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The Jenkins Strebel differentials with one cylinder are dense

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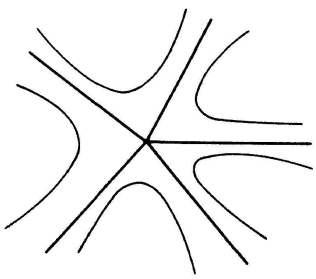
§1. Introduction and statement of the theorem

Let X be a compact Riemann surface of genus $g>1$ and $q \in H^0(X, \Omega^{\otimes 2})$ a non-zero holomorphic quadratic differential on X. A horizontal trajectory is a curve γ on X such that $q>0$ along γ. The horizontal trajectories give a foliation of X singular at the zeros of q. At a zero of order k, there are $k+2$ *critical leaves* emanating from the zero. The following drawing illustrates the situation for $k=3$.



A differential q is *Jenkins–Strebel* if the union of the critical leaves and zeroes of q is compact. It forms a graph Γ_q whose complement in X is a disjoint union of pairwise non-homotopic open cylinders. Each cylinder is swept out by homotopic closed leaves. The maximum number of cylinders is $3g-3$.

These differentials were discovered by Jenkins [3] as solutions of an extremal problem. A general existence and uniqueness theorem was later found by Strebel [4]. Douady and Hubbard [1] proved the important result that the Jenkins–Strebel forms are dense in $H^0(X, \Omega^{\otimes 2})$. The purpose of this paper is to give an improvement of that result.

THEOREM 1. *The Jenkins–Strebel differentials with one cylinder are dense in* $H^0(X, \Omega^{\otimes 2})$.

We will show any Jenkins–Strebel differential can be approximated by differentials with one cylinder. The idea is to find a *particular* Jenkins–Strebel

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differential equivalent to the given one (c.f. §2 for the definition) for which an approximating sequence can be explicitly found. Then results of Hubbard and Masur [2] show that a corresponding sequence approximating the given differential must exist as well.

As an application we will prove a result announced by Thurston [6]; for a C^∞ compact surface M, there is a natural topology such that the set of homotopy classes of simple closed curves is dense in a space of foliations. The precise statement is given in §2.

§2. Measured foliations

In this section we recall definitions given in [6] and [2] that are necessary for this paper.

We suppose M is a compact C^∞ surface of genus $g>1$. Then a Riemann surface X is given by a complex structure on M. A *measured foliation F on M with singularities* at $x_1, \ldots, x_n$ *of orders* $k_1, \ldots, k_n$ is given by the following:

(i) open sets U_α which cover $M-\{x_1, \ldots, x_n\}$ together with C^∞ closed real-valued 1-forms φ_α on each U_α such that $\varphi_\alpha = \pm\varphi_\beta$ in $U_\alpha \cap U_\beta$.

(ii) a neighborhood V_i of x_i with coordinates (x, y) such that $\varphi_\alpha = \operatorname{Im} z^{k_i/2}\, dz$ in $U_\alpha \cap V_i$, $z = x+iy$.

A nonzero quadratic differential q on X gives a measured foliation F_q on M by setting $\varphi_\alpha = \operatorname{Im} q^{1/2}\, dz$ in simply connected open sets away from the zeroes. At a zero x_i at which q vanishes to the order k_i, local coordinates z such that $q = z^{k_i}\, dz^2$ satisfy (ii). The horizontal trajectories are the leaves of the foliation, the curves along which the 1- form vanishes.

For each measured foliation F and curve γ, the *transverse length* $V_F(\gamma)$ is defined by $\int_\gamma |\varphi_\alpha|$. The condition $\varphi_\alpha = \pm\varphi_\beta$ in $U_\alpha \cap U_\beta$ guarantees $V_F(\gamma)$ is well-defined.

We define an equivalence relation on measured foliations to be the minimal one that includes the following two equivalences.

(i) $F_1 \sim F_2$ if there is a diffeomorphism h of M to itself, homotopic to the identity such that $h^*F_2 = F_1$.

(ii) $F_1 \sim F_2$ if there is a compact critical leaf γ joining two singularities of F_1 (resp. F_2), a continuous map h of M homotopic to the identity, which is a diffeomorphism restricted to $M-\gamma$, satisfies $h^*F_2 = F_1$ (resp. $h^*F_1 = F_2$) on $M-\gamma$, and collapses γ to a zero of F_2 (resp. F_1).

Let S be the set of homotopy classes of simple closed curves on M, $\mathbf{R}^S$ the linear space of maps of S to the reals with the product topology and $p: \mathbf{R}^S - \{0\} \to P\mathbf{R}^S$ the map to the corresponding projective space. There is a map j of S into

$\mathbf{R}^S - \{0\}$ given by

$$[\gamma_1] \mapsto ([\gamma_2] \mapsto i([\gamma_1], [\gamma_2])).$$

Here $i(\cdot,\cdot)$ is the geometric intersection number. On the other hand, each measured foliation F determines an element in $\mathbf{R}^S$ by sending $[\gamma]$ to $\inf_{\gamma \in [\gamma]} V_F(\gamma)$. It turns out that F gives a nonzero element of $\mathbf{R}^S$ ([2], [6]). It is easy to see that two equivalent foliations determines the same element in $\mathbf{R}^S - \{0\}$, and therefore also in $P\mathbf{R}^S$. We now compare the images of the measured foliations in $P\mathbf{R}^S$ with $p.j(S)$.

THEOREM 2. (*Thurston* [6]). *The image in* $P\mathbf{R}^S$ *of the measured foliations is the closure of* $p \cdot \gamma(S)$.

§3. Jenkins–Strebel differentials with $3g-3$ cylinders

Forms with $3g-3$ cylinders are easiest to handle, and it is known that they are already dense. Suppose $q \in H^0(X, \Omega^{\otimes 2})$ defines $3g-3$ cylinders with heights $h_1, \ldots, h_{3g-3}$ with respect to the vertical metric $|\operatorname{Im} q^{1/2}\, dz|$. The complement of the central curves of the cylinders is $2g-2$ spheres each with three discs removed. The complement of the critical graph Γ_q in each sphere is three cylinders; one boundary component of each cylinder is a central curve. The only possibilities up to homotopy are given in the following drawings.

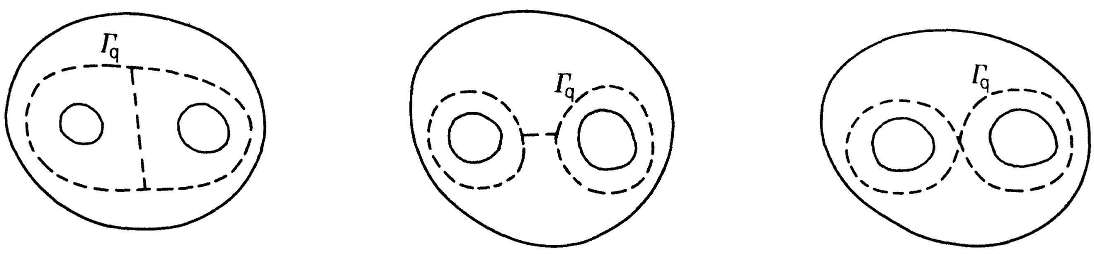

Figure 1 is symmetric and occurs precisely when the length with respect to the metric $|q^{1/2}\, dz|$ of each central curve is less than the sum of the other two. In Figure 2 the length of one central curve is greater than the sum of the other two and in Figure 3 there is equality. Figures 2 and 3 may or may not occur in three cases. They will not if for instance, two of the central curves coincide on the surface.

Conversely, we start with a graph Γ on the C^∞ surface M such that the complement of Γ is $3g-3$ topological cylinders homotopic to the prescribed curves and such that the complement of the $3g-3$ curves is $2g-2$ spheres with graphs all of the type in Figure 1. Each boundary component of each cylinder contains two edges of Γ. To each edge assign a positive number t_i such that for each cylinder the sum of the numbers on the boundary components are equal. This gives $3g-3$ equations in $6g-6$ unknowns. Each t_i appears exactly twice and each equation is of the form

$$t_i+t_j=t_k+t_l$$

with the possibility that t_i may appear on both sides of the equation.

We take the particular solution $t_i=\frac{1}{2}$ for all i. Next we take $3g-3$ Euclidean cylinders in $\mathbf{R}^3$ with heights h_i and circumferences one. Each boundary component is divided into two pieces of length $\frac{1}{2}$ and the cylinders are glued together along their edges according to the prescription of the graph Γ. This gives a Riemann surface X' and a Jenkins–Strebel differential q'. For local coordinates away from the dividing points take the Euclidean coordinates w_i of the cylinder. Then $q'=dw_i^2$. The dividing points are simple zeroes of q'. Here three edges come together and $z=(\frac{3}{2}w)^{2/3}$ gives a local coordinate so that $q'=z\,dz^2$. It is readily seen that the measured foliations $F_{q'}$ and F_q are equivalent as any one of the drawings 1, 2, or 3 can be deformed into any other by collapsing and expanding segments of the graph.

In the construction of q' there remains a choice for each cylinder of the position of the dividing points on one boundary component with respect to the points on the other. We are interested only in the cases where two vertical lines in each cylinder connect two pairs of dividing points. There are then exactly two possibilities which differ by a rotation through angle π. The horizontal trajectories of $-q'$ are the vertical lines in the cylinders of q'. The union of the critical vertical lines is compact, so $-q'$ will always have closed trajectories.

LEMMA 1. *There is a choice of rotation in each cylinder so that $-q'$ determines only one cylinder.*

Proof. To prove the lemma it is enough to position the zeroes so that one trajectory of $-q'$ intersects every segment of $\Gamma_{q'}$. For then suppose $-q'$ determines two cylinders C_1 and C_2 and a trajectory of C_1 intersects every edge of $\Gamma_{q'}$. Then there is a segment γ contained in an edge of $\Gamma_{q'}$ which has one endpoint on a trajectory of C_1 and the other on a trajectory of C_2. Then γ passes from C_1 to C_2 so it must intersect a critical trajectory of $-q'$. But the critical trajectories intersect $\Gamma_{q'}$ only at the zeros by construction.

We show that we can modify q' one cylinder at a time so that eventually a trajectory of $-q'$ has the above property. Start with any closed trajectory β of $-q'$

that intersects the midpoint of a boundary-edge. Suppose first that there is no cylinder of q' such that β intersects two of the boundary edges but not all four. Then for a subset of the $3g-3$ cylinders, β intersects all four boundary edges of each cylinder in the subset and does not intersect any cylinder not in the subset. This means each edge of a cylinder in the subset is identified with an edge of another cylinder in the same subset which in turn implies the union of these cylinders is a connected component in the surface. The subset can not be proper and β has the required property.

Therefore let C be a cylinder such that β intersects two but not four edges and let $\hat{\beta}$ be the closed trajectory through the midpoint of the remaining two edges. Rotate a boundary component of C through angle π and glue according to the graph. This gives a new Jenkins–Strebel form q'. The corresponding β now intersects all four edges of C and moreover contains every arc in the complement of C that was part of either the previous β or $\hat{\beta}$. We continue rotating one cylinder at a time as necessary until β has the required property.

LEMMA 2. *If the differential q' of lemma 1 has integer heights the differential $q'_n = (1 - i/n)^2\, q'$ is Jenkins–Strebel with one cylinder for each positive integer n.*

Proof. An easy calculation shows that the trajectories of q'_n are lines with slope $1/n$ with respect to the Euclidean coordinates of q'. Since the ratio of height to circumference of each cylinder is an integer, a trajectory of q'_n leaving a boundary edge next intersects the opposite component at the point on the same vertical line. Therefore the trajectories of q'_n are closed because those of $-q'$ are, and since each trajectory intersects every edge of $\Gamma_{q'}$ by the construction of q', there can again be only one cylinder.

Proof of Theorem 1. The Jenkins–Strebel differentials with $3g-3$ cylinders and rational heights are dense in $H^0(X, \Omega^{\otimes 2})$. This follows from the density theorem [1] and the fact that any differential with $p < 3g-3$ can be approximated by ones with $p+1$ cylinders by letting either the height or the modulus of the extra cylinder approach zero. ([2], [4]). The statement about rational heights is obvious. Now if any quadratic differential with $3g-3$ cylinders and integer heights can be approximated so can any differential with rational heights. Therefore we suppose q is a Jenkins–Strebel differential with integer heights. Let q' be the equivalent differential found in lemma 1 on the surface X'. By the main theorem of [2] there are differentials q_n on X equivalent to the differentials q'_n given by lemma 2 and q_n must necessarily be Jenkins–Strebel with one cylinder. We may also appeal to the existence theorem in [4] and [5] to find q_n. The transverse length of a homotopy class varies continuously in $H^0(X', \Omega^{\otimes 2})$ so for each $[\gamma]$

$$\lim_{n\to\infty} V_{F_{q_n}}[\gamma] = \lim_{n\to\infty} V_{F_{q'_n}}[\gamma] = V_{F_{q'}}[\gamma] = V_{F_q}[\gamma].$$

By Theorem 4 and the main theorem of [2], $\lim_{n\to\infty} q_n = q$ q.e.d.

Proof of Theorem 2. By the main theorem of [2], the set of measured foliations can be identified with the non-zero differentials in $H^0(X, \Omega^{\otimes 2})$. Differentials which differ by real scalar multiples have the same image in $P\mathbf{R}^S$. Let q be a Jenkins–Strebel differential with one cylinder of homotopy class $[\gamma]$ and height one. For any $[\beta] \in S$,

$$V_{F_q}([\beta]) = i([\gamma], [\beta]).$$

Therefore the Jenkins–Strebel differentials and S have the same image in $P\mathbf{R}^S$. Together with Theorem 1 this proves Theorem 2.

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