X^{(k)})$. Since ϕ_k is the lift of a map $X^{(k)} \to X^{(k)}$, it follows that $(\psi\phi_k\psi^{-1})_*(\pi_1(M_0)) \subset \pi_1(M_0)$, and so $f_*(\pi_1(M)) \subset \pi_1(M)$.

Since the universal cover of M is R^n by hypothesis, this contradicts Proposition 4.1. Hence $K^{(k)}$ and $K^{(k)*}$ are inequivalent.

Proof of Theorem 1.1. After Theorem 3.1 and Proposition 4.2, it suffices to find infinitely many knots of the form $K^{(k)}$, with k odd and K a knot of S^1 in S^3 whose k-fold branched cyclic cover has universal cover R^3.

An easy class of knots to handle are the twist-spun torus knots. So let $K_{p,q}$ be the torus knot of type p, q, with $p, q > 1$ and coprime, and let $M_{p,q}^{(k)}$ be its k-fold branched cyclic cover, $k > 1$. Then $M_{p,q}^{(k)}$ is a Seifert fibre space, whose invariants can be computed, using [13, §9]. In particular, it may be readily shown that the universal cover of $M_{p,q}^{(k)}$ is R^3 except in the cases

$$\begin{aligned}(k, \{p, q\}) = (2, \{2, q\}),\ &(2, \{3, 4\}),\ (2, \{3, 5\}),\\ &(3, \{2, 3\}),\ (3, \{2, 5\}),\ (4, \{2, 3\}),\\ &\text{and}\ \ (5, \{2, 3\}).\end{aligned}$$

(These have S^3 as universal cover.)

Thus there are infinitely many suitable $K_{p,q}^{(k)}$, distinguished (for example) by their commutator subgroups $\pi_1(M_{p,q}^{(k)})$.

Remarks. (1) One can use other knots besides the torus knots. It can be shown, for example, that the k-fold branched cyclic cover ($k>1$) of any doubled (non-trivial) knot [17] is always irreducible and sufficiently large (no doubt ad hoc arguments could be found to establish this for other classes of knots), and hence by [16, Theorem 8.1] has universal cover R^3. The k-twist-spins of these (with k odd and >1) will therefore also do.

(2) We can summarize our present knowledge about the status of the twist-spun torus knots $K_{p,q}^{(k)}$, with regard to the property of being determined by their exterior, as follows:

$K_{p,q}^{(k)}$ is determined by its exterior if $k=0$ ([6], [3], Proposition 3.3), $k=1$ ([18, Corollary 2]), or $k=2$ and $p(\text{say})=2$ (for $M_{2,q}^{(2)}$ is the lens space $L(q, 1)$, which can be spun in the sense of [6, §17]).

$K_{p,q}^{(k)}$ is not determined by its exterior if k is odd and >1, and $(k, \{p, q\}) \neq (3, \{2, 3\}), (3, \{2, 5\})$, or $(5, \{2, 3\})$.

We do not know what happens in the other cases. We venture to suggest, however, that a refinement of the present argument, together with a good understanding of the action of the fundamental group of $M_{p,q}^{(k)}$ on its universal cover, might enable one to prove that $K_{p,q}^{(k)}$ is also not determined by its exterior if k is even (provided it is not one of the exceptions listed in the proof of Theorem 1.1 above).[1]

The remaining cases $K_{3,4}^{(2)}$, $K_{2,3}^{(3)}$, $K_{2,3}^{(4)}$, $K_{3,5}^{(2)}$, $K_{2,5}^{(3)}$, $K_{2,3}^{(5)}$ pose problems of a different kind. (We might mention in passing that the last three form a rather interesting triple. Their complements fibre over S^1 with fibre punctured dodecadedral space [18], each has group $G\times\mathbf{Z}$, where G is the binary dodecahedral group of order 120 [18], [7], but no two complements are homotopy equivalent [8].) Since $K_{2,3}^{(5)}$ has a certain historical significance [18], we recall specifically:

QUESTION (Zeeman [18, p. 494, problem 2]). *Is the 5-twist-spun trefoil determined by its exterior?*

(3) The examples of Cappell and Shaneson [4] have exteriors which fibre over S^1 with fibre $T_0^n = n$-torus–open disc, and their proof uses special properties of this fibre. The proofs of Propositions 4.1 and 4.2, however, show that the situation is simplified if the bundles have finite (odd order) bundle group. (It is not

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[1] Added in proof. On the contrary, R. A. Litherland has shown (unpublished) that for any $K=(S^n, S^{n-2})$, $n\geq 3$, $K^{(2)}$ and $K^{(2)*}$ are always equivalent.

hard to show that this can never be achieved with T_0^n as fibre, if $n \geq 3$.) In view of Proposition 4.2, and the apparent difficulty in showing that suitable examples as in [4] exist for all n, we therefore conclude with:

QUESTION. *Can a branched cyclic cover of a knot of S^{n-2} in S^n ever be a $K(\pi, 1)$, if $n \geq 4$?*

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