

Zeta Functions and Their Asymptotic Expansions for Compact Symmetric Spaces of Rank One

Autor(en): **Cahn, Robert S. / Wolf, Joseph A.**

Objekttyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **51 (1976)**

PDF erstellt am: **22.07.2024**

Persistenter Link: <https://doi.org/10.5169/seals-39426>

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

Zeta Functions and Their Asymptotic Expansions for Compact Symmetric Spaces of Rank One

by ROBERT S. CAHN (University of Miami, Coral Gables, Florida)

and

JOSEPH A. WOLF (University of California, Berkeley, California)

§0. Introduction

In this paper we apply É. Cartan's theory of class 1 representations [3] to derive explicit formulae for the ζ -functions of the compact riemannian symmetric spaces of strictly positive curvature. We then combine those formulae with an asymptotic expansion of Mulholland [5] and evaluate the coefficients in the Minakshisundaram asymptotic expansion (see [1]) of the ζ -function.

§1. Generalities on Compact Symmetric Spaces

We assemble the basic facts required to discuss ζ -functions of compact symmetric spaces from the representation-theoretic viewpoint. In principle, everything here in §1 is contained in Garth Warner's book [6], and we refer to Warner [6] and Helgason [4] for the original sources (of which Cartan [3] is the principal one).

Fix a compact riemannian symmetric space M and let G be the largest connected group of isometries. Thus G is a compact connected Lie group with an involutive automorphism σ , and $M = G/K$ where K is an open subgroup of $G^\sigma = \{g \in G : \sigma(g) = g\}$, and the riemannian metric on M derives from a positive definite invariant bilinear form on the Lie algebra of G .

$\hat{G}$ denotes the set of all equivalence classes $[\pi]$ of irreducible unitary representations π of G . Given $[\pi]$, V_π denotes the (finite dimensional complex Hilbert) space on which π represents G . A class $[\pi] \in \hat{G}$ is of *class 1* relative to K if there exists

$$0 \neq v \in V_\pi \quad \text{such that} \quad \pi(k)v = v \quad \text{for all } k \in K,$$

that is if V_π has a nonzero K -fixed vector. Let us write

$$\hat{G}_K = \{[\pi] \in \hat{G} : [\pi] \text{ is of class 1 relative to } K\}. \tag{1.1}$$

G acts on $L_2(M)$ through its left regular representation, that is

$$[l(g)f](x) = f(g^{-1}x) \quad \text{for } f \in L_2(M), \quad g \in G \quad \text{and } x \in M = G/K.$$

This action decomposes over $\hat{G}_K$ as follows.

1.2. THEOREM (É. Cartan [3]). $L_2(M) = \sum_{[\pi] \in \hat{G}_K} V_\pi$ as unitary left G -module.

Proof. $L_2(G) = \sum_G V_\pi \otimes V_{\pi^*}$ according to the Peter-Weyl Theorem. Here $V_\pi \otimes V_{\pi^*}$ is identified with the space of all matrix coefficient functions

$$f_{v,w}(g) = \langle v, \pi(g)w \rangle \quad \text{for } v, w \in V_\pi \quad \text{and } g \in G$$

of $[\pi]$. The left and right actions of G on $L_2(G)$ are

$$[l(g_1) \otimes r(g_2)f](x) = f(g_1^{-1}xg_2),$$

so the action on coefficients of $[\pi]$ is

$$\{l(g_1) \otimes r(g_2)\} f_{v,w} + f_{\pi(g_1)v, \pi(g_2)w},$$

which is $\pi \otimes \pi^*$.

View $L_2(M = G/K)$ as $\{f \in L_2(G) : f(gk) = f(g) \text{ for } g \in G \text{ and } k \in K\}$. Writing superscripts for invariants and 1_K for the trivial 1-dimensional representation of K , now

$$\begin{aligned} L_2(M) &= L_2(G)^{r(K)} = \sum_G V_\pi \otimes V_{\pi^*}^K \\ &= \sum_G \text{mult}(1_K, \pi^* \mid_K) V_\pi = \sum_G \text{mult}(1_K, \pi \mid_K) V_\pi \\ &= \sum_{\hat{G}_K} \text{mult}(1_K, \pi \mid_K) V_\pi \end{aligned}$$

as unitary left G -module. The latter multiplicities all = 1; for example see Helgason [4, p. 408] for a proof of Gelfand's theorem that a certain algebra $C^*(G)$, which is W^* -dense in the commuting algebra of $l(G)$ on $L_2(M)$, is abelian. *q.e.d.*

Now let $\mathfrak{g}$ denote the Lie algebra of G , $\mathfrak{G}$ the universal enveloping algebra of $\mathfrak{g}$, and $\mathfrak{Z}$ the center of $\mathfrak{G}$. Every class $[\pi] \in \hat{G}$ maps every element of $\mathfrak{Z}$ to a scalar, giving an associative algebra homomorphism that we denote

$$\pi : \mathfrak{Z} \rightarrow \mathbb{C}, \quad \text{infinitesimal character of } [\pi].$$

Recall that the riemannian metric on M is derived from an invariant positive definite inner product on $\mathfrak{g}$. If $\{x_1, \dots, x_n\}$ is an orthonormal basis then $\sum x_i^2 \in \mathfrak{Z}$ and depends

only on the inner product, and as differential operator

$$-\sum x_i^2 = \Delta, \quad \text{the Laplace-Beltrami operator on } M. \quad (1.3)$$

If we use the negative of the Cartan-Killing form of $\mathfrak{g}$ for the inner product, then $-\sum x_i^2 = \Omega$, the Casimir element of $\mathfrak{G}$, and so $\Delta = l(\Omega)$ on $L_2(M)$.

Define $\zeta_M(t) = \sum_{\lambda} e^{-\lambda t}$ where λ ranges over the eigenvalues (with multiplicity) of the Laplace-Beltrami operator (1.3). This is the trace of the heat kernel. The Minakshisundaram-Pleijel zeta function $\sum_{\lambda} \lambda^{-s}$ is related to ζ_M by a Mellin transform.

1.4. COROLLARY. *If the riemannian metric on M is defined by the negative of the Cartan-Killing form of $\mathfrak{g}$ then M has ζ -function given by*

$$\zeta(t) = \sum_{[\pi] \in \hat{G}_K} (\text{degree of } \pi) e^{-t\pi(\Omega)}$$

where $\Omega \in \mathfrak{Z}$ is the Casimir element of $\mathfrak{G}$.

To specify the ζ -function of M we now have to describe $\hat{G}_K$, and specify degree (π) and $\pi(\Omega)$ for every class $[\pi] \in \hat{G}_K$.

The Lie algebra $\mathfrak{g}$ decomposes under the automorphism σ as $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ where $\mathfrak{k}$ is the $(+1)$ -eigenspace and $\mathfrak{s}$ is the (-1) -eigenspace. Of course, $\mathfrak{k}$ is the Lie algebra of K . Choose

$$\mathfrak{a}: \text{maximal abelian subspace of } \mathfrak{s}, \quad (1.5a)$$

and

$$\Sigma_{\mathfrak{a}}^+ \text{ positive } \mathfrak{a}_{\mathbb{C}}\text{-root system on } \mathfrak{g}_{\mathbb{C}}. \quad (1.5b)$$

Define $\mathfrak{m}' = \{x \in \mathfrak{k}: [x, \mathfrak{a}] = 0\}$ and let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{m}$. Then

$$\mathfrak{h} = \mathfrak{t} + \mathfrak{a} \text{ is a Cartan subalgebra of } \mathfrak{g}. \quad (1.6a)$$

Any choice of positive $\mathfrak{t}_{\mathbb{C}}$ -root system on $\mathfrak{m}_{\mathbb{C}}$ specifies a choice of

$$\begin{aligned} \Sigma^+ &: \text{positive } \mathfrak{h}_{\mathbb{C}}\text{-root system on } \mathfrak{g}_{\mathbb{C}} \text{ such that} \\ \Sigma_{\mathfrak{a}}^+ &= \{\phi|_{\mathfrak{a}}: \phi \in \Sigma^+ \text{ and } \phi|_{\mathfrak{a}} \neq 0\}. \end{aligned} \quad (1.6b)$$

Each class $[\pi] \in \hat{G}$ is specified by its highest weight relative to $(\mathfrak{h}, \Sigma^+)$, and the class 1 representations have a certain remarkable property.

1.7. THEOREM (É. Cartan [3]). *If $[\pi] \in \hat{G}_K$ has highest weight λ relative to $(\mathfrak{h}, \Sigma^+)$, then $\lambda(\mathfrak{t})=0$, that is $\lambda \in i\mathfrak{a}^*$.*

Proof. The noncompact dual $\tilde{\mathfrak{g}} = \mathfrak{k} + i\mathfrak{s}$ of $\mathfrak{g}$ has Iwasawa decomposition $\tilde{\mathfrak{g}} = \mathfrak{n} + i\mathfrak{a} + \mathfrak{k}$ where $\mathfrak{n}$ is the sum of its $\Sigma_{\mathfrak{a}}^+$ -negative ($i\mathfrak{a}$)-root spaces. Writing capital German letters for universal enveloping algebras of complexifications, now $\mathfrak{G} = \mathfrak{N}\mathfrak{A}\mathfrak{K}$. Decompose

$$V_{\pi} = \sum V_{\pi, \nu} \text{ sum of weight spaces.}$$

Let w be a nonzero K -fixed vector and decompose

$$w = \sum w_{\nu} \text{ where } w_{\nu} \in V_{\pi, \nu}.$$

Then

$$\begin{aligned} V_{\pi} &= \pi(\mathfrak{G}) w = \pi(\mathfrak{N}) \pi(\mathfrak{A}) \pi(\mathfrak{K}) w = \pi(\mathfrak{N}) \pi(\mathfrak{A}) w \\ &\subset \pi(\mathfrak{N}) \sum_{w_{\nu} \neq 0} V_{\pi, \nu} \subset \sum_{\mu \leq \nu} \sum_{w_{\nu} \neq 0} V_{\pi, \mu}. \end{aligned}$$

We conclude that $w_{\lambda} \neq 0$. As $\pi(\mathfrak{t}) w = 0$ now $\lambda(\mathfrak{t}) = 0$. As the weights are in $i\mathfrak{h}^*$ now $\lambda \in i\mathfrak{a}^*$. *q.e.d.*

1.8 THEOREM (É. Cartan [3]; S. Helgason [4], [7]). *Define*

$$\Lambda^+ = \{ \lambda \in i\mathfrak{a}^* : \langle \lambda, \psi \rangle / \langle \psi, \psi \rangle \text{ integer } \geq 0 \text{ for all } \psi \in \Sigma_{\mathfrak{a}}^+ \} \quad (1.9)$$

where $\langle \cdot, \cdot \rangle$ is the Cartan-Killing form. Then

$$[\pi] \rightarrow \text{highest weight relative to } (\mathfrak{h}, \Sigma_{\mathfrak{a}}^+)$$

is an injective map from $\hat{G}_K$ into Λ^+ . If K is connected and G is simply connected then it is a bijection.

The proof is technical and we refer to Chapter III of Warner [6].

1.10. COROLLARY. *If M is simply connected and if its riemannian metric derives from the negative of the Cartan-Killing form of $\mathfrak{g}$, then M has ζ -function given by*

$$\zeta_M(t) = \sum_{\lambda \in \Lambda^+} P(\lambda) e^{-tq(\lambda)} \quad (1.11)$$

where

$$q = \frac{1}{2} \sum_{\phi \in \Sigma^+} \phi \text{ and } q_{\mathfrak{a}} = q|_{\mathfrak{a}}; \quad (1.12a)$$

$$P(\lambda) = \prod_{\phi \in \Sigma^+} \frac{\langle \lambda + \varrho, \phi \rangle}{\langle \varrho, \phi \rangle}; \quad (1.12b)$$

and

$$q(\lambda) = \|\lambda + \varrho\|^2 - \|\varrho\|^2 = \|\lambda + \varrho_{\alpha}\|^2 - \|\varrho_{\alpha}\|^2. \quad (1.12c)$$

Proof. Write $[\pi_{\lambda}]$ for the class with highest weight λ . Simple connectivity and Theorem 1.8 insure that

$$\hat{G}_K \ni [\pi_{\lambda}] \rightarrow \lambda \in \Lambda^+$$

is bijective. The Hermann Weyl Degree Formula says that $[\pi_{\lambda}]$ has degree $P(\lambda)$ as in (1.12), and it is standard that π_{λ} acts on the Casimir element by

$$\pi_{\lambda}(\Omega) = \|\lambda + \varrho\|^2 - \|\varrho\|^2 \quad \text{for all } [\pi_{\lambda}] \in \hat{G}.$$

Here $\varrho = \varrho_{\alpha} + \varrho_t$ with $\langle \lambda, \varrho_t \rangle = 0$ by Theorem 1.7, so also

$$\pi_{\lambda}(\Omega) = \|\lambda + \varrho_{\alpha}\|^2 - \|\varrho_{\alpha}\|^2.$$

Now $\pi_{\lambda}(\Omega) = q(\lambda)$ as in (1.12) and our formula for $\zeta(t)$ follows from Corollary 1.4. *q.e.d.*

In the sequel we will explicitly calculate the ingredients (1.12) for symmetric spaces of rank 1 (that is, where $\dim \alpha = 1$), obtaining explicit formulae for their ζ -functions, and then study the asymptotic behaviour of these ζ -functions.

§2. Odd Dimensional Spheres and Real Projective Spaces

We work out explicit formulae for the ζ -functions of the spheres and real projective spaces of odd dimension $2n-1$,

$$S^{2n-1} = SO(2n)/SO(2n-1), \quad n \geq 1, \quad (2.1a)$$

and

$$P^{2n-1}(R) = S^{2n-1}/\{\pm I\} = SO(2n)/O(2n-1). \quad (2.1b)$$

If $n=1$, both are circles $S^1 = \{z \in \mathbb{C} : |z|=1\} = \{e^{i\theta} : \theta \text{ real}\}$. $L_2(S^1) = \sum_{-\infty}^{\infty} V_m$ where V_m is the 1-dimensional span of

$$f_m(e^{i\theta}) = e^{im\theta}, \quad m \text{ integer.}$$

Normalize the riemannian metric so that the circle has length l . Then the metric is $ds^2 = (l/2\pi)^2 d\theta^2$, so the circle has Laplace-Beltrami operator

$$\Delta = -(2\pi/l)^2 \frac{\partial^2}{\partial \theta^2} : f_m \mapsto (2\pi/l)^2 m^2 f_m.$$

We conclude that the circle of length l has ζ -function

$$\zeta_{S^1}(t) = 1 + 2 \sum_{m=1}^{\infty} e^{-t \{2\pi m/l\}^2} \quad (2.2)$$

If $n=2$ then $G=SO(4)$ has Dynkin diagram D_2 : $\underset{\alpha_1}{\circ} \quad \underset{\alpha_2}{\circ}$. Then $\Sigma^+ = \{\alpha_1, \alpha_2\}$ and $\Sigma_a^+ = \{\alpha\}$ where $\varrho = \varrho_a = \alpha = \frac{1}{2}(\alpha_1 + \alpha_2)$, so $\Lambda^+ = \{m\alpha : m \geq 0 \text{ integer}\}$ and $q(m\alpha) = \|m\alpha + \varrho_a\|^2 - \|\varrho_a\|^2 = (m^2 + 2m) \|\varrho_a\|^2$ and we calculate

$$P(m\alpha) = \frac{\langle m\alpha + \varrho, \alpha_1 \rangle}{\langle \varrho, \alpha_1 \rangle} \cdot \frac{\langle m\alpha + \varrho, \alpha_2 \rangle}{\langle \varrho, \alpha_2 \rangle} = (m+1)^2.$$

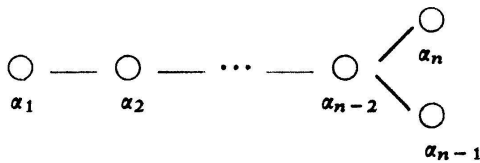
Using the negative of the Killing form to specify the riemannian metrics of S^3 and $P^3(R)$, the tables at the end of Bourbaki [2] show $\langle \alpha_i, \alpha_i \rangle = \frac{1}{2}$, so $\|\varrho_a\|^2 = \frac{1}{4} \langle \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 \rangle = \frac{1}{4}$, and

$$\zeta_{S^3}(t) = \sum_{m=0}^{\infty} (m+1)^2 e^{-t(m^2+2m)/4}. \quad (2.3a)$$

It is classical that $\pi_{m\alpha}(-I) = 1$ just when m is even, so also

$$\zeta_{P^3(R)}(t) = \sum_{r=0}^{\infty} (2r+1)^2 e^{-t(r^2+r)}. \quad (2.3b)$$

Now we assume $n \geq 3$ in (2.1) so that $G=SO(2n)$ is a simple group of type D_n , and denote its Dynkin diagram



with

$$\Sigma_a^+ = \{\alpha\}, \quad \alpha_1|_a = \alpha \quad \text{and} \quad \alpha_i|_a = 0 \quad \text{for } i > 1.$$

Relative to an appropriate positive multiple of the Cartan-Killing form, $i\mathfrak{h}^*$ has orthonormal basis $\{\varepsilon_1, \dots, \varepsilon_n\}$ such that

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1} \quad \text{for } 1 \leq i < n \quad \text{and} \quad \alpha_n = \varepsilon_{n-1} + \varepsilon_n.$$

Thus Σ^+ consists of the roots $\varepsilon_i \pm \varepsilon_j$ for $1 \leq i < j \leq n$, and so $\alpha = \varepsilon_1$ and $\varepsilon_1 \pm \varepsilon_j$ ($1 < j \leq n$) are the roots that restrict to α .

Now

$$\Lambda^+ = \{m\varepsilon_1 : m \geq 0 \text{ integer}\}, \quad \varrho_\alpha = (n-1)\varepsilon_1 \quad \text{and} \quad \varrho = \sum_{j=1}^{n-1} (n-j)\varepsilon_j.$$

If $1 \leq i < j \leq n$ then $\langle \varrho, \varepsilon_i \pm \varepsilon_j \rangle = \{(n-i) \pm (n-j)\} \|\varepsilon_1\|^2$, so

$$\frac{\langle m\varepsilon_1 + \varrho, \varepsilon_i \pm \varepsilon_j \rangle}{\langle \varrho, \varepsilon_i \pm \varepsilon_j \rangle} = 1 \quad \text{if } i > 1, = \frac{m + (n-1) \pm (n-j)}{(n-1) \pm (n-j)} \quad \text{if } i = 1.$$

That gives us

$$P(m\varepsilon_1) = \prod_{j=2}^n \frac{m+2n-j-1}{2n-j-1} \cdot \frac{m-1+j}{j-1} = \frac{m+n-1}{n-1} \prod_{k=1}^{2n-3} \frac{m+k}{k}.$$

Recall from the tables at the end of Bourbaki [2] that $\|\varepsilon_1\|^2 = 1/4(n-1)$. Now

$$q(m\varepsilon_1) = \|m\varepsilon_1 + \varrho_\alpha\|^2 - \|\varrho_\alpha\|^2 = \{m^2 + 2m(n-1)\}/4(n-1).$$

Now Corollary 1.10 gives us

2.4. THEOREM. *Let S^{2n-1} denote the sphere of odd dimension $2n-1$, $n \geq 3$, with riemannian metric of constant positive curvature induced by the negative of the Cartan-Killing form of $SO(2n)$. It has ζ -function*

$$\zeta_{S^{2n-1}}(t) = \sum_{m=0}^{\infty} \left\{ \frac{m+n-1}{n-1} \cdot \prod_{k=1}^{2n-3} \frac{m+k}{k} \right\} e^{-t \{m^2 + 2m(n-1)\}/4(n-1)}. \quad (2.5)$$

The real projective space $P^{2n-1}(R) = S^{2n-1}/\{\pm I\}$ has ζ -function given by summing the summands of (2.5) whose representations $[\pi_{m\varepsilon_1}]$ occur in $L_2(P^{2n-1}(R))$, that is the ones with a vector fixed under the subgroup $SO(2n-1) \cup (-I_{2n}) \cdot SO(2n-1)$. These are the $[\pi_{m\varepsilon_1}]$ whose kernel contains $-I_{2n}$, which are easily seen to be the ones for which m is even.

2.6. COROLLARY. *Let $P^{2n-1}(R)$ denote the real projective space of odd dimension $2n-1$, $n \geq 3$, with riemannian metric of constant positive curvature induced by the negative of the Cartan-Killing form of $SO(2n)$. It has ζ -function*

$$\zeta_{P^{2n-1}(R)}(t) = \sum_{r=0}^{\infty} \left\{ \frac{2r+n-1}{n-1} \cdot \prod_{k=1}^{2n-3} \frac{2r+k}{k} \right\} e^{-t \{r^2 + r(n-1)\}/(n-1)}. \quad (2.7)$$

§3. Even Dimensional Spheres and Real Projective Spaces

We work out explicit formulae for the ζ -functions of the spheres and real projective spaces of even dimension $2n$,

$$S^{2n} = SO(2n+1)/SO(2n), \quad n \geq 1, \quad (3.1a)$$

and

$$P^{2n}(R) = S^{2n}/\{\pm I\} = SO(2n+1)/SO(2n) \times O(1). \quad (3.1b)$$

$G = SO(2n+1)$ has Dynkin diagram

$$B_n: \underset{\alpha_1}{\bigcirc} \text{ --- } \underset{\alpha_2}{\bigcirc} \text{ --- } \cdots \text{ --- } \underset{\alpha_{n-1}}{\bigcirc} \text{ = } \underset{\alpha_n}{\bullet}$$

with

$$\Sigma_{\alpha}^+ = \{\alpha\}, \quad \alpha_1|_{\alpha} = \alpha \quad \text{and} \quad \alpha_i|_{\alpha} = 0 \quad \text{for } i > 1.$$

Arguing as in §2 one proves

3.2. THEOREM. *Let S^{2n} denote the sphere of even dimension $2n$ with riemannian metric of constant positive curvature induced by the negative of the Cartan-Killing form of $SO(2n+1)$. It has ζ -function*

$$\zeta_{S^{2n}}(t) = \sum_{m=0}^{\infty} \left\{ \frac{2m+2n-1}{2n-1} \prod_{k=1}^{2n-2} \frac{m+k}{k} \right\} e^{-t \{m^2 + m(2n-1)\}/(4n-2)}. \quad (3.3)$$

and

3.4. COROLLARY. *Let $P^{2n}(R)$ denote the real projective space of even dimension $2n$ with riemannian metric of constant positive curvature induced by the negative of the*

Cartan-Killing form $SO(2n+1)$. It has ζ -function

$$\zeta_{P^{2n}(R)}(t) = \sum_{r=0}^{\infty} \left\{ \frac{4r+2n-1}{2n-1} \prod_{k=1}^{2n-2} \frac{2r+k}{k} \right\} e^{-t \{2r^2 + r(2n-1)\}/(2n-1)}. \quad (3.5)$$

§4. Complex Projective Spaces

We state the formula for the ζ -function of the complex projective spaces

$$P^n(C) = U(n+1)/U(n) \times U(1) = SU(n+1)/S(U(n) \times U(1)) \quad (4.1)$$

of complex dimension n , real dimension $2n$. Since $P^1(C)$ is the sphere S^2 , already considered in §3, we will work under the hypothesis $n > 1$. Then a glance at the case $n=1$ of (3.3) will show our conclusion valid in general.

$G = SU(n+1)/\{e^{2\pi i k/(n+1)} I\}$ has Dynkin diagram

$$A_n: \underset{\alpha_1}{\bigcirc} \text{ --- } \underset{\alpha_2}{\bigcirc} \text{ --- } \cdots \text{ --- } \underset{\alpha_n}{\bigcirc}$$

with

$$\Sigma^+ = \{\alpha, 2\alpha\}, \quad \alpha_1|_{\alpha} = \alpha = \alpha_n|_{\alpha}, \quad \alpha_i|_{\alpha} = 0 \quad \text{for } 1 < i < n.$$

Arguing as before and using the case $n=1$ of Theorem 3.2,

4.2. THEOREM. *Let $P^n(C)$ denote the complex projective n -space with riemannian metric induced by the negative of the Cartan-Killing form of $SU(n+1)$. It has ζ -function*

$$\zeta_{P^n(C)}(t) = \sum_{m=0}^{\infty} \left\{ \frac{2m+n}{n} \prod_{k=0}^{n-1} \left(\frac{m+k}{k} \right)^2 \right\} e^{-t \{m^2 + mn\}/(n+1)} \quad (4.3)$$

§5. Quaternionic Projective Spaces

Here is the formula for the ζ -functions of the quaternionic projective spaces

$$P^{n-1}(Q) = Sp(n)/Sp(n-1) \times Sp(1), \quad n \geq 2, \quad (5.1)$$

of real dimension $4(n-1)$. Here note that $P^1(Q) = S^4$.

$G = Sp(n)/\{\pm I\}$ has Dynkin diagram

$$C_n: \underset{\alpha_1}{\bullet} \text{ --- } \underset{\alpha_2}{\bullet} \text{ --- } \cdots \text{ --- } \underset{\alpha_{n-1}}{\bullet} \text{ --- } \underset{\alpha_n}{\bigcirc}$$

with

$$\Sigma_a^+ = \{\alpha, 2\alpha\}, \quad \alpha_2|_a = \alpha, \alpha_i|_a = 0 \quad \text{for } i \neq 2.$$

An argument similar to that of §2 gives

5.2. THEOREM. *Let $P^{n-1}(Q)$ denote the quaternionic projective $n-1$ space, with riemannian metric induced by the negative of the Cartan-Killing form of $Sp(n)$. It has ζ -function*

$$\zeta_{P^{n-1}(Q)}(t) = \sum_{m=0}^{\infty} \left\{ \frac{2m+2n-1}{2n-1} \cdot \prod_{r=2}^{2n-2} \frac{m+r}{r} \cdot \prod_{s=1}^{2n-3} \frac{m+s}{s} \right\} e^{-t(m^2+2mn-m)/2(n+1)}.$$

Notice that the case $n=2$ is $P^1(Q) = S^4$, where both (3.3) and (5.3) provide the same ζ -function

$$\zeta_{S^4}(t) = \sum_{m=0}^{\infty} \left\{ \frac{2m+3}{3} \cdot \frac{m+2}{2} \cdot \frac{m+1}{1} \right\} e^{-t(m^2+3m)/6} \quad (5.4)$$

§6. The Cayley Projective Plane

Finally, we work out the ζ -function for the Cayley projective plane

$$P^2(\text{Cay}) = F_4/\text{Spin}(9), \quad \text{real dimension } 16. \quad (6.1)$$

$G = F_4$ has Dynkin diagram

$$\begin{array}{cccc} \bigcirc & \text{---} & \bigcirc & \text{---} & \bullet & \text{---} & \bullet \\ \alpha_1 & & \alpha_2 & & \alpha_3 & & \alpha_4 \end{array}$$

with $\Sigma^+ = \{\alpha, 2\alpha\}$ where $\alpha_4|_a = \alpha$ and the other three $\alpha_i|_a = 0$. Relative to an appropriate multiple of the Cartan-Killing form, $i\hbar^*$ has orthonormal basis $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$ with

$$\alpha_1 = \varepsilon_2 - \varepsilon_3, \alpha_2 = \varepsilon_3 - \varepsilon_4, \alpha_3 = \varepsilon_4 \quad \text{and} \quad \alpha_4 = \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4).$$

Thus Σ^+ consists of the roots

$$\varepsilon_i (1 \leq i \leq 4), \varepsilon_i \pm \varepsilon_j (1 \leq i < j \leq 4), \quad \frac{1}{2}(\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4).$$

Now $\alpha = \frac{1}{2}\varepsilon_1$ and the $\frac{1}{2}(\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)$ are the roots restricting to α , $2\alpha = \varepsilon_1$ and the roots restricting to it are $\varepsilon_1, \varepsilon_1 \pm \varepsilon_2, \varepsilon_1 \pm \varepsilon_3, \varepsilon_1 \pm \varepsilon_4$.

Thus

$$\Lambda^+ = \{m\varepsilon_1 : m \geq 0 \text{ integer}\}, \quad \varrho_\alpha = 11\alpha = \frac{11}{2}\varepsilon_1, \quad \varrho = \frac{1}{2}(11\varepsilon_1 + 5\varepsilon_2 + 3\varepsilon_3 + \varepsilon_4).$$

Now calculate

$$P(m\varepsilon_1) = \frac{2m+11}{11} \cdot \prod_{q=1}^3 \frac{m+q}{q} \prod_{r=4}^7 \left(\frac{m+r}{r}\right)^2 \cdot \prod_{s=8}^{16} \frac{m+s}{s}.$$

From the tables at the end of Bourbaki [2], $\|\varepsilon_1\|^2 = 1/18$, so $\|\alpha\|^2 = 1/72$, and thus

$$q(m\varepsilon_1) = \|(2m+11)\alpha\|^2 - \|11\alpha\|^2 = (m^2 + 11m)/18.$$

Now Corollary 1.10 says

6.2. THEOREM. *Let P^2 (Cay) denote the Cayley projective plane with riemannian metric induced by the negative of the Cartan-Killing form of F_4 . It has ζ -function*

$$\zeta_{P^2(\text{Cay})}(t) = \sum_{m=0}^{\infty} \left\{ \frac{2m+11}{11} \cdot \prod_{q=1}^3 \frac{m+q}{q} \cdot \prod_{r=4}^7 \left(\frac{m+r}{r}\right)^2 \cdot \prod_{s=8}^{10} \frac{m+s}{s} \right\} e^{-t(m^2+11m)/18}. \quad (6.3)$$

§ 7. The Asymptotic Expansion for Compact Riemannian Manifolds

We wish to give a brief account of the properties of the eigenvalues of the Laplacian, Δ , of a compact riemannian manifold. We will assume (M, g) is a compact riemannian manifold without boundary of dimension d . Then Δ will be a self-adjoint elliptic operator with eigenvalues $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$. It is known that these eigenvalues contain a great deal of geometric information about (M, g) and a tool to recover some of this information is the zeta-function, $\zeta_M(t) = \sum_{k=0}^{\infty} e^{-\lambda_k t}$. The interest of the zeta-function comes from the following theorem.

7.1. THEOREM (Minakshisundaram). *If (M, g) is a compact riemannian manifold without boundary of dimension d then there exist constants a_n , $n \geq 0$, such that*

$$\zeta_M(t) = (4\pi t)^{-d/2} (a_0 + a_1 t + \dots + a_k t + o(t^{k+1}))$$

as $t \downarrow 0$.

Proof. See Berger [1].

In the remainder of this paper we will compute the coefficients, a_m , of the asymptotic expansion of $\zeta_M(t)$ when M is a symmetric space of rank one.

§8. Summation Lemmas

We wish to analyze the zeta-functions derived in the first part of this paper using certain classical summation formulas. The formulas we need will be gathered together in this section.

8.1. LEMMA. Let $f(t) = \sum_{s \in \mathbf{Z}} e^{-s^2 t}$. Then $f(t) = \pi^{1/2} t^{-1/2} + O(e^{-1/t})$ as $t \downarrow 0$ and $(-1)^k f^{(k)} = \sum_{s \in \mathbf{Z}} s^{2k} e^{-s^2 t} = (\frac{1}{2}) (\frac{3}{2}) \cdots (2k-1)/2 \pi^{1/2} t^{-(2k+1)/2} + O(e^{-1/t})$ as $t \downarrow 0$.

Proof. If $r(x) = e^{-x^2 t}$ we may apply the Poisson summation formula to derive

$$\sum_{s \in \mathbf{Z}} e^{-s^2 t} = \pi^{1/2} t^{-1/2} \sum_{s \in \mathbf{Z}} e^{-\pi^2 s^2 / t}. \quad (8.2)$$

The first part of the Lemma now follows by noting that $\sum_{s \neq 0} e^{-\pi^2 s^2 / t}$ is $O(e^{-1/t})$. We may now take derivatives with respect to t to derive the second formula. *q.e.d.*

For the remainder of the paper we will define $b_0 = 1$, $b_k = (\frac{1}{2}) (\frac{3}{2}) \cdots ((2k-1)/2)$, $k \geq 1$. Note $\pi^{1/2} b_k = \Gamma((2k+1)/2)$.

8.3. LEMMA. Let $g(t) = \sum_{j=0}^{\infty} (2j+1) e^{-(j+1/2)^2 t}$. Then

$$g(t) = \frac{1}{t} + c_0 + c_1 t + \frac{c_2}{2!} t^2 + \cdots + \frac{c_n}{n!} t^n + O(t^{n+1})$$

and

$$g^{(k)}(t) = \frac{(-1)^k k!}{t^{k+1}} + c_k + c_{k+1} t + \cdots + \frac{c_{k+n}}{n!} t^n + O(t^{n+1})$$

as $t \downarrow 0$, where

$$c_n = \frac{(-1)^n}{(n+1)} B_{2n+2} (1 - 2^{-2n-1}),$$

B_n is the n th Bernoulli number.

Proof. See Mulholland [5].

Before proceeding we wish to note that $\sum_{n=0}^{\infty} (c_n/n!) t^n$ is not convergent. Therefore Lemma 8.3 gives an asymptotic series.

8.4. LEMMA. Let $g_1(t) = \sum_{j=0}^{\infty} (4j+1) e^{-(2j+1/2)^2 t}$ and $g_2(t) = \sum_{j=0}^{\infty} (4j+3) e^{-(2j+3/2)^2 t}$. Then $g_1(t) + g_2(t) = g(t)$ and

$$g_i^{(k)}(t) = \frac{1}{2} \left(\frac{1}{t} + c_0 + c_1 t + \cdots + \frac{c_n}{n!} t^n + O(t^{n+1}) \right) \quad i = 1, 2$$

$$g_i^{(k)}(t) = \frac{1}{2} \left(\frac{(-1)^k k!}{t^{k+1}} + c_k + c_{k+1} t + \cdots + \frac{c_{k+n}}{n!} t^n + O(t^{n+1}) \right) \quad i = 1, 2$$

as $t \downarrow 0$.

Proof. See Mulholland [5].

The last lemma of this section is similar to Lemma 8.3.

8.5. LEMMA. Let $h(t) = \sum_{j=0}^{\infty} 2j e^{-j^2 t}$. Then

$$h(t) = \frac{1}{t} + d_0 + d_1 t + \frac{d_2}{2!} t^2 + \cdots + \frac{d_n}{n!} t^n + O(t^{n+1})$$

and

$$h^{(k)}(t) = \frac{(-1)^k k!}{t^{k+1}} + d_k + d_{k+1} t + \cdots + \frac{d_{n+k}}{n!} t^n + O(t^{n+1})$$

as $t \downarrow 0$ with $d_n = [(-1)^n / (n+1)] B_{2n+2}$.

Proof. A slight modification of Mulholland's method gives the desired result.

§9. The Asymptotic Expansion for Odd Dimensional Spheres and Real Projective Spaces

Starting in this section we will analyze the zeta-functions developed in §1 through §6 using the results in the previous section. The goal of this section is to calculate the coefficients, a_n , for the symmetric spaces analyzed in §2.

$M = S^1$. $\zeta_M(t) = f((4\pi^2/l^2)t)$. Consequently

$$\begin{aligned} \zeta_M(t) &= \pi^{1/2} \left(\frac{4\pi^2}{l^2} t \right)^{-1/2} + O(e^{-1/t}) \\ &= l(4\pi t)^{-1/2} + O(e^{-1/t}). \end{aligned}$$

Thus we conclude $a_0 = l$ and $a_m = 0$, $m \geq 1$.

$$M = S^3. \quad \zeta_M(t) = \sum_{m=0}^{\infty} (m+1)^2 e^{-t(m^2+2m)/4}$$

$$\begin{aligned}
&= \sum_{p=0}^{\infty} p^2 e^{-t(p^2-1)/4} \quad (\text{where } p=m+1) \\
&= \frac{1}{2} e^{t/4} \sum_{p=-\infty}^{\infty} p^2 e^{-p^2 t/4} \\
&= \frac{1}{2} e^{t/4} (-1) f' \left(\frac{t}{4} \right) \\
&= 2\pi^{1/2} t^{-3/2} e^{t/4} + ES = \frac{16\pi^2 e^{t/4}}{(4\pi t)^{3/2}} + ES,
\end{aligned}$$

therefore $a_m = 16\pi^2/4^m m!$. ES is an error which is exponentially small as $t \downarrow 0$.

$$M = P^3(\mathbf{R}).$$

$$\begin{aligned}
\zeta_M(t) &= \sum_{r=0}^{\infty} (2r+1)^2 e^{-(r^2+r)t} \\
&= \sum_{r=0}^{\infty} (2r+1)^2 e^{-[(2r+1)^2-1]t/4} \\
&= \frac{1}{2} e^{t/4} \left[\sum_{s \in \mathbf{Z}} s^2 e^{-s^2 t/4} - \sum_{s \in 2\mathbf{Z}} s^2 e^{-s^2 t/4} \right] \\
&= \frac{1}{2} e^{t/4} \left[(-1) f' \left(\frac{t}{4} \right) + 4f'(t) \right] \\
&= \frac{\pi^{1/2}}{2} e^{t/4} [4t^{-3/2} - 2t^{-3/2}] + ES \\
&= \pi^{1/2} t^{-3/2} e^{t/4} + ES = \frac{8\pi^2 e^{t/4}}{(4\pi t)^{3/2}} + ES
\end{aligned}$$

so $a_m = 8\pi^2/4^m m!$.

$M = S^{2a-1}$, $n \geq 3$. At this point we wish to make a general comment about the procedure we will employ in this and following sections. Though we have established a bijection between the representations of $\hat{G}_K$ and functionals $\lambda \in \Lambda^+$ the proper variable to use is not λ but $\lambda + \varrho_\alpha$. This is the guiding principle behind all changes of variable which are used in this and following sections.

$$\zeta_M(t) = \sum_{m=0}^{\infty} \left\{ \frac{m+n-1}{n-1} \prod_{k=0}^{2n-3} \frac{m+k}{k} \right\} e^{-t\{m^2+2m(n-1)\}/4(n-1)}.$$

We will let $s = m + n - 1$. Then

$$\begin{aligned}
\zeta_M(t) &= \frac{1}{(n-1)(2n-3)!} \sum_{s=n-1}^{\infty} \left\{ \prod_{j=0}^{n-2} (s^2 - j^2) \right\} e^{-\{s^2 - (n-1)^2\} t/4 (n-1)} \\
&= \frac{e^{((n-1)/4)t}}{2(n-1)(2n-3)!} \sum_{s \in \mathbf{Z}} \left\{ \prod_{j=0}^{n-2} (s^2 - j^2) \right\} e^{-s^2 t/4 (n-1)}
\end{aligned} \tag{9.1}$$

We now define $\alpha_{k,n}$ by

$$\prod_{j=0}^{n-2} (s^2 - j^2) = \sum_{k=0}^{n-1} \alpha_{k,n} s^{2k}.$$

Then

$$\begin{aligned}
\zeta_M(t) &= \frac{e^{(n-1)t/4}}{(2n-2)!} \sum_{s \in \mathbf{Z}} \sum_{k=0}^{n-1} \alpha_{k,n} s^{2k} e^{-s^2 (t/4 (n-1))} \\
&= \frac{e^{(n-1)t/4}}{(2n-2)!} \sum_{k=0}^{n-1} \alpha_{k,n} (-1)^k f^{(k)}\left(\frac{t}{4(n-1)}\right) \\
&= \frac{\pi^{1/2} e^{(n-1)t/4}}{(2n-2)!} \sum_{k=0}^{n-1} \alpha_{k,n} b_k \left(\frac{t}{4(n-1)}\right)^{-(2k+1)/2} + ES.
\end{aligned}$$

Now by convolving the series we conclude that

$$a_m = \frac{2^{2n-1} \pi^n}{(2n-2)!} \sum_{k=0}^m \frac{(n-1)^{n-1/2}}{k!} \alpha_{n-1-k,n} b_{n-1-k} 4^{n-1/2-2k} \quad \text{if } m < n$$

and

$$a_m = \frac{2^{2n-1} \pi^n}{(2n-2)!} \sum_{k=m-n}^m \frac{(n-1)^{m-1/2}}{k!} \alpha_{m-k-1,n} b_{m-k-1} 4^{m-1/2-2k}$$

if $m \geq n$.

$M = P^{2n-1}(\mathbf{R})$. To compute the asymptotic expansion for the projective spaces we take $s = r + (n-1)/2$. Then

$$\begin{aligned}
\frac{2r+n-1}{n-1} \prod_{j=1}^{2n-3} \frac{2r+k}{k} &= \frac{2}{(2n-2)!} \prod_{j=0}^{n-2} 4(s^2 - (j/2)^2) = \frac{4^{n-1/2}}{(2n-2)!} \sum_{j=0}^{n-1} \alpha'_{j,n} s^{2j}. \\
\zeta_M(t) &= \frac{4^{n-1}}{(2n-2)!} e^{(n-1)t/4} \sum_{s \in 1/2\mathbf{Z}} \sum_{j=0}^{n-1} \alpha'_{j,n} s^{2j} e^{-s^2 t/n-1}
\end{aligned}$$

$$\begin{aligned}
&= \frac{4^{n-1}}{(2n-2)!} e^{(n-1)t/4} \sum_{j=0}^{n-1} \alpha'_{j,n} 4^{-j} f^{(j)}\left(\frac{t}{4(n-1)}\right) \\
&= \frac{4^{n-1} \pi^{1/2}}{(2n-2)!} e^{(n-1)t/4} \sum_{j=0}^{n-1} \alpha'_{j,n} b_j 4^{-j} \left(\frac{t}{4(n-1)}\right)^{-(2j+1)/2} + ES.
\end{aligned}$$

Therefore

$$a_m = \frac{4^{n-1} \pi^n}{(2n-2)!} \sum_{j=0}^m \frac{(n-1)^{n-1/2}}{j!} \alpha'_{n-1-j,n} b_{n-1-j} 4^{-j-1/2} \quad \text{if } m < n$$

and

$$a_m = \frac{4^{n-1} \pi^n}{(2n-2)!} \sum_{j=m-n}^m \frac{(n-1)^{m-1/2}}{j!} \alpha'_{m-j-1,n} b_{m-j-1} 4^{-j-1/2} \quad \text{if } m \geq n.$$

§10. The Asymptotic Expansion for Even Dimensional Spheres and Real Projective Spaces

$M = S^{2n}$. As in the preceding section we will change variables to utilize the lemmas of §8. Recall that

$$\zeta_M(t) = \sum_{m=0}^{\infty} \left\{ \frac{2m+2n-1}{2n-1} \prod_{k=1}^{2n-2} \frac{m+k}{k} \right\} e^{-t \{m^2 + m(2n-1)\}/4n-2}.$$

We will let $s = m + (n-1/2)$. Then

$$\frac{2m+2n-1}{2n-1} \prod_{k=1}^{2n-2} \frac{m+k}{k} = \frac{2s}{(2n-1)!} \prod_{j=1/2}^{n-3/2} (s^2 - j^2) = \frac{2s}{(2n-1)!} \sum_{j=0}^{n-1} \beta_{j,n} s^{2j},$$

where the product runs through the half-integers which are not integers. Also $t \{m^2 + m(2n-1)\}/4n-2 = t \{s^2 - (n-1/2)^2\}/4n-2$. Therefore

$$\begin{aligned}
\zeta_M(t) &= \frac{e^{(n-1/2)t/4}}{(2n-1)!} \sum_{s \geq 1/2} \sum_{j=0}^{n-1} \beta_{j,n} 2s^{2j+1} e^{-s^2 t/4(n-1/2)} \\
&= \frac{e^{(n-1/2)t/4}}{(2n-1)!} \sum_{j=0}^{n-1} \beta_{j,n} (-1)^j g^{(j)}\left(\frac{t}{4(n-1/2)}\right)
\end{aligned}$$

Thus

$$a_m = \frac{(4\pi)^2}{(2n-1)!} \sum_{k=0}^m \frac{(n-1-m+k)!}{k!} 4^{n-m-1} (n-1/2)^{m-n+2k+1} \beta_{n-1-m+k,n} \quad \text{if } m < n$$

and

$$a_m = \frac{(4\pi)^n}{(2n-1)!} \left(\sum_{k=0}^{n-1} \frac{k!}{(m-n+k+1)!} (n-1/2)^{m-n+2k+2} \beta_{k,n} 4^{n-m} \right. \\ \left. + \sum_{k=0}^{m-n} \sum_{j=0}^{n-1} \frac{(-1)^j c_{j+k} \beta_{j,n} (n-1/2)^{m-n-2k}}{k! (m-n-k)!} 4^{n-m} \right)$$

if $m \geq n$.

$M = P^{2^n}(\mathbf{R})$. Recall that

$$\zeta_M(t) = \sum_{r=0}^{\infty} \left\{ \frac{4r+2n-1}{2n-1} \prod_{k=0}^{2n-2} \frac{2r+k}{k} \right\} e^{-t \{2r^2 + r(2n-1)\}/(2n-1)}.$$

We will let $s = r + ((2n-1)/4)$. Then

$$\frac{4r+2n-1}{2n-1} \prod_{k=1}^{2n-2} \frac{2r+k}{k} = \frac{4s}{(2n-1)!} \prod_{j=1/2}^{n-3/2} \left(4s^2 - \frac{j^2}{4} \right) = \frac{4s}{(2n-1)!} \sum_{j=0}^{n-1} \beta'_{j,n} (2s)^{2j}$$

and

$$t \{2r^2 + r(2n-1)\}/(2n-1) = t \left(2s^2 - 2 \left(\frac{2n-1}{4} \right)^2 \right) / (2n-1).$$

Then

$$\zeta_M(t) = \frac{e^{t((2n-1)/8)}}{(2n-1)!} \sum_{s \geq 1/2} \sum_{j=0}^{n-1} \beta'_{j,n} 4s (2s)^{2j} e^{-2s^2 t / (2n-1)} \\ = \frac{e^{t(n-1/2)/4}}{(2n-1)!} \sum_{j=0}^{n-1} \beta'_{j,n} (-1)^j g_i^{(j)} \left(\frac{t}{2(2n-1)} \right)$$

where $i=1$ if n is odd and $i=2$ if n is even.

Thus

$$a_m = \frac{(4\pi)^n}{2(2n-1)!} \sum_{k=0}^m \frac{(n-1-m+k)!}{k!} 4^{n-m-1} (n-1/2)^{n-m+2k+1} \beta'_{n-1-m+k,n}$$

if $m < n$ and

$$a_m = \frac{(4\pi)^m}{2(2n-1)!} \left(\sum_{k=0}^{n-1} \frac{k!}{(m-n+k+1)!} (n-1/2)^{m-n+2k+2} \beta'_{k,n} 4^{n-m} \right. \\ \left. + \sum_{k=0}^{m-n} \sum_{j=0}^{n-1} \frac{(-1)^j c_{j+k} \beta'_{j,n} (n-1/2)^{m-n-2k}}{k! (m-n-k)!} 4^{n-m} \right) \text{ if } m \geq n.$$

§11. The Asymptotic Expansion for Complex Projective Spaces

We will have to treat $P^n(\mathbb{C})$ with 2 separate arguments according to whether n is odd or even. The reason for this division is that when n is even $\varrho_a \in \Lambda^+$ while when n is odd $\varrho \notin \Lambda^+$. The treatment of the 2 cases will then differ only in that when n is even we will use Lemma 8.3 while for n odd we will use Lemma 8.5.

$M = P^n(\mathbb{C}), n$ odd

$$\zeta_M(t) = \sum_{m=0}^{\infty} \left\{ \frac{2m+n}{n} \prod_{k=1}^{n-1} \left(\frac{m+k}{k} \right)^2 \right\} e^{-t \{m^2 + mn\}/(n+1)}.$$

Let $s = m + (n/2)$. Then

$$\frac{2m+n}{n} \prod_{k=1}^{n-1} \left(\frac{m+k}{k} \right)^2 = \frac{2s}{n! (n-1)!} \prod_{j=1/2}^{n/2-1} (s^2 - j^2)^2 = \frac{2s}{n! (n-1)!} \sum_{k=0}^{n-2} \gamma_{k,n} s^{2k}$$

and $t \{m^2 + mn\}/n+1 = t \{s^2 - (n/2)^2\}/n+1$. Therefore

$$\begin{aligned} \zeta_M(t) &= \frac{e^{tn^2/4(n+1)}}{n! (n-1)!} \sum_{k=0}^{n-2} \gamma_{k,n} \sum_{s \geq 1/2} 2s^{2k+1} e^{-s^2 t/(n+1)} \\ &= \frac{e^{tn^2/4(n+1)}}{n! (n-1)!} \sum_{k=0}^{n-2} (-1)^k \gamma_{k,n} g^{(k)}(t/(n+1)). \end{aligned}$$

Thus

$$a_m = \frac{(4\pi)^{n-1}}{n! (n-1)!} \sum_{k=0}^m (n+1)^{n-1-m} \left(\frac{n}{2} \right)^{2k} \frac{(n-m+k-2)!}{k!} \gamma_{n-m+k-2,n}$$

if $m < n-1$ and

$$\begin{aligned} a_m &= \frac{(4\pi)^{n-1}}{n! (n-1)!} \left(\sum_{k=0}^{n-2} \frac{k!}{(m-n+2+k)!} \left(\frac{n}{2} \right)^{2(m-n+2+k)} \gamma_{k,n} (n+1)^{n-m-1} \right. \\ &\quad \left. + \sum_{k=0}^{m-n+1} \left(\frac{n^2}{4(n+1)} \right)^k \frac{1}{k!} \sum_{j=0}^{n-2} \frac{(-1)^j \gamma_{j,n} c_{m-n+1-k+j}}{(m-n+1-k)!} (n+1)^{m-n+1-k} \right) \\ &\quad \text{if } m \geq n-1. \end{aligned}$$

$M = P^n(\mathbb{C}), n$ even. Since n is even we will let $n = 2n_0$. If $s = m + n_0$ then we will write

$$\prod_{k=1}^{n-1} (m+k)^2 = \prod_{k=0}^{n_0-1} (s^2 - j^2)^2 = \sum_{k=0}^{n-2} \gamma_{k,n} s^{2k}.$$

Then

$$\begin{aligned}\zeta_M(t) &= \frac{e^{tn_0^2/(n+1)}}{n!(n-1)!} \sum_{k=0}^{n-2} \gamma_{k,n} \sum_{s \geq 0} 2s^{2k+1} e^{-s^2 t/(n+1)} \\ &= \frac{e^{tn_0^2/(n+1)}}{n!n-1!} \sum_{k=0}^{n-2} (-1)^k \gamma_{k,n} h^{(k)}(t/(n+1)).\end{aligned}$$

Thus

$$a_m = \frac{(4\pi)^{n-1}}{n!(n-1)!} \sum_{k=0}^m (n+1)^{n-1-m} n_0^{2k} \frac{(n-m+k-2)!}{k!} \gamma_{n-m+k-2,n} \quad \text{if } m < n-1$$

and

$$\begin{aligned}a_m &= \frac{(4\pi)^{n-1}}{n!(n-1)!} \left(\sum_{k=0}^{n-2} \frac{k!}{(m-n+2+k)!} n_0^{2(m-n+2+k)} \gamma_{k,n} (n+1)^{n-m-1} \right. \\ &\quad \left. + \sum_{k=0}^{m-n-1} \left(\frac{n_0^2}{n+1} \right)^k \frac{1}{k!} \sum_{j=0}^{n-2} \frac{(-1)^j \gamma_{j,n} d_{m-n+1-k+j}}{(m-n+1-k)!} (n+1)^{m-n+1-k} \right) \quad \text{if } m \geq n-1.\end{aligned}$$

§12. The Asymptotic Expansion of Quaternionic Projective Spaces

If $M = P^{n-1}(Q)$, $n \geq 2$ then recall that

$$\zeta_M(t) = \sum_{m=0}^{\infty} \frac{2m+2n-1}{2n-1} \prod_{r=2}^{2n-2} \frac{m+r}{r} \prod_{P=1}^{2n-2} \frac{m+P}{P} e^{-t(m^2+2mn-m)/2(n+1)}.$$

We will let $s = m + (n-1/2)$. Then

$$\begin{aligned}\frac{2m+2n-1}{2n-1} \prod_{r=2}^{2n-2} \frac{m+r}{r} \prod_{P=1}^{2n-2} \frac{m+P}{P} &= \frac{2s}{(2n-1)!(2n-3)!} \prod_{j=1/2}^{n-3/2} (s^2-j^2) \prod_{j=1/2}^{n-5/2} (s^2-j^2) \\ &= \frac{2s}{(2n-1)!(2n-3)!} \sum_{k=0}^{2n-3} \delta_{k,n} s^{2k} \\ t(m^2+2mn-m)/2(n+1) &= t(s^2-(n-1/2)^2)/2(n+1).\end{aligned}$$

Therefore

$$\begin{aligned}\zeta_M(t) &= \frac{e^{t(n-1/2)^2/2(n+1)}}{(2n-1)!(2n-3)!} \sum_{k=0}^{2n-3} \sum_{s \geq 1/2} \delta_{k,n} 2s^{2k+1} e^{-s^2} \\ &= \frac{e^{t(n-1/2)^2/2(n+1)}}{(2n-1)!(2n-3)!} \sum_{k=0}^{2n-3} \delta_{k,n} (-1)^k g^{(k)}(t)\end{aligned}$$

Thus

$$a_m = \frac{(4\pi)^{2n-2}}{(2n-1)!(2n-3)!} \sum_{k=0}^m \left(\frac{(n-1/2)^2}{2(n+1)} \right)^k \frac{(2n-3-m+k)!}{k!} \delta_{2n-3-m+k, n}$$

if $m < 2n-2$

and

$$a_m = \frac{(4\pi)^{2n-2}}{(2n-1)!(2n-3)!} \left(\sum_{k=0}^{2n-3} \left(\frac{(n-1/2)^2}{2(n+1)} \right)^{2(m+2n-3-k)} \frac{k!}{(m+2n-3-k)!} \delta_{k, n} \right. \\ \left. + \sum_{k=0}^{m-2n+2} \frac{(n-1/2)^{2k}}{2^k (n+1)^k k!} \sum_{j=0}^{2n-3} \frac{(-1)^j \delta_{j, n} c_{j+m-k}}{(m-k)!} \right) \text{ if } m \geq 2n-2.$$

§13. The Asymptotic Expansion of the Cayley Projective Plane

We will deal with the Cayley projective plane by letting $s = m + 11/2$. Then

$$P(m\varepsilon_1) = \frac{3!}{11!7!} 2s(s^2 - (1/2)^2)^2 (s^2 - (3/2)^2)^2 (s^2 - (5/2)^2) (s^2 - (7/2)^2) (s^2 - (9/2)^2) \\ = \frac{3!}{11!7!} 2s \sum_{j=0}^7 \eta_j s^{2j}$$

with

$$\eta_7 = 1, \quad \eta_6 = -\frac{170}{4}, \quad \eta_5 = \frac{10,437}{16}, \quad \eta_4 = -\frac{262,075}{64}, \quad \eta_3 = \frac{2,858,418}{256}, \\ \eta_2 = -\frac{13,020,525}{1024}, \quad \eta_1 = \frac{18,455,239}{4096}, \quad \eta_0 = -\frac{8,037,225}{16,384}.$$

$$t(m^2 + 11m)/18 = t(s^2 - (11/2)^2)/18 \text{ so}$$

$$\zeta_{P^2(\text{Cay})} = \frac{3!}{7!11!} e^{(121/72)t} \sum_{j=0}^7 \sum_{s \geq 1/2} \eta_j 2s^{2j+1} e^{-s^2 t} \\ = \frac{3!}{7!11!} e^{(121/72)t} \sum_{j=0}^7 \eta_j (-1)^j g^{(j)}(t).$$

Therefore

$$a_m = \frac{3!}{7!11!} (4\pi)^8 \sum_{k=0}^m \left(\frac{121}{72} \right)^k \eta_{7-m+k} \frac{(7-m+k)!}{k!} \text{ if } m \leq 7 \\ a_m = \frac{3!}{7!11!} (4\pi)^8 \left(\sum_{k=0}^7 \left(\frac{121}{72} \right)^{(m+7-k)} \eta_k \frac{k!}{(m+7-k)!} \right. \\ \left. + \sum_{k=0}^{m-8} \frac{(121/72)^k}{k!} \sum_{j=0}^7 \frac{(-1)^j \eta_j c_{j+m-k}}{(m-k)!} \right), \text{ if } m > 7.$$

BIBLIOGRAPHY

- [1] BERGER, M. GAUDUCHON, P. and MAZET, E., *Le Spèctre d'une Variété Riemannienne*, Springer-Verlag, Berlin, 1971.
- [2] BOURBAKI, N., *Éléments de Mathématique, Groupes et algèbres de Lie*, Chapitres 4–6, Hermann, Paris, 1968.
- [3] CARTAN, É., *Sur la Détermination d'un Système Orthogonal Complet dans un Espace de Riemann Symétrique Clos*, Rendiconti del Circolo Mate. de Palermo, 53, (1929), pp. 217-252.
- [4] HELGASON, S., *Differential Geometry and Symmetric Spaces*, Academic Press, New York, 1962.
- [5] MULHOLLAND, H. P., *An Asymptotic Expansion for $\sum_0^\infty (2n+1) e^{-\sigma(n+1/2)^2}$* , Proc. Cam. Phil. Soc., 24 (1928), pp. 280–289.
- [6] WARNER, G., *Harmonic Analysis on Semi-Simple Lie Groups I*, Springer-Verlag, Berlin, 1972.
- [7] HELGASON, S., *A duality for symmetric spaces with applications to group representations*, Advances in Math., 5 (1970), pp. 1–154.

Received January 27/April 13, 1975

