# G Maps and the Projective Class Group

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# G Maps and the Projective Class Group

TED PETRIE[1]

## 0. Introduction and Motivation

Let $G$ be a compact Lie group and $f: X \to Y$ be a $G$ normal map (see §1) between smooth closed $G$ manifolds $X$ and $Y$. We are interested in the relation between the homological dimension over $H_*(G, R)$ of $K_*(f, R) = \ker(H_*(X, R) \to H_*(Y, R))$ and Smith theory. The latter states that if $f$ is a $G$ map between two $G$ spaces (not necessarily manifolds) which induces an isomorphism in mod $p$ homology, then for each $p$ subgroup $K$ of $G$, the fixed point mapping $f^K$ also induces an isomorphism in mod $p$ homology.

To study this relationship we introduce an invariant $\chi(f) \in \tilde{K}_0(Z(G/G_0))$ (the reduced projective class group of the group ring of $G/G_0$) for a $G$ map $f: X \to Y$ which satisfies the conclusions of Smith theory for each $p$ subgroup $K$ of $G$. Here $X$ and $Y$ need not be manifolds.

We expect that $\chi(f)$ will be a useful tool in other areas of $G$ homotopy theory. Since our application is in $G$ normal cobordism theory, we emphasize the relationship mentioned in the first paragraph.

In order to motivate the ideas, let $X$ and $Y$ be smooth closed oriented $G$ manifolds. The singular set of $X$ written ${}^sX$ is the set of points of $X$ whose isotropy groups are not principle. If $G$ acts freely on $X$, then ${}^sX = \phi$ and $X/G$ is a manifold of dimension $m - g$ if $\dim X = m$ and $\dim G = g$.

The following results serve as a starting point for our study.

THEOREM 0.1. (Folklore) *If $G$ is connected and acts freely on $Y$ and $K_i(f) = 0$ for $i < \lambda = [(m-g)/2]$ and $m - g$ is even, then $K_*(f) = H_*(G) \otimes K_\lambda(f)$ as an $H_*(G)$ module and $K_\lambda(f)$ is free over $Z$.*

THEOREM 0.2 [5] and [12]. *If $G$ is finite, so $H_*(G) = Z(G)$, and acts freely on $Y$ with $K_i(f) = 0$ for $i < \lambda$ and $m$ is even, then $K_i(f) = 0$ for $i \neq \lambda$ and $K_\lambda(f)$ is $Z(G)$ projective and zero in $\tilde{K}_0(Z(G))$. If $m$ is odd and $K_i(f) = 0$ for $i < \lambda$ and $K_\lambda(f)$ is a $Z$ torsion module, then $K_\lambda(f)$ has homological dimension $\leq 1$ over $Z(G)$ and gives zero in $\tilde{K}_0(Z(G))$.*

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Observe that the condition that $G$ act freely on $Y$ implies ${}^sY=\phi$ a very restrictive condition; however, examples show that some restrictions on ${}^sX$ and ${}^sY$ are necessary for conclusions like those of (0.1) and (0.2). The conclusions of Smith theory are restrictions on ${}^sX$ and ${}^sY$ and together with the assumption that $K_i(f, Z)=0$ for $i<\lambda$ are just the conditions necessary to establish the analog (6.1) of 0.1 and 0.2. Of course some condition on ${}^sX$ e.g. $\dim {}^sX/G<\frac{1}{2}\dim X/G$ is necessary to achieve $K_i(f, Z)=0$ for $i<\lambda$. Not only do the singular sets appear implicitly in the definition of $\chi(f)$ (5.2), but also in its calculation (5.4) where $\chi(f)=\chi({}^sf)$.

The relation between $\chi(f)$, $K_*(f)$ and Smith theory is (6.1) which under the conditions there gives $\chi(f)=\pm[K_\lambda(f, Z)^*]$. One of the interesting consequences of this is that $\chi(f)$ (and so $K_\lambda(f, Z)$) depends not only on the $p$ subgroups of $G$ but on all subgroups (§9). This is certainly a new feature in $G$ homotopy theory.

This paper is organized as follows: The first four sections are technical. In §5 we define $\chi(f)$. A key ingredient here is a paper of Rim [9]. In §6 we give the main result, the structure of $K_*(f, Z)$ as an $H_*(G)$ module. In §7 we give a very brief outline of the application of $\chi(f)$ to the $G$ normal cobordism problem. In §8 we discuss the Swan homomorphism $\sigma_G: Z_n^* \to \tilde{K}_0(Z(G))$, relate it to $\chi(f)$ and prove geometrically a theorem of [11]. In §9 we give examples where $\chi(f)\neq 0$ and in §10 we give an application to equivariant homotopy groups of spheres.

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## 1. *Notation*

Throughout we consider only compact Lie Groups. Let $G$ be such a group and g its dimension. Its connected component is denoted by $G_0$, its maximal torus by $T$ and $N$ is the normalizer of $T$. If $p$ is a prime, $G_p$ is the inverse image in $G$ of the Sylow $p$ subgroup $(G/G_0)_p$ of $G/G_0$.

$$\mathcal{H}(N_p)=\{H\subset N_p\, H\neq 1, \quad H/H_0 \text{ is a } p \text{ group}\}. \tag{1.1}$$

The sets $\mathcal{H}(N_p)$ play a central role and have two important properties

(i) If $G$ is finite or abelian and $H$ and $K$ are in $\mathcal{H}(N_p)$, so is $H\cdot K$.

(ii) If $L \subset N_p$, it has a finite normal subgroup $F \subset L \cap N_0$ with $F_p = 1$, $L/F \cong L_p$ and if $N(L)$ denotes the normalizer of $L$, $N(L) \subset N(L_p) \cdot L$ (by Sylow's theorem on the conjugacy of Sylow subgroups). Note $N(L)_p \subset N(L_p)_p$. These normalizers are taken in $N_p$.

If $X$ and $Y$ are $G$ spaces and $f: X \to Y$ is a $G$ map, $M_f$ is the mapping cone of $f$. It is a $G$ space with a canonical fixed point $q \in (M_f)^G = M_f^G$ corresponding to the point obtained by identifying $X$ to a point. Here $f^G: X^G \to Y^G$ is the induced map of the fixed point sets. The equality

$$(M_f)^G = M_f^G \tag{1.3}$$

is important and includes the convention that $M_h = \text{point}$ if $h$ is a map of the empty set. The isotropy group of a point $x \in X$ is denoted by $G_x$ and the singular set of $X$ denoted by ${}^sX$ is defined as

$${}^sX = \{x \in X \mid G_x \neq \text{principal isotropy group}\}. \tag{1.4}$$

Let ${}^sf: {}^sX \to {}^sY$ denote the restriction of $f$ to ${}^sX$. Then

$${}^s(M_f) = M_{{}^sf}. \tag{1.5}$$

Suppose that $E$ is a contractible $G$ space on which $G$ acts freely so the orbit space $E/G$ is the classifying space $B_G$ of $G$. Let $C_*^G(X)$ denote the chain complex of $X \times_G E$. If $M$ is a module over the group ring $\Lambda = Z(G/G_0)$, we write $H_*^G(X, M)$ and $H_G^*(X, M)$ for the homology of the chain complexes $C_*^{G_0}(X) \otimes_\Lambda M$ and $\text{Hom}_\Lambda (C_*^{G_0}(X), M)$. In particular

$$H_G^*(X, \Lambda) = H_{G_0}^*(X, Z), \qquad H_G^*(X, Z) = H^*(X \times_G E, Z). \tag{1.6}$$

If $A$ is an algebra over $\Lambda$ and $M$ is an $A$ module, then $H_G^*(X, M)$ is an $H_G^*(X, A)$ module. Set $\tilde{H}_G(X, A) = \ker (H_G^*(X, A) \to H^*(X, A)$.

When $G$ is a finite group, $\tilde{K}_0(\Lambda)$ is the reduced projective class group of $\Lambda$. That is the Grothendieck group of $\Lambda$ modules of finite homological dimension modulo the subgroup generated by free modules. The involution of $\tilde{K}_0(\Lambda)$ defined by $M \to \text{Hom}_Z (M, Z) = M^*$ is denoted by $*$.

In what follows, all manifolds are smooth and *oriented* and all $G$ spaces have only a finite number of conjugacy classes of isotropy subgroups. Let $X$ and $Y$ be smooth closed $G$ manifolds of dimension $m$.

DEFINITION

A $G$ normal map $f: X \to Y$ consists of a $G$ map $f$ whose degree is 1 together with a specific $G$ bundle map $F: \nu_X \to \xi$ covering $f$ from the stable $G$ normal bundle $\nu_X$ of a $G$ imbedding of $X$ in a real $G$ module to some $G$ vector bundle $\xi$ over $Y$. Briefly this is denoted by $(X, f)$. Note $F$ defines an isomorphism $\nu_X \cong f^*\xi$.

The definition of a $G$ normal cobordism between two $G$ normal maps $(X_i, f_i)$ $i = 0, 1$ ($Y$ is fixed) is straightforward. This generalizes the definition of [4] where $G = 1$.

*The $G$ normal cobordism problem:* Given a $G$ normal map $(X, f)$ to $Y$. When is $(X, f)$ $G$ normally cobordant to $(X', f')$ with $f'$ a homotopy equivalence?

Define $K_*(f, R) = \ker(H_*(X, R) \to H_*(Y, R))$, $K^*(f, R) = \operatorname{coker}(H^*(Y, R) \to H^*(X, R))$. These groups satisfy duality $K^i(f, R) \cong K_{m-i}(f, R)$ and a universal coefficient theorem $K_i(f, R) = K_i(f, Z) \otimes_Z R \oplus \operatorname{Tor}(K_{i-1}(f, Z), R)$ and similarly for $K^*(f, R)$. When $R$ is $Z$, we abbreviate $K_*(f, Z)$ and $K^*(f, Z)$ by $K_*(f)$ and $K^*(f)$. For a $G$ normal map $(X, f)$ to $Y$ we have

$$K^i(f, R) = H^{i+1}(M_f, q, R) \quad \text{and} \quad K_i(f, R) = H_{i+1}(M_f, q, R). \tag{1.9}$$

## 2. Behavior of $H_G(X, M)$ for Subgroups

LEMMA 2.1. *Suppose $X$ is an $N$ space and $M$ is a $Z_p(N/N_0)$ module. Then $H^*_N(X, M) \to H^*_{N_p}(X, M)$ is a monomorphism.*

*Proof.* The composition of restriction $H^*_N(X, M) \to H^*_{N_p}(X, M)$ and transfer $H^*_{N_p}(X, M) \to H^*_N(X, M)$ is multiplication by the index of $N_p$ in $N$.

LEMMA 2.2. *Let $A$ be a $\Lambda$ algebra on which $G/G_0$ acts as the identity. Then $H^*_G(X, A)$ is a subalgebra of $H^*_N(X, A)$.*

*Proof.* This follows from [2] applied to the fibration $G/N \to X \times_N E \to X \times_G E$. There is a homomorphism $t: H^*_N(X, A) \to H^*_G(X, A)$ with $\pi^* t(x) = \chi(G/N) \cdot x$ for $x \in H^*_G(X, A)$. Since the Euler number of $G/N$ is 1, the result follows.

We need two results about finite generation over $H^*_G(q, Z_p)$.

LEMMA 2.3. *Suppose $G$ is connected and $X$ is a $G$ space whose total $Z_p$ cohomology is finite dimensional over $Z_p$. Then $H_G^*(X, Z_p)$ is a finitely generated $H_G^*(q, Z_p)$ module.*

*Proof.* $H_G^*(q, Z_p)$ is Noetherian and there is a spectral sequence of $H_G^*(q, Z_p)$ algebras $E_2 = H_G^*(q, Z_p) \otimes_{Z_p} H^*(X, Z_p) \Rightarrow H_G^*(X, Z_p)$.

Since $E_2$ is finitely generated, the result follows.

LEMMA 2.4. *Suppose $G$ is a finite $p$ group and $M$ is a finitely generated $Z_p(G)$ module. Then $H_G^*(q, M)$ is a finitely generated $H_G^*(q, Z_p)$ module.*

*Proof.* Let $I$ be the kernel of the augmentation $Z_p(G) \to Z_p$. Then $I$ is nilpotent, say $I^n = 0$ [1]. Filter $M$ as $M \supset IM \supset \cdots \supset I^nM = 0$. We have an exact triangle

$$\begin{array}{ccc} H_G^*(q, I^{k+1}M) & \longrightarrow & H_G^*(q, I^kM) \\ \nwarrow & & \swarrow \\ & H_G^*(q, I^kM/I^{k+1}M) & \end{array}$$

and each $I^kM/I^{k+1}M$ is a $Z_p$ vector space with trivial action of $G$. The result follows by induction.

LEMMA 2.5. *Suppose $M$ is a (graded) finitely generated $H_G^*(q, Z_p)$ module and for each multiplicative subset $s \in \tilde{H}_G(q, Z_p)$, $s^{-1}M = 0$. Then $M$ is zero for large $i$.*

*Proof.* Suppose $\Gamma = H_G^*(q, Z_p)$ has one algebra generator $y$ of positive dimension and $M$ has one generator $m$ as a $\Gamma$ module. Let $s$ be the set of powers of $y$. Since $s^{-1}M = 0$, $y^km = 0$ for some $k$. Then $M^i = 0$ for $i > k$ dimension $(y) \cdot$ dimension $(m)$. The general case is similar.

## 3. $K^*(f)$ as an $H^*(G)$ module-$G$ connected

LEMMA 3.1. *Let $W$ be a $G$ space with $q \in W^G \neq \phi$ and $H^*(W^H, q, Z_p) = 0$ for all $H \in \mathcal{H}(N_p)$. Then $H^*({}_pW, q, Z_p) = 0$ where ${}_pW = \cup_{H \subset \mathcal{H}(N_p)} W^H$.*

*Proof.* If $N_p$ is finite or abelian, this follows from Meyer–Vietoris and induction by (1.2)(i). In general we show $H_G^*({}_pW, q) = 0$ implying $H^*({}_pW, q) = 0$. ($Z_p$ coefficients understood.) We can suppose $G = N_p$ and choose $P \subset \mathcal{H}(G)$ with $W^P \neq W^G$ and contained in no other $P'$ in $\mathcal{H}(G)$ with this property. Order the conjugacy classes of isotropy groups $Q_i$ containing $P$ so that $G = Q_0$ and if some conjugate of $Q_i$ contains $Q_j$ then $i<j$. Note $Q_{ip}=P$ for $i\neq 0$. As a matter of notation, let $r$ be the largest index and $Q_r=P$ (eventhough $P$ may not be an isotropy subgroup). Define $W_0=W^G$ and $W_{n+1}=GW^{Q_{n+1}}\cup W_n$. The $W_i$ give a $G$ filtration of $GW^P$ and the $W_i^P$ give an $N(P)$ filtration of $W^P$. These filtrations produce spectral sequences $E_r\Rightarrow H_G^*(GW^P, W^G)$ and $E_r'\Rightarrow H_{N(P)}^*(W^P, W^G)$ and the inclusion of spaces a map of spectral sequences $E_r\to E_r'$ which is an isomorphism of $E_1$ to $E_1'$ because $H_G(W_i, W_{i-1})\to H_{N(P)}(W_i^P, W_{i-1}^P)$ is an isomorphism for all $i$. In fact this map is the composition of these isomorphisms:

$$\begin{aligned}H_G^*(W_i, W_{i-1}) &= H_G^*(Gx_{N(Q_i)}(W^{Q_i}, W_{i-1}^{Q_i}))\\ &\cong H_{N(Q_i)}^*(W^{Q_i}, W_{i-1}^{Q_i}) =_\alpha H_{N(P)\cap N(Q_i)}^*(W^{Q_i}, W_{i-1}^{Q_i})\\ &\cong H_{N(P)}^*(N(P)\times_{N(P)\cap N(Q_i)}(W^{Q_i}, W_{i-1}^{Q_i}))\\ &\cong H_{N(P)}^*(N(P)W^{Q_i}, N(P)W_{i-1}^{Q_i}) =_\beta H_{N(P)}^*(W_i^P, W_{i-1}^P).\end{aligned}$$

Only steps $\alpha$ and $\beta$ require comment. Since $Q_{ip}=P$, $N(Q_i)_p=(N(Q_i)\cap N(P))_p$ by (1.2)(ii). Since $H_L^*(A, B)=H_{L_p}^*(A, B)$ for $L$ in $N_p$ by (1.2)(ii), this shows $\alpha$ is true. For $\beta$ the key facts are $(GW^{Q_j})^P=N(P)Q^{Q_i}$ and $N(P)(GW^{Q_j})^{Q_i}=N(P)W^{Q_i}$ if some conjugate of $Q_j$ contains $Q_i$. For $gQ_jg^{-1}\supset Q_i\supset P$ implies $g\in N(P)Q_j$ by Sylow's theorem.

This argument shows the natural map $H_G^*(GW^P, W^G)\to H_{N(P)}^*(W^P, W^G)$ is an isomorphism, but the latter group is zero because $P\in\mathcal{H}(N_p)$. The proof now follows by induction considering ${}_pW/GW^P$.

LEMMA 3.2. *Let $W$ satisfy the hypothesis of* (3.1). *Then for each multiplicative set $s\in\tilde{H}_{N_p}^*(q, Z_p)$ (the kernel of $H_{N_p}^*(q, Z_p)\to H^*(q, Z_p)$), $s^{-1}\tilde{H}_{N_p}^*(W, q, Z_p)=0$. If $s\in\tilde{H}_{N_p/N_0}^*(q)$, then $s^{-1}H_{N_p}(W, q)=0$.*

*Proof.* $s^{-1}H_{N_p}^*(W, q, Z_p)\to s^{-1}H_{N_p}^*({}_pW, q, Z_p)$ is an isomorphism. To see this note that each $x\in W-{}_pW$ has isotropy group $(N_p)_x$ which is finite or order prime to $p$ by 1.2(ii). This means that $s$ maps to zero in $H_{(N_p)_x}^*(q, Z_p)$; so $s^{-1}H_{(N_p)_x}^*(q, Z_p)=0$. This implies $s^{-1}H_{N_p}^*(W, {}_pW, Z_p)=0$. Since $H^*({}_pW, q, Z_p)=0$ by (3.1), $H_{N_p}^*({}_pW, q, Z_p)=0$. For the second statement, note that each $x\in W-p^W$ has isotropy group $(N_p)_x\in N_0$, $\tilde{H}_{N_p/N_0}^*(q)\to\tilde{H}_{N_0}^*(q)$ is zero and $H_{N_p}^*({}_pW, q)\to H_{N_0}^*({}_pW, q)$ is an isomorphism by (3.1).

COROLLARY 3.3. *Let $G$ be connected and $W$ satisfy the hypothesis of* (3.1) *and have its total mod $p$ cohomology finite dimensional over $Z_p$. Then $H_G^i(W, q, Z_p)=0$ for large $i$.*

*Proof.* By (2.1) and (2.2), $H_G^*(W, q, Z_p)$ is a subalgebra of $H_{N_p}^*(W, q, Z_p)$. Let

$s \in \tilde{H}_G(q, Z_p) \subset \tilde{H}_{N_p}(q, Z_p)$ be any multiplicative set. Then $s^{-1}H^*_{N_p}(W, q, Z_p) = 0$ (3.2); so $s^{-1}H^*_G(W, q, Z_p) = 0$. But $H^*_G(W, q, Z_p)$ is a finitely generated $H^*_G(q, Z_p)$ module by (2.3). The result follows from (2.5).

THEOREM 3.4. *Let $G$ be a compact connected Lie group with $H_*(G)$ $Z$ torsion free and $W$ a $G$ space with $q \in W^G \neq \phi$. Suppose that* (i) *for some integer $m$, $H^i(W, q, R) \cong H_{m-i+2}(W', q, R)$ for all $i$ and every $R$,* (ii) *if $\lambda = [(m-g)/2]+1$, $H_i(W, q) = 0$ for $i < \lambda$,* (iii) *$H_\lambda(W, q)$ is a $Z$ torsion module if $m-g$ is odd and* (iv) *for each prime $p$ and for each $K \in \mathcal{H}(N_p)$ $H^*(W^K, q, Z_p) = 0$. Then there is a filtration of $H_*(W, q)$ such that $E_0(H_*(W, q)) = H_*(G) \otimes H^G_*(W, q)$; moreover, $H^G_i(W, q) = 0$ for $i \neq \lambda$ and if $m-g$ is even $H^G_\lambda(W, q)$ is $Z$ free and is $Z$ torsion if $m-g$ is odd. In particular for $m-g$ even, $H_*(W, q)$ is a free $H_*(G)$ module and the hypothesis $H_*(G)$ is torsion free is superfluous.*

*Proof.* First note that $H^i_G(W, q, Z_p) = 0$ for large $i$ (3.3). Let $d$ be the largest $i$ such that $H^i_G(W, q, Z_p) \neq 0$. The spectral sequence $H^*_G(W, q, Z_p) \otimes H^*(G, Z_p) \Rightarrow H^*(W, q, Z_p)$ has a non zero term in $E_2$ of bidegree $(d, g)$ as $E_2^{d,g} = H^d_G(W, q, Z_p) \otimes H^g(G, Z_p)$. This term survives to $E_\infty$ and shows $H^{g+d}(W, q, Z_p) \neq 0$. But then $H_{m-g-d+2}(W, q, Z_p) \neq 0$ so $m-g-d+2 \geq [(m-g)/2]+1$ and $d \leq m-g-[(m-g)/2]+1$. Also $H^i_G(W, g, Z_p) = 0$ for $i < [(m-g)/2]+1$ since the same is true of $H^i(W, q, Z_p)$. Thus $H^i_G(W, q, Z_p) = 0$ for $i \neq \lambda$ if $m-g$ is even and for $i \neq \lambda$, $\lambda+1$ if $m-g$ is odd. This shows that $H^i_G(W, q) = 0$ for $i \neq \lambda$ and $H^\lambda_G(W, q)$ is $Z$ free if $m-g$ is even. If $m-g$ is odd $H^{\lambda+1}_G(W, q)$ is a $Z$ torsion module and $H^i_G(W, q) = 0$ $i \neq \lambda+1$. In either case the spectral sequence $H^*(G) \otimes H^*_G(W, q) \Rightarrow H^*(W, q)$ collapses implying the homology spectral sequences collapses giving $E_0(H_*(W, q)) = H_*(G) \otimes H^G_*(W, q)$ as an $H_*(G)$ module.

THEOREM 3.5. *Let $G$ be connected and $H_*(G)$ be $Z$ torsion free. Let $f: X \to Y$ a $G$ normal map between oriented smooth closed $G$ manifolds of dimension $m$. Suppose for each prime $p$ for each $H \in \mathcal{H}(N_p)$, $K^*(f^H, Z_p) = 0$, $K_i(f) = 0$ for $i < [(m-g)/2] = \lambda$ and if $m-g$ is odd $K_\lambda(f)$ is a $Z$ torsion module. Then there is a filtration of $K_*(f)$ such that $E_0K_*(f) = H_*(G) \otimes H^G_*(M_f, q)$; moreover, $H^G_i(M_f, q) = 0$ for $i \neq \lambda$ and if $m-g$ is even $K_\lambda(f) = H^G_{\lambda+1}(M_f, q)$ is $Z$ torsion free and is $Z$ torsion if $m-g$ is odd. In particular for $m-g$ even, $K_*(f)$ is a free $H_*(G)$ module and the hypothesis $H_*(G)$ is torsion free is superfluous.*

*Proof.* Since the degree of $f^K$ (for each component of $X^K$) is a unit of $Z_p$ [6], for each $H \in \mathcal{H}(N_p)$, $K^i(f^H, Z_p) = H^{i+1}(M^H_f, q, Z_p)$. Since $(M_f)^H = M_{f^H}$ (1.3), $H^*(M^H_f, q, Z_p) = 0$ for all $p$ and all $H \in \mathcal{H}(N_p)$. Now apply (3.4) with $W = M_f$ noting $K^{m-i}(f) \cong K_i(f)$ and (1.9).

*Remark* 3.6. Certainly the hypothesis that $H_*(G)$ be torsion free can be removed from the hypothesis with only minor changes in the conclusion.

## 4. Localization in $H^*_{G_p}(q, Z_p)$ and homological dimension of $Z(G)$ modules

Throughout this section $G$ is finite. Using [9], we show a relation between homological dimension of $Z(G)$ modules and localization in $H_{G_p}(q, Z_p)$. The first result is an easy consequence of the universal coefficient theorem and [9] (4.11):

THEOREM 4.1 [9]. *A finitely generated $Z(G)$ module $M$ which is $Z$ torsion free is projective iff for each prime $p$ $M \otimes_Z Z_p$ is $Z_p(G)$ projective.*

This together with the results of [9] and a few elementary lemmas gives

THEOREM 4.2. *A finitely generated $Z(G)$ module $M$ has homological dimension $\leq 1$ if for each prime $p$, $H^i_{G_p}(q, M \otimes Z_p) = 0$ for large $i$. If in addition $M$ is $Z$ torsion free, then $M$ is projective over $Z(G)$. (Moreover if $M \otimes Z_p$ is replaced by $M$, the condition is necessary and sufficient.)*

Using the fact that $H^*_{G_p}(q, M \otimes Z_p)$ is an $H^*_{G_p}(q, Z_p)$ module, we have this more convenient statement:

THEOREM 4.3. *Let $M$ be a finitely generated $Z(G)$ module (which is $Z$ free) then the homological dimension of $M$ is $\leq 1$ ($\leq 0$) if for each prime $p$ and each multiplicative set $s \in \tilde{H}^*_{G_p}(q, Z_p)$, $s^{-1}H^*_{G_p}(q, M \otimes Z_p) = 0$. Moreover if $Z_p$ is replaced by $Z$, the condition is necessary and sufficient for zero homological dimension.*

*Proof.* This is immediate from (4.2) and (2.5).

Our principle application occurs when $M$ is a (graded) module arising from the cohomology of a $G$ space. Say $M = H^*(W, q)$. The universal coefficient theorem $H^*(W, q, Z_p) = H^*(W, q) \otimes Z_p \oplus \mathrm{Tor}\,(H^{*+1}(W, q), Z)$ clearly implies

COROLLARY 4.4. *Let $W$ be a $G$ space, with $q \in W^G$. If $H^i(W, q)$ is a finitely generated $\Lambda$ module (with each $H^i(W, q)$ $Z$ free) then the homological dimension of each $H^i(W, q)$ is $\leq 1$ ($\leq 0$) if for each prime $p$ and multiplicative set $s \in \tilde{H}^*_{G_p}(q, Z_p)$, $s^{-1}H^*_{G_p}(q, H^i(W, Z_p)) = 0$ or if for each $s \in \tilde{H}^*_{G_p}(q)$, $s^{-1}H^*_{G_p}(q, H^i(W, q)) = 0$.*

## 5. Defining $\chi(f)$

Throughout this section $f: X \to Y$ is a $G$ map between $G$ spaces whose total cohomology is finitely generated over $Z$. Then $H^i_{G_0}(M_f, q)$ is a finitely generated $Z(G/G_0)$ module for each $i$. We give conditions insuring that the definition

$$\chi(f) = \Sigma(-1)^i H^i_{G_0}(M_f, q) \in \tilde{K}_0(Z(G/G_0)) \tag{5.1}$$

makes sense. Clearly $\chi(f) = 0$ if $f$ is a homotopy equivalence. It measures the deviation from being a homotopy equivalence.

THEOREM 5.2. *Suppose for each prime $p$ and each $K \in \mathcal{H}(N_p)$ that $H^*(M_f^K, q, Z_p) = 0$. Then $H^i_{G_0}(M_f, q) = 0$ for $i$ large. If also the spectral sequence $H^*_{G_p/G_0}(q, H^*_{G_0}(M_f, q, Z_p)) \Rightarrow H^*_{G_p}(M_f, q, Z_p)$ collapses for each $p$, then each $H^i_{G_0}(M_f, q)$ has homological dimension $\leq 1$ ($\leq 0$ if $H^i_{G_0}(M_f, q)$ is $Z$ free) over $Z(G/G_0)$ and $\chi(f)$ makes sense. Alternatively if the spectral sequence collapses with integral coefficients the same conclusion is valid.*

*Proof.* The total cohomology of $M_f$ is a finitely generated $Z$ module. Suppose $H^i(M_f, q) = 0$ for $i > N$. Then $H^i(M_f, q, Z_p) = 0$ for $i > N+1$ for each prime $p$. By (3.3) $H^i_{G_0}(M_f, q, Z_p) = 0$ for $i$ large. Examining the spectral sequence $H^*(G_0, Z_p) \otimes H^*_{G_0}(M_f, q, Z_p) \Rightarrow H^*(M_f, q, Z_p)$, we see that if $d$ is the largest integer with $H^d_{G_0}(M_f, q, Z_p) \neq 0$ then $g + d \leq N+1$. Since this holds for each $p$, $H^i_{G_0}(M_f, q) = 0$ for $i > N+1-g$.

Now suppose the spectral sequence in the statement of the theorem collapses. Then there is a filtration (of $Z(G_p/G_0)$ modules) of $H^*_{G_p}(M_f, q, Z_p)$ with $E_0 H^*_{G_p}(M_f, q, Z_p)$ equal to $H^*_{G_p/G_0}(q, H^*_{G_0}(M_f, q, Z_p))$. Let $s$ be any multiplicative set in $\bar{H}_{G_p/G_0}(q, Z_p)$. This gives rise to a multiplicative set again called $s$ in $H^*_{G_p}(q, Z_p)$ under the obvious algebra homomorphism. By (3.2), $s^{-1} H^*_{G_p}(M_f, q, Z_p) = 0$. Since localization is exact, $s^{-1}$ and $E_0$ commute; thus $s^{-1} H^*_{G_p/G_0}(q, H^*_{G_0}(M_f, q, Z_p)) = 0$. Apply (4.4) replacing $G$ by $G/G_0$ and $W$ by $M_f \times_{G_0} E/q \times_{G_0} E$. This shows $H^i_{G_0}(M_f, q)$ has homological dimension $\leq 1$ over $Z(G/G_0)$.

*Remark* 5.3. The spectral sequence of 5.2 certainly collapses if $H^i_{G_0}(M_f, q, Z_p) = 0$ for all but one value of $i$. This is a frequent situation of application. See e.g. (3.5).

THEOREM 5.4. *Suppose $G$ is a finite group and there is a point $y \in Y$ with $G_y = 1$. Then $\chi(f) = \chi({}^s f)$ provided both are defined.*

*Proof.* $G$ operates freely on $M_f - {}^sM_f$ which is $M_f - M_{s_f}$ by (1.5). Thus the cellular cochain complex $C^*(M_f, M_{s_f}) = C^*$ is a free $\Lambda$ module. Clearly $\chi(C^*) = \Sigma(-1)^i C^i$ is zero in $\tilde{K}_0(\Lambda)$.

The exact sequence of cochain complexes $0 \to C^*(M_f, M_{s_f}) \to C^*(M_f, q) \to C^*(M_{s_f}, q) \to 0$ gives rise to an exact triangle

$$\begin{array}{ccc} H^*(M_f, M_{s_f}) & \longrightarrow & H^*(M_f, q) \\ & \nwarrow \quad \swarrow & \\ & H^*(M_{s_f}, q) & \end{array}$$

which implies that $H^i(M_f, M_{s_f})$ has finite homological dimension over $Z(G)$ so $\chi(f, {}^sf) = \Sigma(-1)^i H^i(M_f, M_{s_f}) = \chi(C^*) = 0$. But $\chi(f) = \chi({}^sf) + \chi(f, {}^sf)$.

LEMMA 5.5. *If $G$ is a $p$ group and the conditions of* 5.2 *are satisfied, $\chi({}^sf)$ is defined.*

*Proof.* Apply (3.1) with $W = M_f$. Then ${}_pW = M_{s_f}$ and $H^*(M_{s_f}, q, Z_p) = 0$; so $H^*(M_{s_f}, q)$ is a $Z$ torsion module with no $p$ torsion and for each $i$, $H^i(M_{s_f}, q)$ has homological dimension $\leq 1$ over $Z(G)$ by (4.3).

## 6. $K_*(f)$ as an $H_*(G)$ module

We are now prepared to discuss the structure of $K_*(f)$ as an $H_*(G)$ module. The homology algebra $H_*(G)$ is the "twisted" tensor product $H_*(G_0) \otimes_t Z(G/G_0)$. In fact $H_*(G_0)$ is a $Z(G/G_0)$ module. $x^g = \bar{g}^{-1}x\bar{g}$ for $x \in H_*(G_0)$, $g \in G/G_0$ and $\bar{g} \in G$ representing $g$. The multiplication in the twisted tensor product is given by $x \otimes w \cdot x' \otimes w' = x \cdot x' \otimes ww'$ for $x$, $x' \in H_*(G_0)$.

THEOREM 6.1. *Let $H_*(G_0)$ be $Z$ torsion free and $f: X \to Y$ be a $G$ normal map between smooth closed oriented $G$ manifolds of dimension $m$. Suppose for each prime $p$ and for each $H \in \mathcal{H}(N_p)$ that $K^*(f^H, Z_p) = 0$, $K_i(f) = 0$ for $i < [(m-g)/2] = \lambda$ and if $m - g$ is odd $K_\lambda(f)$ is a $Z$ torsion module. Then there is a filtration of $K_*(f)$ by $H_*(G_0)$ modules such that $E_0K_*(f) = H_*(G_0) \otimes K_\lambda(f)$ and $K_\lambda(f)$ is a projective $Z(G/G_0)$ module if $m - g$ is even and has homological dimension $\leq 1$ if $m - g$ is odd; moreover, when $m - g$ is even, the hypothesis on $H_*(G_0)$ is superfluous, $\chi(f) = \pm[K_*(f)^*]$ and $K_*(f)$ is a stably free $H_*(G)$ module iff $\chi(f) = 0$.*

*Proof.* The first conclusion is a restatement of (3.5) noting $H_\lambda^{G_0}(M_f, q) = K_\lambda(f)$.

For the second, note that $H^i_{G_0}(M_f, q) = 0$ unless $i = \lambda$ when $m - g$ is even or

$i=\lambda+1$ when $m-g$ is odd by the universal coefficient theorem. Thus the spectral sequence of (5.2) collapses and $H^i_{G_0}(M_f, q)=K^i(f)$ has homological dimension $\leq 1$ for $i=\lambda$ ($m-g$ even) or $i=\lambda+1$ ($m-g$ odd). In the first case $K^\lambda(f)$ is $Z$ torsion free since $K_i(f)=0$ for $i<\lambda$; so in this case $K^\lambda(f)$ is a projective $Z(G/G_0)$ module. When $m-g$ is odd, $K_\lambda(f)=\mathrm{Ext}^1_Z(K^{\lambda+1}(f), Z)$; so it too has homological dimension $\leq 1$.

Since $H^i_{G_0}(M_f, q)=0$ for $i\neq\lambda$ or $\lambda+1$ depending on $m-g$, $\chi(f)=\pm[K^\lambda(f)]$ or $\pm[K^{\lambda+1}(f)]$. Moreover, in the first case $K_\lambda(f)=\mathrm{Hom}_Z(K^\lambda(f), Z)=K^\lambda(f)^*$ by the universal coefficient theorem; so $K_\lambda(f)$ is also $Z(G/G_0)$ projective. If it is free over $Z(G/G_0)$, then $K_*(f)$ is free over $H_*(G)$.

## 7. Application to the $G$ normal cobordism problem

Let $\gamma\in H_*(G_0)$ denote the orientation class and define a homomorphism $w_1: G/G_0\to Z_2=\{\pm 1\}$ by

$$\gamma^g=w_1(g)\gamma \quad \text{for} \quad g\subset G/G_0 \tag{7.1}$$

Let $[X]\in H_*(X)$ denote the orientation class for $X$ and define $w_2: G/G_0\to Z_2$ by

$$g[X]=w_2(g)[X] \tag{7.2}$$

When the hypothesis of (6.1) hold and $m-g$ is even, we can define an integral valued non singular bilinear form $\langle\ \rangle$ on $K_\lambda(f)$ using the intersection pairing $\circ$ in $H_*(X)$;

$$\langle x, y\rangle=x\circ(\gamma\cdot y)\in Z; \qquad x, y\in K_\lambda(f) \tag{7.3}$$

Then for $g\in G/G_0$, $\langle gx, gy\rangle=w(g)\langle x, y\rangle$ where $w(g)=w_1(g)w_2(g)$. This follows from the fact that $\gamma\cdot(gy)=(\gamma g)\cdot y=(g\gamma^g)\cdot y=g(\gamma^g\cdot y)$ and $g\alpha\circ g\beta=w_2(g)(\alpha\circ\beta)$. The fact that $\langle\ \rangle$ is non singular i.e. induces an isomorphism $K_\lambda(f)\cong\mathrm{Hom}_Z(K_\lambda(f), Z)$ of $Z(G/G_0)$ modules follows from the fact that the intersection pairing $K_\lambda(f)\otimes K_{\lambda+g}(f)\to Z$ is non singular and the isomorphism of $H_*(G_0)$ modules of $H_*(G_0)\otimes K_\lambda(f)$ and $K_*(f)$ is defined by $\alpha\otimes\beta\to\alpha\cdot\beta$ i.e. by the structure of $K_*(f)$ as an $H_*(G_0)$ module. Thus we have

COROLLARY 7.4. *If the hypothesis of* (6.1) *hold,* $K_\lambda(f)$ *is a projective* $Z(G/G_0)$ *module supporting a* $Z$ *valued non singular bilinear form* $\langle\ \rangle$ *satisfying* $\langle gx, gy\rangle=w(g)\langle x, y\rangle$ *for* $g\in G/G_0$, $x, y\in K_\lambda(f)$ *and* $w(g)=w_1(g)w_2(g)$.

Of course we can also view $\langle\ \rangle$ as a bilinear form (over $\Lambda$) on $K_\lambda(f)$ with values in $\Lambda$ by setting

$$(x, y) = \sum_{g \in G/G_0} \langle x, g^{-1}y \rangle g$$

This is to conform to the standard notation for this situation when $G = G/G_0$ acts freely on $Y$ [13]. Under certain hypothesis on ${}^sX$ e.g. $\dim {}^sX/G < \frac{1}{2} \dim X/G$, it is possible to define a self intersection form $\mu : K_\lambda(f) \to \Lambda/I$ where $I$ is the subgroup of $\Lambda$ consisting of $\nu + (-1)^{\lambda-1}\bar{\nu}$ for $\nu \in \Lambda$ and $\nu \to \bar{\nu}$ the automorphism of $\Lambda$ defined by $\sum \overline{a_g g} = \sum_{g\varepsilon} w(g) a_g g^{-1}$.

When $\chi(f) = 0$, so $K_\lambda(f)$ is $\Lambda$ free,

$$\sigma(f) = (K_\lambda(f), ( \ , \ ), \mu) \in L_{2\lambda}(G/G_0, w) \tag{7.5}$$

represents an element of the group $L_{2\lambda}(G/G_0, w)$ of Wall [13]. Under suitable hypothesis e.g. trivial principle isotropy group, $\pi_1(Y) = 0$ and $\dim {}^sX/G < \frac{1}{2} \dim X/G$, $\sigma(f)$ is the only obstruction to finding a $G$ normal cobordism between $(X, f)$ and $(X', f')$ where $f' : X' \to Y$ is a homotopy equivalence. Thus $\chi(f)$ is a primary obstruction and $\sigma(f)$ a secondary obstruction to making $f$ a homotopy equivalence. Of course this is all relative to the hypothesis of (6.1).

To achieve the full obstruction theory for the $G$ normal cobordism problem (1.8), we first generalize $\chi(f)$ and $\sigma(f)$ slightly by introducing $\chi(f, Z_{(p)}) \in \tilde{K}_0(Z_{(p)}(G/G_0))$ and $\sigma(f, Z_{(p)}) \in L_{2\lambda}(Z_{(p)}(G/G_0), w)$ where $Z_{(p)}$ is $Z$ localized at $p$. This is to be able to treat maps whose degree is a unit in $Z_{(p)}$. For each $p$, partially order the conjugacy classes of groups in $\mathcal{H}(N_p)$ by setting $K \leq H$ if $K$ contains a conjugate of $H$. Roughly each conjugacy class $K$ in $\mathcal{H}(N_p)$ contributes two obstructions $\chi_K(f) = \chi(f^K, Z_{(p)})$ and $\sigma_K(f) = \sigma(f^K, Z_{(p)})$ as $K/K_0$ is a $p$ group. In fact $\chi_K(f)$ is defined only if $\chi_L(f) = 0$ and $\sigma_L(f) = 0$ for $L < K$ and corresponds to replacing $G$ by $N(K)/K$ and $f$ by $f^K : X^K \to Y^K$ in our preceeding discussion. Here $N(K)$ is the normalizer of $K$ and $\chi_L(f) \in \tilde{K}_0(Z_{(p)}(L''))$, $\sigma_L(f) \in L_\alpha(Z_{(p)}(L''), w_L)$ where $L/L_0$ is a $p$ group, $L' = N(L)/L$ and $L'' = L'/L'_0$.

This very brief discussion illustrates the obstruction theory for dealing with the hypothesis of (6.1) and shows how the Smith theory conditions show up in a constructive manner for handling the $G$ normal cobordism problem.

For a complete discussion of the application of the obstruction theory for $G = S^1$ see [6]. There all the obstructions $\chi_L(f)$ vanish because $L''$ is 1.

## 8. The homomorphism $\sigma_G : Z_n^* \to \tilde{K}_0(Z(G))$

As a consequence of (4.3), we see that if the order of $G$ is $n$ and $q$ is prime to

$n$, $Z_q$ viewed as a $Z(G)$ module has homological dimension $\leq 1$; so represents an element $[Z_q] \in \tilde{K}_0(Z(G))$. Swan showed [11] that this gives rise to a homomorphism $\sigma_G : Z_n^* \to \tilde{K}_0(Z(G))$ from the multiplicative group of units of the ring $Z_n$ to $\tilde{K}_0(Z(G))$. He proved the

THEOREM 8.1 [11]. *$\sigma_G$ is zero if $G$ is cyclic.*

Since this is important for our study, we give a very simple geometric proof.

*Proof.* Let $G' = S^1$ and $G = Z_p \subset S^1$ be the cyclic group of order $p$ (not necessarily a prime). Let $N$ and $M$ be the complex two dimensional $G'$ modules defined by

(i) $N : t(z_0, z_1) = (t^p z_0, t^q z_1), \qquad z = (z_0, z_1) \in N$

(ii) $M : t(z_0, z_1) = (t z_0, t^{pq} z_1), \qquad z = (z_0, z_1) \in M$

Here $t \in S^1 \subset C$ and $q$ is an integer prime to $p$. Choose integers $a$ and $b$ so that $-ap + bq = 1$. Define a $G'$ map $w : N \to M$ by

$$\omega(z_0, z_1) = (\bar{z}_0^a z_1^b, z_0^q + z_1^p) \tag{6.2}$$

This gives rise to a $G'$ map from the unit sphere of $N$ to the unit sphere of $M : f : S(N) \to S(M)$ by $f(z) = \omega(z)/\|\omega(z)\|$.

Restrict the action to $G$ and set $X = S(N)$, $Y = S(M)$. Since the degree of $f$ is 1 [8], [7], $f$ is a homotopy equivalence so $\chi(f)$ is zero. Note that $G$ acts semi-freely on $X$ and $Y$ with $X^G = \{(z_0, 0) \mid |z_0| = 1\}$ and $Y^G = \{(0, z_1) \mid |z_1| = 1\}$; moreover, $f^G(z_0, 0) = (0, z_0^q)$ is a map of degree $q$. Clearly $H^2(M_{f^G}) = Z_q$ and $H^i(M_f, q) = 0$ for $i \neq 2$. Since $M_{f^G} = (M_f)^G$, $G$ acts trivially on $Z_q$. Since $G$ acts semi-freely on $X$ and $Y$, ${}^s f = f^G$. Thus $\chi({}^s f) = \chi(f^G) = [Z_q] = \sigma_G(q)$. Since $\chi(f) = \chi({}^s f)$ by (5.4), $0 = \chi(f) = \chi({}^s f) = \sigma_G(q)$.

COROLLARY 8.3. *Let $G$ be an arbitrary finite group of order $n$ acting semi-freely on $X$ and $Y$ and $f : X \to Y$ a $G$ map. Suppose each $H^i(M_{f^G}, q, Z_n) = 0$. Then $\chi({}^s f)$ is defined. If $\chi(f)$ is also defined $\chi(f) \varepsilon$ image $\sigma_G$.*

*Proof.* Each $H^i(M_{f^G}, q)$ is a $Z$ torsion module of order prime to $n$ and hence has homological dimension $\leq 1$ over $Z(G)$. Since $G$ acts trivially on $H^i(M_{f^G}, q)$, the class it represents in $\tilde{K}_0(Z(G))$ is in the image of $\sigma_G$. Since $G$ acts semi-freely on $X$ and $Y$, $f^G = {}^s f$; so $\chi(f) = \chi(f^G) \varepsilon$ image $\sigma_G$.

COROLLARY 8.4. *Suppose $G$ is $Z_p$ with $p$ prime. Suppose also the hypothesis of (5.2). Then $\chi(f) = 0$.*

*Proof.* The hypothesis of (5.2) guarantee $H^*(M_{f^G}, q, Z_p)=0$. The result now follows from (8.3) and (8.1).

## 9. An example with $\chi(f)\neq 0$

Let $G=Q$ be the quaternion group; so $Q=\{\pm 1, \pm i, \pm j, \pm k\}\subset \mathbf{H}$ where $\mathbf{H}$ is the quaternion skew field. Viewing $\mathbf{H}$ as a left complex vector space, it is a complex $Q$ module with $Q$ acting by right multiplication. Note that the function $h:\mathbf{H}\to C$ defined by $h(x+yj)=x^4+y^4$ is $Q$ invariant if $Q$ acts trivially on $C$ and $x$ and $y$ are the complex coordinates of $x+yj\in\mathbf{H}$. This shows that for each integer $\lambda$, the variety

$$V_\lambda=\{(z_0, z_1, z_2, x, y)\in C^3\times\mathbf{H} \mid h_\lambda=0\}$$
$$h_\lambda(z_0, z_1, z_2, x, y)=z_0^\lambda+z_1^2+z_2^2+x^4+y^4$$

is $Q$ invariant. Here $Q$ acts on $C^3\times\mathbf{H}$ by $(u, v)q=(u, vq)$ for $q\in Q$, $\mu\in C^3$ and $v\in\mathbf{H}$. Set

$$L_\lambda = V_\lambda\cap S(C^3\times\mathbf{H})$$

where $S(C^3\times\mathbf{H})$ is the unit sphere in $C^3\times\mathbf{H}$. Clearly $L_\lambda$ is $Q$ invariant.

The subvariety $W_\lambda=\{(z_0, z_1, z_2, x, y)\in L_\lambda \mid x=y=0\}$ is the fixed point set $L_\lambda^Q$ and its homology is given by

$$H_1(W_\lambda)=Z_\lambda, \qquad H_i(W_\lambda)=Z, \qquad i=0, 3$$

and $H_2(W_\lambda)=0$. See [3], p. 275. The action of $Q$ on $L_\lambda$ is semi-free so the singular set ${}^sL_\lambda$ is $L_\lambda^Q=W_\lambda$.

Let $\lambda$ be an odd integer and choose integers $a$ and $b$ such that $-2a+\lambda b=1$. Define a $Q$ map $f:L_\lambda\to S(C^2\times\mathbf{H})$ by

$$f(z_0, z_1, z_2, x, y)=\frac{(\bar{z}_0^a\cdot z_1^b, x_2, x, y)}{\|(\bar{z}_0^a z_1^b, z_2, x, y)\|}.$$

Then

(i) Both $f$ and $f^Q$ have degree 1
(ii) $f_*^Q:H_*(L_\lambda^Q, Z_2)\to H_*(S(C^2\times\mathbf{H})^Q, Z_2)$ is an isomorphism
(iii) $H^i(M_f, q)=0$ for $i\neq 5$ and $H^5(M_f, q)\cong H^4(L_\lambda)$ is a $Z$ torsion module of

odd order [3], p. 279.
(iv) $H^i(M_{f^Q}, q)=0$ for $i\neq 3$ and $H^3(M_{f^Q}, q)=H^2(W_\lambda)=Z_\lambda$

These facts insure that both $\chi(f)$ and $\chi(f^Q)$ are defined and

THEOREM 9.1. $\chi(f)=\chi(f^Q)=\sigma_Q(\lambda)$. *For* $\lambda=3$, $\chi(f)\neq 0$.

*Proof.* Since the actions are semi-free, the first equality follows from (5.4) while the second follows from (iv). The fact that $\sigma_Q(3)\neq 0$, is a result of Swan [11].

*Remark* 9.2. The map $f: L_\lambda \to S(C^2\times \mathbf{H})$ is a $Q$ normal map. The $Q$ normal bundle of $L_\lambda \subset C^3\times \mathbf{H}$ is $L_\lambda\times R^3$ with trivial $Q$ action on $R^3$.

One might suspect that the invariant $\chi(f)$ is completely determined by the Sylow subgroups, a phenomenon which occurs for example for the cohomology of a group. This is not the case. To see this let $J_\lambda\subset S(C^3\times\mathbf{H})$ be the subvariety $z_0^\lambda+z_1^2+z_2^2+z_3^{12}+z_4^{12}=0$. The group $G=Z_3\times Q$ acts semi-freely on $J_\lambda$. The action is induced by the action of $Z_3\times Q$ on $\mathbf{H}$ defined by viewing $Z_3$ as the multiplicative subgroup of $C$ of 3rd roots of unity and allowing $Z_3$ to act via left multiplication on $\mathbf{H}$ and $Q$ via right multiplication. The same map $f$ as above gives a $G$ normal map $f: J_\lambda\to S(C^2\times\mathbf{H})$ and again $\chi(f)=\pm[Z_\lambda]=\sigma_G(\lambda)\in \tilde{K}_0(Z(G))$. The order of $G$ is 24 and $\sigma_G(17)\neq 0$ but $\sigma_{Z_3}(17)=0$ and $\sigma_Q(17)=0$. See [11].

*Remark* 9.3. The $Q$ variety $L_\lambda$ has higher dimensional analogs generated by the functions $z_0^\lambda+z_1^2+\cdots+z_{2k}^2+x_1^4+\cdots+x_{2l}^4$ as $k$ and $l$ vary.

*Remark* 9.4. The fact that $\chi(f)=\chi(f^Q)=\sigma_G(3)$ when $\lambda=3$, shows that $(L_\lambda, f)$ is never $Q$ normally cobordant rel $L_\lambda^Q$ to $(X', f')$ with $f'$ a homotopy equivalence even though $f_*^Q: H_*(L^Q, Z_2)\to H_*(S(\mathbf{C}^2\times\mathbf{H})^Q, Z_2)$ is an isomorphism.

## 10. Application to Equivariant Homotopy Groups of Spheres

If $\Sigma_i$ $i=0, 1$ are homotopy spheres supporting an action of $G$ and $f:\Sigma_0\to\Sigma_1$ is a $G$ map of degree 1, then $f^H:\Sigma_0^H\to\Sigma_1^H$ is a map whose degree is non zero mod $p$ for every $p$ group $H$ in $G$ (Smith theory). In particular this means that if $G$ acts semi-freely on $\Sigma_i$ (i.e. the only isotropy groups are $G$ and 1) then deg $f^G$ is a unit in $Z_n$ where $n=$ order $G$. For cyclic groups, deg $f^G$ can be an arbitrary element of $Z_n^*$. See e.g. the example of (8.1). In general there are additional restrictions, namely

PROPOSITION 10.1. *Let* $f:\Sigma_0\to\Sigma_1$ *be a degree* 1 $G$ *map where* $G$ *acts* *semi-freely on* $\Sigma_i$ *and suppose* $\Sigma_i^G$ *is a homotopy sphere for* $i=0$, 1. *Then* $\sigma_G(\deg f^G)=0$ *in* $\tilde{K}_0(Z(G))$.

*Proof.* $\sigma_G(\deg f^G)=\chi(f^G)=\chi(f)=0$ because $f$ is a homotopy equivalence.

For example if $G=Q$ is the quaternion group of section 8, then $\deg f^G \not\equiv \pm 3(8)$.

Proposition 10.1 is an example of the relation between the homological invariants of $G$ manifolds and $G$ maps. For another example, if $\Sigma_i$ $i=0$, 1 are rational homotopy spheres supporting an $S^1$ action with $\Sigma_i^{S^1}=\phi$ and $f:\Sigma_0\to\Sigma_1$ is an $S^1$ map, then deg $f$ is uniquely determined by the $S^1$ manifolds $\Sigma_i$.

## REFERENCES


[1] Bass, H., *Algebraic K Theory*, Benjamin (1968).
[2] Becker, J. and Gottlieb, D., *The transfer in fiber spaces*, Topology *14* (1975), 1-12.
[3] Bredon, G., *Introduction to compact transformation groups*, Academic Press (1972).
[4] Browder, W., *Surgery on simply connected manifolds*, Springer Verlag (1972).
[5] Connolly, F., *Linking numbers and surgery*, Topology *12* (1973).
[6] Petrie, T., *G surgery—G transversality*, to appear.
[7] ——, *A setting for smooth $S^1$ actions with applications to real algebraic actions on $P(C^{4n})$*, Topology, *13* (1974), 363–374.
[8] ——, *Exotic $S^1$ actions on $CP^3$ and related topics*, Inventiones Math. *17* (1972), 317–327.
[9] Rim, D. S., *Modules over finite groups*, Ann. of Math. *69* (1959), 700–712.
[10] Swan, R. G., *Induced representations and projective modules*, Ann. of Math. (2) *71* (1960), 552–578.
[11] ——, *Periodic resolutions for finite groups*, Ann. of Math. *72* (1960), 267–291.
[12] Wall, C. T. C., *Surgery of non-simply connected manifolds*, Ann. of Math. *84* (1966), 217–276.
[13] ——, *Surgery on compact manifolds*, Academic Press (1970).
[14] ——, *Finiteness conditions for CW-complex*, Ann. of Math. *81* (1965), 56–69.
[15] Mislin, G., *Walls obstruction for nilpotent spaces*, to appear in Topology.


*3 Alta Vista Drive,*
*Princeton, N.J.* 08540
*U.S.A.*

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