Comment. Math. Helvetici 54 (1979) 105–110 Birkhäuser Verlag, Basel

The normality of closures of orbits in a Lie algebra

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Abstract. Let X be the closure of a G-orbit in the Lie algebra of a connected reductive group G. It seems that the variety X is always normal. After a reduction to nilpotent orbits, this is proved for some special cases. Results on determinantal schemes are used for Gl_n. If X is small enough we use a resolution and Bott's theorem on the cohomology of homogeneous vector bundles. Our results are conclusive for groups of type A_1, A_2, A_3 and B_2.

0. Introduction

Let G be a connected reductive algebraic group over an algebraically closed field k of characteristic zero. G has an adjoint action on its Lie algebra $\mathfrak{g}$. Let $a \in \mathfrak{g}$ and let X be the closure of the G-orbit of a. If a is semi simple the orbit is closed so that X is a smooth variety. If a is regular X is normal cf. [17] Theorem 16.

PROBLEM. *Is the variety X always normal?*

This problem was brought to our attention by Walter Borho in the fall of 1975. A positive solution would have applications in the theory of the infinite dimensional representations of $\mathfrak{g}$, see [2] (2.6) and [3]. After a reduction we give two more cases where we have an (affirmative) answer. The method used in the second case is the more interesting one. It involves a resolution and some cohomology.

1. Reductions

We have the additive Jordan decomposition $a = a_s + a_n$. Let G' and $\mathfrak{g}'$ be the centralizers of a_s in G and $\mathfrak{g}$ respectively. Now $a_n \in \mathfrak{g}'$ and $\mathfrak{g}'$ is the Lie algebra of G' cf. [1] (9.1). Let X' be the closure of the G'-orbit of a_n in $\mathfrak{g}'$.

PROPOSITION. *The morphism $f: G \times X' \to X$ given by $f(g, x) = Ad(g)(a_s + x)$, is a smooth surjective morphism.*

The proof is standard and may be omitted. The only assumption needed here is that G is a linear algebraic group.

By [9] (IV 17.5.7) normality of X is now equivalent to normality of X'. The group G' is connected and reductive, cf. [19] (3.11) and (3.7). So we may replace G, a, X by G', a_n, X', i.e. we may assume that a is a nilpotent element of $\mathfrak{g}$.

It is easy to see that we may replace G by a reductive or semi simple group of the same type. As a product of normal varieties over k is normal we may assume that G has an irreducible root system.

2. *Case I.* Assume $G = Gl(V)$ where V is a vector space of dimension n. Now $\mathfrak{g} = \mathrm{End}(V)$ and a is a nilpotent endomorphism of V. Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be the partition of the blocks of the Jordan normal form of a. So $\lambda_1 \geq \cdots \geq \lambda_r \geq 1$, there are $e_1, \ldots, e_r \in V$ such that the elements $a^m e_i$ with $o \leq m < \lambda_i$ form a basis of V and that $a^m e_i = 0$ if $m \geq \lambda_i$. Clearly $n = \lambda_1 + \cdots + \lambda_r$.

PROPOSITION. *If $\lambda_2 = 1$ then X is Cohen-Macaulay and normal.*

Proof. Put $q = \lambda_1$ so that $n = q + r - 1$. The dimension of X is $(q-1)(2n-q)$, cf. [10] (3.8). Let N be the variety of the nilpotent endomorphisms of V, let D be the variety of the endomorphisms of V of rank $<q$, and let X' be the schematic intersection of N and D. It follows from [10] (3.10) that $X = X'_{\mathrm{red}}$, i.e. that X is the reduced variety with the same points as X'. For $x \in \mathrm{End}(V)$ let

$$\det(x - T.id) = (-T)^n + \sum_{i=1}^{n} (-T)^{n-i} \sigma_i(x)$$

be its characteristic polynomial. The subvariety N of *End* (V) is defined by the ideal generated by $\sigma_1, \ldots, \sigma_n$. As $\sigma_i \mid D = 0$ for $i \geq q$, the subscheme X' of D is defined by the ideal generated by $\sigma_1, \ldots, \sigma_{q-1}$. The variety D is Cohen-Macaulay of dimension $(q-1)(2n-q+1)$, cf. [7] Theorem 1 and [15] (4.13). So X' is Cohen-Macaulay by [9] (0_{IV} 16.5.6). Using the cross section of [10] (3.7) one verifies that the orbit of a is contained in the regular locus of X', so that X' is non-singular in codimension one. By Serre's criterion [9] (IV 5.8.6) it follows that X' is normal and hence equal to X.

3. Some cohomology

The results in this section are due to Kempf [12], [13]. The language used is closer to [5] (1.5) and [11]. Let G be a connected reductive group and P a parabolic subgroup of G. Let E be a P-module, i.e. a finite dimensional vector space with a given representation $P \to Gl(E)$. Consider the variety $Z = G \times^P E$ which is the quotient of $G \times E$ under the right P-action given by $(g, x)p = (gp, p^{-1}x)$. Let $\psi : Z \to G/P$ be given by $\psi(g, x)P = gP$, it is a locally trivial vector bundle. The locally free $\mathcal{O}_{G/P}$-module $\mathcal{L}(E)$ is defined as the sheaf of sections of ψ. We write $H^n(E) = H^n(G/P, \mathcal{L}(E))$, these groups are G-modules.

LEMMA. *Let V be a G-module and E a completely reducible P-module. Let $\pi : V \to E$ be a surjective morphism of P-modules. Then $H^n(E) = 0$ for $n \geq 1$ and the canonical G-morphism $\pi' : V \to H^0(E)$ is surjective.*

Proof. We may consider $H^0(E)$ as the G-module of the morphisms $f : G \to E$ satisfying $f(gp) = p^{-1}f(g)$. Now π' is given by $\pi'(v)(g) = \pi(g^{-1}v)$. Clearly $\pi = q \circ \pi'$ where $q : H^0(E) \to E$ is given by $q(f) = f(1)$. Write $E = \oplus_i E_i$ where each E_i is an irreducible P-module. As q is surjective we have $H^0(E_i) \neq 0$ for all i. Now Bott's theorem, cf. [16] (6.4), which holds in our algebraic situation by theorem 5 of [4] exp. II, implies that $H^n(E) = 0$ for all $n \geq 1$ and that the G-modules $H^0(E_i)$ are irreducible. The image of π' has a non-zero intersection with each $H^0(E_i)$, so π' is surjective.

Construction. Let V be a G-module and E a P-invariant subspace. Put $Z = G \times^P E$. Let $\tau : Z \to V$ be given by $\tau(g, x)P = gx$. The group G acts on Z and τ is G-equivariant. Identifying Z with the closed subvariety of $(G/P) \times V$ of the pairs (gP, x) with $g^{-1}x \in E$, one verifies that τ is a projective morphism. So the image of τ is the irreducible closed subvariety of V defined by the ideal $ker\,(\tau^0)$ where $\tau^0 : \Gamma(V, \mathcal{O}_V) \to \Gamma(Z, \mathcal{O}_Z)$ is the comorphism.

THEOREM. (Kempf [12]). *If E is a completely reducible P-module then $H^n(Z, \mathcal{O}_Z) = 0$ for $n \geq 1$, and τ^0 is surjective.*

Proof. The ring $\Gamma(V, \mathcal{O}_V)$ is the graded symmetrical algebra $\oplus_{m \geq 0} S_m(V^*)$ on the dual V^* of V. As $\psi_*(\mathcal{O}_Z) = \oplus_m \mathcal{L}(S_m(E^*))$, we have $H^n(Z, \mathcal{O}_Z) = \oplus_m H^n(S_m(E^*))$ for all $n \geq 0$ by [9] (III 1.3.3) and [8] chap. II (3.10). A P-module F is completely reducible if and only if the unipotent radical of P acts trivially on F. So the P-modules $S_m(E^*)$ are completely reducible. Now the assertions follow from the lemma applied on the projections from $S_m(V^*)$ to $S_m(E^*)$.

4. The resolution

Let G be connected and reductive with an irreducible root system. Let a be a non-zero nilpotent element of $\mathfrak{g}$. There is a uniquely determined parabolic subgroup P of G associated to a, see [18] (III, 4). The closure of the P-orbit of a

is a normal subalgebra, called $\mathfrak{u}_2$, of the Lie algebra $\mathfrak{p}$ of P. We form $Z = G \times^P \mathfrak{u}_2$ and $\tau : Z \to \mathfrak{g}$ as above.

PROPOSITION. *The morphism τ induces a G-equivariant, projective, birational and surjective morphism $\tau : Z \to X$. The variety X is normal if and only if the comorphism $\tau^0 : \Gamma(X, \mathcal{O}_X) \to \Gamma(Z, \mathcal{O}_Z)$ is bijective.*

Proof. Consider $b = (1, a)P$ in Z. The centralizer of a in G is contained in P, cf. loc. cit., and hence equal to the centralizer of b. So τ induces a bijection between the orbits of b and a. Using [18] (I, 5.6) and [1] (6.7) one shows that this bijection is an isomorphism. The orbits of b and a are dense and open in Z and X respectively, so $\tau : Z \to X$ is birational. The other properties of τ follow immediately. Since the variety Z is regular and the morphism τ is proper and birational, the ring $\Gamma(Z, \mathcal{O}_Z)$ is the integral closure of $\Gamma(X, \mathcal{O}_X)$ in its field of fractions. This concludes the proof.

Consider the following cases.

Case II. The P-module $\mathfrak{u}_2$ is completely reducible.

Case III. The nilpotent element a is regular.

THEOREM. *In the cases II and III the variety X is normal and $H^n(Z, \mathcal{O}_Z) = 0$ for $n \geq 1$.*

Remark. So in these cases X has rational singularities cf. [14] p. 51.

Proof. Case II is immediate from the above proposition and the theorem in **3**. For case III see [17] theorem 16 and [11] theorem A.

5. Applications

We follow [18] (III, 4). There are $h, b \in \mathfrak{g}$ with $[h, a] = 2a$, $[h, b] = -2b$, $[a, b] = h$. For $i \in \mathbf{Z}$ put $\mathfrak{g}(i) = \{x \in \mathfrak{g} \mid [h, x] = ix\}$. We have $\mathfrak{g} = \bigoplus \mathfrak{g}(i)$, $\mathfrak{p} = \bigoplus_{i \geq 0} \mathfrak{g}(i)$, $u_2 = \bigoplus_{i \geq 2} \mathfrak{g}(i)$. Let T be a maximal torus which leaves each $\mathfrak{g}(i)$ invariant. Let R be the root system of G with respect to T. For $\alpha \in R$ let d_α be given by $\mathfrak{g}_\alpha \subset \mathfrak{g}(d_\alpha)$. Let S be a set of simple roots with $d_\alpha \geq 0$ for all $\alpha \in S$. Then $d_\alpha \in \{0, 1, 2\}$ for all $\alpha \in S$. The G-orbit of a is characterized by the numbers d_α, $\alpha \in S$, attached to the corresponding nodes of the Dynkin diagram. Let $\sum_{\alpha \in S} n_\alpha \alpha$ be the highest root. As the unipotent radical of P has Lie algebra

$\mathfrak{u}_1 = \bigoplus_{i \geq 1} \mathfrak{g}(i)$, we obtain

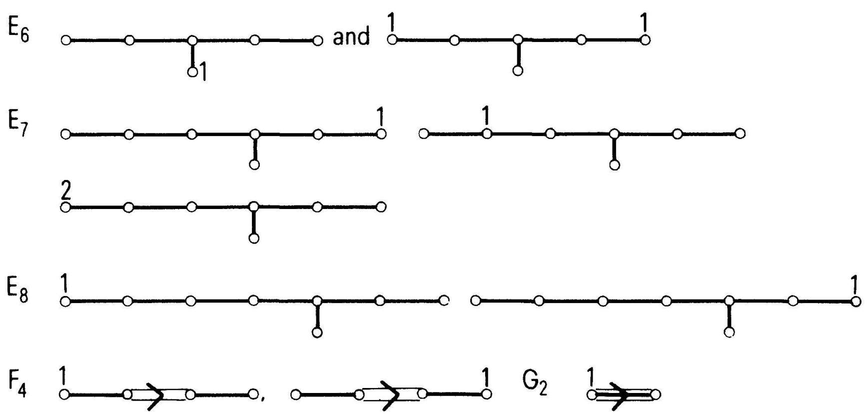
Criterion 1. *Case II applies if and only if* $\sum_{\alpha \in S} n_\alpha d_\alpha \leq 2$.

Let G be one of the classical groups Gl_n, Sp_n, SO_n and let $\rho : G \hookrightarrow Gl(V)$ be its usual representation in a vector space V of dimension n. Let λ be the partition of the nilpotent endomorphism $d\rho(a)$ of V, cf section 2. Using [18] (IV 1.13 and 2.32) we obtain

Criterion 2. *If G is Gl_n or Sp_n then case II applies if and only if $\lambda_1 \leq 2$. If G is SO_n then case II applies if and only if $\lambda_1 + \lambda_2 \leq 4$.*

Remark 1. By inspection of the tables in [10] (4.9) it follows that X is normal if G is of type A_1, A_2, A_3, B_2 and that X has rational singularities if G is of type A_1, A_2, B_2.

Remark 2. For the exceptional groups inspection of the tables 16–20 in [6] yields that case II applies for nilpotent elements with the following weighted Dynkin diagrams (here the numbers $d_\alpha = 0$ are suppressed).



Remark 3. Let k be a field of positive characteristic p. The propositions in **1** and **2** still hold. For the reductions in **1**, the proposition in **4** and the normality of X in case III we need some restrictions on p, cf. [19], [18], [20]. Although the theorems fail, a case-by-case analysis shows that X is normal if $p \neq 2, 3$ and G is of type A_1, A_2, A_3 and B_2.

REFERENCES

[1] BOREL, A. *Linear algebraic groups.* New York: Benjamin 1969.

[2] BORHO, W. *Recent advances in enveloping algebras of semi-simple Lie algebras.* Sém. Bourbaki, 29e année, 1976/77, no 489.

[3] BORHO, W. und KRAFT, H. *über Bahnen und deren Deformationen bei linearen Aktionen reduktiver Gruppen.* This volume.
[4] SÉMINAIRE H. CARTAN, vol. *3,* 1956–1957. New York: Benjamin 1967.
[5] DEMAZURE, M. *Désingularisation des variétés de Schubert généralisées.* Ann. Sc. de l'Ec. Norm. Sup. 4e serie, *7,* (1974) 53–88.
[6] DYNKIN, E. B. *Semisimple subalgebras of semisimple Lie algebras.* Am. Math. Soc. Transl. Ser. 2,6 (1957), 111–245 (=Mat. Sbornik N.S. *30,* (1952)) 349–462.
[7] EAGON, J. A., and HOCHSTER, M., *Cohen-Macaulay rings, invariant theory, and the generic perfection of determinantal loci.* Am. J. of Math. *93* (1971) 1020–1058.
[8] GODEMENT, R., *Théorie des faisceaux.* Paris: Hermann 1958.
[9] GROTHENDIECK, A., et DIEUDONNÉ, J. A., *Eléments de géométrie algébrique.* Publ. Math. de l'I.H.E.S. *11, 20, 24, 32* (Paris 1961–1967).
[10] HESSELINK, W. H. *Singularities in the nilpotent scheme of a classical group.* Trans. Am. Math. Soc. *222* (1976) 1–32.
[11] ——, *Cohomology and the resolution of the nilpotent variety.* Math. Ann. *223* (1976) 249–252.
[12] KEMPF, G. R., *Images of homogeneous vector bundles and varieties of complexes.* Bull. Am. Math. Soc. *81* (1975) 900–901.
[13] ——, *On the collapsing of homogeneous bundles.* Inventiones Math. *37* (1976) 229–239.
[14] KEMPF, G. R., et al., *Toroidal embeddings I. Lecture notes in math. 339.* Berlin, Springer 1973.
[15] KLEIMAN, S. L., and LANDOLFI, J., *Geometry and deformation of special Schubert varieties.* Compositio Math. *23* (1971) 407–434.
[16] KOSTANT, B., *Lie algebra cohomology and the generalized Borel-Weil theorem.* Ann. of Math. *74* (1961) 329–387.
[17] ——, *Lie group representations on polynomial rings.* Am. J. of Math. *85* (1963) 327–404.
[18] SPRINGER, T. A., and STEINBERG, R., *Conjugacy classes.* In A. BOREL et al.: *Seminar on algebraic groups and related finite groups.* Lecture notes in math. *131.* Berlin: Springer 1970.
[19] STEINBERG, R., *Torsion in reductive groups.* Advances in Math. *15* (1975) 63–92.
[20] VELDKAMP, F. D., *The center of the universal enveloping algebra in characteristic* p. Ann. Sc. de l'Ec. Norm. Sup., 4e série, *5* (1972) 217–240.

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Received November 1, 1977.