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Abelian group extensions and the axiom of constructibility

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Introduction

Throughout this paper the word "group" will mean "abelian group". The results of Shelah's remarkable work on Whitehead's problem ([Sh_1], [Sh_2]) suggested the investigation of the structure of Ext $(A, \mathbf{Z})$ for torsion-free A under the hypothesis of the Axiom of Constructibility, $V = L$. Applying Shelah's methods, H. Hiller, Shelah and the second-named author obtained a surprisingly simple description of the torsion-free part of Ext $(A, \mathbf{Z})$ in terms of A[H–H–S].

In this paper we study, in the same spirit, the group Ext (A, G) in the case where A is torsion-free and G is any group satisfying suitable cardinality conditions. We are interested in characterizing pairs (A, G) such that Ext $(A, G) = 0$ as well as in determining the structure of Ext (A, G). Herein we restrict our attention to its torsion-free part. Since in our case Ext (A, G) is always divisible, the structure of its torsion-free part is completely determined by its torsion-free rank. (Following [F_1] we denote the torsion-free rank of a group B by $r_0(B)$.)

Our first task is to settle the case where A is countable. For this, of course, we do not need any additional axiom of set theory. We assume G to be a group of countable torsion-free rank, thus unifying the known cases $G = \mathbf{Z}[J, §2]$ and $G = T$, a torsion group ([B_1], [B_2]). In Section 1 we consider the crucial case where A is of rank 1. For such A we give a group-theoretical characterization of pairs (A, G) such that Ext $(A, G) = 0$ and show that Ext $(A, G) \neq 0$ implies $r_0(\text{Ext}\,(A, G)) \geq 2^{\aleph_0}$ (Theorem 1.2). In Section 2 we study Ext (A, G) in case A is any coutable torsion-free group. Applying Theorem 1.2 we obtain various conditions that are necessary and (or) sufficient for the vanishing of Ext (A, G) (Theorems 2.1 and 2.6, Corollaries 2.4 and 2.7). In particular we have the following analogue of Pontryagin's criterion: If Ext $(B, G) = 0$ for every subgroup B of A of finite rank, then Ext $(A, G) = 0$ (Corollary 2.7). Using Theorem 1.2 we conclude that also in this case, Ext $(A, G) \neq 0$ implies $r_0(\text{Ext}\,(A, G)) \geq 2^{\aleph_0}$ (Theorem 2.8).

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Section 3 is devoted to the vanishing of Ext (A, G) for uncountable A. From now on we have to assume $V = L$ in order to be able to apply Shelah's methods. The main theorem of this section (Theorem 3.2) generalizes earlier results of the first-named author (see [E_3]). In particular it contains the following singular compactness theorem for Ext: $(V = L)$. Let A be a group of singular cardinality κ and let G be a group of cardinality $< \kappa$. If Ext $(B, G) = 0$ for every subgroup B of A of cardinality $< \kappa$, then Ext $(A, G) = 0$. The proof of this is based on a new version [Sh_3] of the principal result of [Sh_2]. Among other consequences of Theorem 3.2 we deduce a vanishing result for Ext (A, G) (Theorem 3.7) which corresponds to a theorem of Hill [Hi].

The final section deals with the structure of Ext (A, G) for uncountable A. We show that the main result of [H–H–S] generalizes to our situation; we proceed along the lines of that proof. Theorem 4.5 may be viewed as the principal result of Section 4: $(V = L)$. Let A be torsion-free and G of countable torsion-free rank such that Ext $(A, G) \neq 0$. Suppose that B is a pure subgroup of A and that Ext $(A/B, G) = 0$. If B is of minimal cardinality, then $r_0(\text{Ext}\,(A, G)) \geq 2^{|B|}$. (As usual $|B|$ denotes the cardinality of B.) The case where A is of singular cardinality relies on a variant of Theorem 3.2 which is of interest in its own right (Theorem 4.3). Finally we deduce some corollaries concerning the torsion-free rank of Ext (A, G) which extend results of [H–H–S] and [Hu].

1. The rank one case

In this section we investigate the group Ext (A, G) in case A is torsion-free of rank 1 and G any group of (at most) countable torsion-free rank.

We first recall some definitions and known facts and state certain exact sequences which will be important tools in the proof of Theorem 1.2. Given a prime p, we denote the p-primary part of a group G by t_pG and the torsion subgroup of G by tG. The p-primary part of $\mathbf{Q}/\mathbf{Z}$ is denoted by $\mathbf{Z}(p^\infty)$ and its full preimage in $\mathbf{Q}$ by $\mathbf{Q}^{(p)}$. A group G is called *p-divisible* if $pG = G$; G is *divisible* if it is p-divisible for every prime p. A group which does not contain any nontrivial divisible subgroup is called *reduced.* It is well known that every group is the direct sum of its maximal divisible subgroup and a reduced group.

Let A be a torsion-free group of rank 1. For a nonzero element $x \in A$ and any prime p let $h_p(x)$ be the largest integer k such that p^k divides x if it exists, or $h_p(x) = \infty$ otherwise; $h_p(x)$ is called the *p-height* of x. Suppose that for every p we are given k_p which is either a nonnegative integer or ∞. Then there exists a nonzero $y \in A$ such that for every p, $h_p(y) = k_p$ if and only if the following two conditions hold:

$$k_p \neq h_p(x) \quad \text{only for finitely many } p\text{'s}; \tag{1.1a}$$

$$k_p = h_p(x) \quad \text{whenever} \quad k_p = \infty \quad \text{or} \quad h_p(x) = \infty \tag{1.1b}$$

(see e.g. (F_2, §85]). By definition of the p-heights we can associate with every nonzero $x \in A$ a short exact sequence

$$0 \to \mathbf{Z} \xrightarrow{\mu} A \to \bigoplus_p \mathbf{Z}(p^{k_p}) \to 0, \tag{1.2}$$

where μ is given by $\mu(1) = x$ and $k_p = h_p(x)$ for every p. Here $k_p = 0$ means that the p-primary part does not occur. For any group G this sequence induces a long exact sequence

$$\begin{aligned} 0 &\to \prod_p \operatorname{Hom}(\mathbf{Z}(p^{k_p}), G) \to \operatorname{Hom}(A, G) \to G \to \\ &\to \prod_p \operatorname{Ext}(\mathbf{Z}(p^{k_p}), G) \to \operatorname{Ext}(A, G) \to 0. \end{aligned} \tag{1.3}$$

In particular we shall make use of the sequence

$$\begin{aligned} 0 &\to \operatorname{Hom}(\mathbf{Z}(p^\infty), G) \to \operatorname{Hom}(\mathbf{Q}^{(p)}, G) \to G \to \\ &\to \operatorname{Ext}(\mathbf{Z}(p^\infty), G) \to \operatorname{Ext}(\mathbf{Q}^{(p)}, G) \to 0 \end{aligned} \tag{1.4}$$

which is induced by $0 \to \mathbf{Z} \to \mathbf{Q}^{(p)} \to \mathbf{Z}(p^\infty) \to 0$.

The following facts will be applied several times; therefore we state them as a lemma.

LEMMA 1.1. (a) *For any group* G, $\operatorname{Ext}(\mathbf{Z}(p^n), G) \cong G/p^nG$. (b) $\operatorname{Ext}(\mathbf{Z}(p^\infty), G) = 0$ *if and only if* G *is* p-*divisible.*

Proof. Statement (a) is well-known (see e.g. [F_1](D), p. 222) while (b) follows from Corollary 4.3 and Theorem 4.5 of [N_1].

Finally we assign to every group G two sets of primes

$$D_1(G) = \{p \mid pG \neq G\} \quad \text{and}$$

$$D_2(G) = \{p \mid p^{k+1}G \neq p^kG \quad \text{for all} \quad k\}.$$

We are now ready to state the result of this section.

THEOREM 1.2. *Let A be a torsion-free group of rank* 1 *and let G be any group of countable torsion-free rank. Then*

(a) $\operatorname{Ext}(A, G)=0$ *if and only if for any nonzero $x \in A$ the following conditions hold:*

  (1) $\{p \in D_1(G) \mid h_p(x) \neq 0\}$ *is finite;*

  (2) *for all* $p \in D_2(G)$, $h_p(x)<\infty$.

(b) *If* $\operatorname{Ext}(A, G) \neq 0$, *then the torsion-free rank of* $\operatorname{Ext}(A, G)$ *is* $\geqslant 2^{\aleph_0}$.

Remarks

1) Combining (a) with (1.1a, b) we obtain that $\operatorname{Ext}(A, G)=0$ if and only if there is a nonzero $x \in A$ such that (1) and (2′) are satisfied, where (2′) means that $h_p(x)=0$ for all $p \in D_2(G)$.
2) Let G be a group satisfying conditions (1) and (2) for any nonzero $x \in \mathbf{Q}$. It is not hard to see that such a G is the direct sum of a divisible group and a bounded torsion group. On the other hand, there are *cotorsion* groups G, i.e. groups satisfying $\operatorname{Ext}(\mathbf{Q}, G)=0$, (of uncountable torsion-free rank) that are not of this form (cf. [F_1], §55). We conclude that the countability hypothesis on G in statement (a) cannot be dropped.
3) In (b) the hypothesis on G cannot be omitted either. By [M−V, p. 119] there are groups A and G, A torsion-free of rank 1 and G torsion-free of rank $2^{\aleph}$, such that $\operatorname{Ext}(A, G) \cong \bigoplus_{\aleph_0} \mathbf{Q}$.

Proof of Theorem 1.2. We observe that it suffices to prove the following two statements:

(a′) If there is a nonzero $x \in A$ such that (1) and (2) are satisfied, then $\operatorname{Ext}(A, G)=0$.

(b′) If there is a nonzero $x \in A$ such that (1) or (2) does not hold, then $r_0(\operatorname{Ext}(A, G)) \geqslant 2^{\aleph_0}$.

Proof of (a′): We consider the associated exact sequence (1.3). Since $\operatorname{Ext}(A, G)$ is divisible, the assertion follows if we can show that $\prod_p \operatorname{Ext}(\mathbf{Z}(p^{k_p}), G)$ is divisible, the assertion follows if we can show that $\prod_p \operatorname{Ext}(\mathbf{Z}(p^{k_p}), G)$ is a bounded torsion group. By conditin (1) and Lemma 1.1 $\operatorname{Ext}(\mathbf{Z}(p^{k_p}, G)$ nonzero only for a finite set of primes, say I. Thus it remains to show that for all $p \in I$, $\operatorname{Ext}(\mathbf{Z}(p^{k_p}), G)$ is a bounded torsion group. If $k_p<\infty$ this is obvious. In case $k_p=\infty$ condition (2) implies that $p^{k+1} G=p^k G$ for some k. Therefore by Lemma 1.1(b) the first term of the exact sequence

$$\operatorname{Ext}(\mathbf{Z}(p^{\infty}), p^k G) \rightarrow \operatorname{Ext}(\mathbf{Z}(p^{\infty}), G) \rightarrow \operatorname{Ext}(\mathbf{Z}(p^{\infty}), G / p^k G) \rightarrow 0$$

is trivial. Consequently, $\mathrm{Ext}(\mathbf{Z}(p^\infty), G)$ is a bounded torsion group. This completes the proof of (a′).

Proof of (b′): Suppose first that (1) does not hold. Again we consider the associated exact sequence (1.3). By hypothesis and Lemma 1.1 the product $\prod_p \mathrm{Ext}(\mathbf{Z}(p^{k_p}), G)$ has infinitely many nontrivial factors. Therefore it has a quotient isomorphic to $\prod_{p \in J} \mathbf{Z}(p)$ for some infinite set of primes J. It follows that $r_0(\prod_p \mathrm{Ext}(\mathbf{Z}(p^{k_p}), G)) \geq 2^{\aleph_0}$. As $r_0(G)$ is countable, we see from (1.3) that $r_0(\mathrm{Ext}(A, G)) \geq 2^{\aleph_0}$ as well.

Now suppose that x does not satisfy (2). So there is a prime p such that A contains a copy of $\mathbf{Q}^{(p)}$ and the chain

$$G \supseteq pG \supseteq \cdots \supseteq p^k G \supseteq \cdots$$

is properly descending. Since $\mathrm{Ext}(\mathbf{Q}^{(p)}, G)$ is then an epimorphic image of $\mathrm{Ext}(A, G)$, it suffices to show that $r_0(\mathrm{Ext}(\mathbf{Q}^{(p)}, G)) \geq 2^{\aleph_0}$. For this purpose we distinguish two cases. First assume that G/tG is not p-divisible. Then its p-adic completion $(G/tG)\hat{}_p$ is torsion-free of cardinality $2^{\aleph_0}$. But by [N_1, p. 233], $(G/tG)\hat{}_p$ is an epimorphic image of $\mathrm{Ext}(\mathbf{Z}(p^\infty), G/tG)$; hence $r_0(\mathrm{Ext}(\mathbf{Z}(p^\infty), G/tG)) \geq 2^{\aleph_0}$. Using the sequence (1.4) for G/tG, we conclude that $r_0(\mathrm{Ext}(\mathbf{Q}^{(p)}, G/tG)) \geq 2^{\aleph_0}$, and hence $r_0(\mathrm{Ext}(\mathbf{Q}^{(p)}, G)) \geq 2^{\aleph_0}$.

In the second case assume that G/tG is p-divisible. Then the reduced part of t_pG must be unbounded. Therefore, by the main result of [Sz], there is an epimorphism

$$tG \twoheadrightarrow \bigoplus_{k<\omega} \mathbf{Z}(p^k) = H.$$

We claim that $r_0(\mathrm{Ext}(\mathbf{Q}^{(p)}, H)) = 2^{\aleph_0}$. By Lemma 1 of [R] we have $|\mathrm{Ext}(\mathbf{Z}(p^\infty), H)| = 2^{\aleph_0}$. Thus, using exactness of (1.4) for $G = H$, we conclude that $|\mathrm{Ext}(\mathbf{Q}^{(p)}, H)| = 2^{\aleph_0}$. But $\mathrm{Ext}(\mathbf{Q}^{(p)}, H)$ is torsion-free, hence the claim is proved. Now $\mathrm{Ext}(\mathbf{Q}^{(p)}, H)$ is an epimorphic image of $\mathrm{Ext}(\mathbf{Q}^{(p)}, tG)$, and the latter fits into an exact sequence

$$\mathrm{Hom}(\mathbf{Q}^{(p)}, G/tG) \to \mathrm{Ext}(\mathbf{Q}^{(p)}, tG) \to \mathrm{Ext}(\mathbf{Q}^{(p)}, G).$$

As $\mathrm{Hom}(\mathbf{Q}^{(p)}, G/tG)$ is countable, it follows that $r_0(\mathrm{Ext}(\mathbf{Q}^{(p)}, G)) \geq 2^{\aleph_0}$ also in this case. This completes the proof of Theorem 1.2.

2. The countable case

Throughout this section G denotes a given group of countable torsion-free rank. We now study the group Ext (A, G) in case A is any countable torsion-free group. We start with

THEOREM 2.1. *For any countable torsion-free group A the following statements are equivalent:*

(a) $\mathrm{Ext}\,(A, G)=0$;

(b) *A is the union of an ascending chain of pure subgroups $\{A_n \mid n<\omega\}$ such that $A_0=0$ and for all n, $r_0(A_{n+1}/A_n)\leq 1$ and* $\mathrm{Ext}\,(A_{n+1}/A_n, G)=0$.

For the proof of this theorem we need the following auxiliary result.

LEMMA 2.2. *Suppose that G is reduced. If A is torsion-free of finite rank such that* $\mathrm{Ext}\,(A, G)=0$, *then the torsion-free rank of* $\mathrm{Hom}\,(A, G)$ *is countable.*

Proof. We proceed by induction on the rank of A. Suppose first that A is of rank 1 and let $x\in A$, $x\neq 0$. Then we consider the associated exact sequences (1.2) and (1.3). Since G is reduced, we have $\mathrm{Hom}\,(\mathbf{Z}(p^{k_p}), G)=0$ if $k_p=\infty$ or if $pG=G$. It remains to consider those primes for which $pG\neq G$ and $k_p<\infty$. But we know from Theorem 1.2(a) that $k_p\neq 0$ only for a finite number of them. Therefore $\prod_p \mathrm{Hom}\,(\mathbf{Z}(p^{k_p}), G)$ is a bounded torsion group, and hence exactness of (1.3) implies that $r_0(\mathrm{Hom}\,A, G))$ is countable.

Now assume that the lemma holds for all torsion-free groups of rank $\leq n$. Let A be torsion-free of rank $n+1$ such that $\mathrm{Ext}\,(A, G)=0$, and let B be a pure subgroup of A of rank n. Then there is an exact sequence

$$0\to \mathrm{Hom}\,(A/B, G)\to \mathrm{Hom}\,(A, G)\to \mathrm{Hom}\,(B, G)\to \mathrm{Ext}\,(A/B, G)\to 0.$$

As by induction hypothesis $r_0(\mathrm{Hom}\,(B, G))$ is countable, we conclude that $r_0(\mathrm{Ext}\,(A/B, G))$ is countable too. But then we have $\mathrm{Ext}\,(A/B, G)=0$ by Theorem 1.2(b). Therefore by the first part, $r_0(\mathrm{Hom}\,(A/B, G))$ is countable; hence r $(\mathrm{Hom}\,(A, G))$ is countable as well.

Proof of Theorem 2.1. The implication (b)$\Rightarrow$(a) is a special case of [E_2, Theorem 1.2]. Conversely, suppose that A is a countable torsion-free group such that $\mathrm{Ext}\,(A, G)=0$. Let A be represented as the union of an ascending chain of pure subgroups $\{A_n \mid n<\omega\}$ such that $A_0=0$ and $r_0(A_{n+1}/A_n)\leq 1$. Clearly such a chain of subgroups exists. Then we have $\mathrm{Ext}\,(A_n, G)=0$ for all n, and therefore there are exact sequences

$$\text{Hom}\,(A_n, G) \to \text{Ext}\,(A_{n+1}/A_n, G) \to 0.$$

Note that we may assume G to be reduced. Thus $r_0(\text{Hom}\,(A_n, G))$ is countable by Lemma 2.2 and hence $r_0(\text{Ext}\,(A_{n+1}/A_n, G))$ is countable too. But then by Theorem 1.2(b) $\text{Ext}\,(A_{n+1}/A_n, G) = 0$ for all n. This completes our proof.

Theorems 1.2 and 2.1 provide a number of interesting consequences. The first generalizes Stein's theorem (see e.g. [E_1, Theorem 4.1]).

COROLLARY 2.3. *Suppose that G is countable torsion-free and for all primes p, G is not p-divisible. If A is any group of countable torsion-free rank, then* $\text{Ext}\,(A, G) = 0$ *implies A free.*

Proof. First it follows from [N_1, Theorem 4.5] that for any A, $\text{Ext}\,(A, G) = 0$ implies A torsion-free. Therefore, if A is of countable rank, we may apply Theorem 2.1. Hence A is the union of an ascending chain $\{A_n \mid n < \omega\}$ of pure subgroups such that $A_0 = 0$ and for all n, $r_0(A_{n+1}/A_n) \leq 1$ and $\text{Ext}(A_{n+1}/A_n, G) = 0$. Now by the hypothesis on G we have $D_1(G) = D_2(G) = P$ (the set of all primes). Thus, by remark 1) from Theorem 1.2, there is for every n a nonzero $x \in A_{n+1}/A_n$ (except that $A_{n+1} = A_n$) such that $h_p(x) = 0$ for all p. But this means that for all n, A_{n+1}/A_n is free; hence by [E_1, Theorem 2.6] A is free.

COROLLARY 2.4. *Let G' be a pure subgroup of G. If A is any countable torsion-free group, then* $\text{Ext}\,(A, G) = 0$ *if and only if* $\text{Ext}\,(A, G') = 0$ *and* $\text{Ext}\,(A, G/G') = 0$.

Proof. The "if" part holds trivially. Conversely, suppose that $\text{Ext}\,(A, G) = 0$. Then clearly $\text{Ext}\,(A, G/G') = 0$, and by Theorem 2.1, A is the union of an ascending chain of pure subgroups $\{A_n \mid n < \omega\}$ such that $A_0 = 0$ and for all n, $r_0(A_{n+1}/A_n) \leq 1$ and $\text{Ext}\,(A_{n+1}/A_n, G) = 0$. Now $D_i(G')$ is contained in $D_i(G)$ for $i = 1, 2$, since G' is pure in G. Therefore by Theorem 1.2(a) we have $\text{Ext}\,(A_{n+1}/A_n, G') = 0$ for all n, and hence $\text{Ext}\,(A, G') = 0$ by Theorem 2.1.

COROLLARY 2.5. *There is a countable quotient H of G such that for any countable torsion-free group A,* $\text{Ext}\,(A, G) = 0$ *if and only if* $\text{Ext}\,(A, H) = 0$.

Proof. By the main result of [Sz] and the proof of [E_3, Theorem 2.2] there is a countable torsion group T and an epimorphism $\varepsilon : tG \twoheadrightarrow T$ such that for any countable torsion-free A, $\text{Ext}\,(A, tG) = 0$ if and only if $\text{Ext}\,(A, T) = 0$. We denote the induced homomorphism Ext $(G/tG, tG) \to$ Ext $(G/tG, T)$ by ε_* and let H be a representative of the class $\varepsilon_*[G]$. Thus there is a commutative diagram

$$\begin{array}{ccccccccc} 0 \to & tG & \to & G & \to & G/tG & \to 0 \\ & \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \eta} & & \| & \\ 0 \to & T & \to & H & \to & G/tG & \to 0 \end{array}$$

with exact rows. We claim that H has the required properties. First it is clear that H is countable. Furthermore we see from the diagram that η is an epimorphism. Hence for any group A, Ext $(A, G) = 0$ implies Ext $(A, H) = 0$. Conversely, suppose that A is countable torsion-free such that Ext $(A, H) = 0$. Then we have Ext $(A, G/tG) = 0$ and Ext $(A, T) = 0$ by Corollary 2.4. Thus Ext $(A, tG) = 0$ as well and hence Ext $(A, G) = 0$. This completes our proof.

THEOREM 2.6. *If A is a countable torsion-free group such that* Ext $(A, G) =$ 0, *then* Ext $(A/B, G) = 0$ *for every pure subgroup B of A of finite rank.*

Proof. Let B be any pure subgroup of A of finite rank. We can choose an ascending chain of pure subgroups $\{B_n \mid n < \omega\}$ of A of finite rank with union A such that $B_0 = B$ and for all n, $r_0(B_{n+1}/B_n) \leq 1$. Then the same argument as in the proof of Theorem 2.1 shows that Ext $(B_{n+1}/B_n, G) = 0$ for all n. Now let $A_n = B_n/B$, so we have $A/B = \bigcup_{n<\omega} A_n$ where $A_0 = 0$ and for all n, A_n is a pure subgroup of A/B of finite rank and $A_{n+1}/A_n \cong B_{n+1}/B_n$. Therefore we have Ext $(A/B, G) = 0$ by Theorem 2.1.

Remark. Combining Theorems 1.2, 2.1 and 2.6 we obtain another proof of Baer's criterion which characterizes pairs of groups (A, T), A countable torsion-free and T a torsion group, such that every extension of T by A splits [B_1].

The following result is an analogue of Pontryagin's criterion (see e.g. [F_1], Theorem 19.1).

COROLLARY 2.7. *If A is a countable torsion-free group such that* Ext $(B, G) = 0$ *for every subgroup B of A of finite rank, then* Ext $(A, G) = 0$.

Proof. Let A be represented as the union of an ascending chain of pure subgroups $\{A_n \mid n < \omega\}$ such that $A_0 = 0$ and for all n, $r_0(A_{n+1}/A_n) \leq 1$. Then for all n, Ext $(A_n, G) = 0$ by hypothesis. Using Theorem 2.6, we conclude that for all n, Ext $(A_{n+1}/A_n, G) = 0$. Hence we have Ext $(A, G) = 0$ by Theorem 2.1.

The next result contains Proposition 5 of [Hu] and, in particular, the well-known fact that for every countable torsion-free nonfree group A, $r_0(\mathrm{Ext}(A, \mathbf{Z}))$ is $2^{\aleph_0}$ (see e.g. [J], Théorème 2.7).

THEOREM 2.8. *Let* A *be a countable torsion-free group such that* $\mathrm{Ext}(A, G) \neq 0$. *Then the torsion-free rank of* $\mathrm{Ext}(A, G)$ *is* $\geqslant 2^{\aleph_0}$.

Proof. Clearly we may assume that G is reduced. If $\mathrm{Ext}(A, G) \neq 0$, then by Corollary 2.7 there exists a subgroup B of A of finite rank such that $\mathrm{Ext}(B, G) \neq 0$. Suppose that B is of minimal rank, and let B' be a pure subgroup of B such that $r_0(B/B') = 1$. Then we consider the exact sequence

$$\mathrm{Hom}(B', G) \to \mathrm{Ext}(B/B', G) \to \mathrm{Ext}(B, G) \to \mathrm{Ext}(B', G).$$

The minimality of $r_0(B)$ implies that $\mathrm{Ext}(B', G) = 0$. Thus the exact sequence yields that $\mathrm{Ext}(B/B', G) \neq 0$, and hence by Theorem 1.2(b), $r_0(\mathrm{Ext}(B/B', G)) \geqslant 2^{\aleph_0}$. On the other hand, by Lemma 2.2, $r_0(\mathrm{Hom}(B', G))$ is countable. It follows that $r_0(\mathrm{Ext}(B, G)) \geqslant 2^{\aleph_0}$, hence $r_0(\mathrm{Ext}(A, G)) \geqslant 2^{\aleph_0}$.

Remark. For A a countable torsion-free group and T an arbitrary torsion group, Baer [B_2] has shown that $\mathrm{Ext}(A, T)$ is torsion-free. If in addition T is countable, we conclude that

$$\mathrm{Ext}(A, T) \cong \prod_{\aleph_0} \mathbf{Q} \quad (\text{cf. } [B_2], \text{ pp. } 229\text{–}230).$$

It is well-known that this group admits a *compact topology*. Furthermore we know from [J, Corollaire 2.8] that the same holds for groups of the form $\mathrm{Ext}(A, \mathbf{Z})$, A being countable torsion-free. These facts led us to ask the following

Question. Does $\mathrm{Ext}(A, G)$ admit a compact topology whenever A is countable torsion-free and G countable of finite torsion-free rank?

3. The uncountable case: vanishing of Ext (A, G)

In order to extend the results of the previous sections to groups of uncountable cardinality, we shall need to assume the Axiom of Constructibility, $V = L$. Before stating the main result of this section, let us recall a definition from [E_3]. For any set U and infinite cardinal λ, let $K_{\lambda^+}(U)$ denote the filter on $\mathcal{P}(U)$, the power set of U, generated by all $X \in \mathcal{P}(\mathcal{P}(U))$ satisfying

(i) X is closed under unions of chains; and
(ii) for all $S \subseteq U$, there exists $H \in X$ such that $S \subseteq H$ and $|H| \leq |S| + \lambda$.

(Such an X will be called a *generating element* of $K_{\lambda^+}(U)$.) We shall say that a property P of subsets of U *holds for almost all subsets* (w.r.t. $K_{\lambda^+}(U)$) if $\{S \subseteq U \mid S$ satisfies $P\}$ belongs to $K_{\lambda^+}(U)$.

LEMMA 3.1. (1) *If $\lambda \leq \mu$, then $K_{\lambda^+}(U) \subseteq K_{\mu^+}(U)$.*
(2) *$K_{\lambda^+}(U)$ is λ^+-complete, i.e. if X_ν, $\gamma < \lambda$ are elements of $K_{\lambda^+}(U)$, then $\bigcap\{X_\gamma \mid \nu < \lambda\}$ belongs to $K_{\lambda^+}(U)$.*
(3) *If $V \subseteq U$ and $X \in K_{\lambda^+}(V)$, then $\{H \subseteq U \mid H \cap V \in X\}$ belongs to $K_{\lambda^+}(U)$.*
(4) *If A is a group, B a subgroup of A and $X \in K_{\lambda^+}(A/B)$, then $\{H \subseteq A \mid (H+B)/B \in X\}$ belongs to $K_{\lambda^+}(A)$.*
(5) *If A is a group, then almost all subsets of A (w.r.t. $K_{\omega_1}(A)$) are pure subgroups of A.*
(6) *If B is a pure subgroup of A, then for almost all subsets H of A (w.r.t. $K_{\omega_1}(A)$), $B+H$ is a pure subgroup of A.*

Proof. (1)–(4) are easy consequences of the definition. Part (5) follows from the facts that (i) the union of a chain of pure subgroups is a pure subgroup, and (ii) every infinite subset of A is contained in a pure subgroup of A of the same cardinality (cf. [F_1], Proposition 26.2). By (5) applied to the group A/B we obtain an element X of $K_{\omega_1}(A/B)$ consisting of pure subgroups of A/B. Then by part (4), $Y = \{H \subseteq A \mid (H+B)/B \in X\}$ belongs to $K_{\omega_1}(A)$; it is readily verified that for all $H \in Y$, $B+H$ is a pure subgroup of A. This proves (6).

For any infinite cardinal κ, let $\mathrm{cf}(\kappa)$ denote the confinality of κ. By definition, κ is *singular* if $\mathrm{cf}(\kappa) < \kappa$ and otherwise κ is *regular*. Recall that a group is said to be *κ-generated* if it has a set of generators of cardinality $< \kappa$.

THEOREM 3.2. ($V = L$). *Let A be a group of uncountable cardinality κ and let G be a group of cardinality $\lambda < \kappa$. Then*
(1) *If κ is singular and* $\mathrm{Ext}(B, G) = 0$ *for every κ-generated subgroup B of A, then* $\mathrm{Ext}(A, G) = 0$.
(2) $\mathrm{Ext}(A, G) = 0$ *if and only if A is the union of a continuous ascending chain $\{A_\nu \mid \nu < \mathrm{cf}(\kappa)\}$ of κ-generated pure subgroups of A such that* $\mathrm{Ext}(A_0, G) = 0$ *and for all $\nu < \mathrm{cf}(\kappa)$,* $\mathrm{Ext}(A_{\nu+1}/A_\nu, G) = 0$.
(3) *If* $\mathrm{Ext}(A, G) = 0$, *then for almost all subgroups H of A (w.r.t. $K_{\lambda^+}(A)$),* $\mathrm{Ext}(A/H, G) = 0$.

The above result is proved in [E_3] for the case of a torsion-free group A and G a *torsion* group, but the same proof works for arbitrary A and G any *countable* group. That proof was based on [Sh_2]; here we shall give a proof based on a new simplified version [Sh_3] of the principal result of [Sh_2]. For the convenience of the reader we shall state a version (for abelian groups) of the main theorem of [Sh_3].

THEOREM 3.3 (Shelah). *Let κ be a singular cardinal and let $\{\kappa_i \mid i<\mathrm{cf}(\kappa)\}$ be an increasing and continuous sequence of cardinals satisfying: $\kappa_0=0$, $\mathrm{cf}(\kappa)\leq\kappa_1$ and $\sup\{\kappa_i \mid i<\mathrm{cf}(\kappa)\}=\kappa$. Let A be a group of cardinality κ; let S_i = the set of all subgroups of A of cardinality κ_i and let $S_i'=\{0\}\cup S_i$. Suppose $\mathcal{F}$ is a class of pairs (C_2, C_1) of subgroups of A such that $C_1\subseteq C_2$. Suppose $\mathcal{F}$ satisfies the following two properties:*

- (HI) *For every $i<\mathrm{cf}(\kappa)$, there is a function $g_i: S_i'\times S_i \to S_i$ such that whenever $A_1 \subsetneqq A_2$ are in S_i' and $A_1\in\{0\}\cup(\text{range } g_i)$, then $A_2\subseteq g_i(A_1, A_2)$ and $(g_i(A_1, A_2), A_1)\in\mathcal{F}$;*
- (HII) *For every $i<\mathrm{cf}(\kappa)$ and every $A_1\subsetneqq A_2$ in S_{i+1}', if $(A_2, A_1)\in\mathcal{F}$ then player II has a winning strategy in the following game: in the n^{th} move $(n<\omega)$, player I chooses $B_n\in S_i'$ such that $C_{n-1}\subseteq B_n$ (where $C_{-1}=0$) and then player II chooses $C_n\in S_i'$ such that $B_n\subseteq C_n$. Player II wins if*

$$\Big(A_2+\bigcup_{n<\omega} C_n,\ A_1+\bigcup_{n<\omega} C_n\Big)\in\mathcal{F}.$$

Then A is the union of a continuous ascending chain $\{A_\nu \mid \nu<\omega\,\mathrm{cf}(\kappa)\}$ of κ-generated subgroups of A such that $A_0=0$ and $(A_{\nu+1}, A_\nu)\in\mathcal{F}$ for all $\nu<\omega\,\mathrm{cf}(\kappa)$.

Proof of Theorem 3.2. Let A and G be as in he hypotheses of 3.2. We shall prove (1), (2) and (3) simultaneously by induction on κ. Part (3) is proved as in the proof of Theorem 3.4 of [E_3]. (In place of Lemma 3.5(3) we require the straightforward generalization in which K_{ω_1} is replaced by K_{λ^+}). The sufficiency of the condition in part (2) is just Theorem 1.2 of [E_2], and when κ is regular, necessity is Theorem 1.5 of [E_2]. (Note that we can assume the chain $\{A_\nu \mid \nu<\kappa\}$ consists of *pure* subgroups by Lemma 3.1(5).) Thus it remains only to prove (1) and necessity in (2) when κ is singular.

Let $\mathcal{F}$ be the class of pairs (C_2, C_1) of subgroups of A such that C_1 is a pure subgroup of C_2 and $\mathrm{Ext}(C_2/C_1, G)=0$. Choose an increasing sequence $\{\kappa_i \mid i<\mathrm{cf}(\kappa)\}$, whose limit is κ such that $\kappa_0=0$ and for all $i\geq 0$, $\kappa_{i+1}\geq \max\{\mathrm{cf}(\kappa), \lambda\}$. We shall show that $\mathcal{F}$ satisfies (HI) and (HII) of Theorem 3.3. First we prove a lemma.

LEMMA 3.4. $(V=L)$. *Let κ be a limit cardinal, let A be a group of cardinality κ and let G be a group of cardinality $\lambda<\kappa$. Suppose that* $\mathrm{Ext}(B, G)=0$ *for every*

κ-generated subgroup B of A. Let C be a subgroup of A of infinite cardinality μ, where $\lambda \leq \mu^+ < \kappa$. Then there is a pure subgroup C^* of A of cardinality μ such that C^* contains C and

(*) Ext $(C'/C^*, G) = 0$ *for all subgroups C' of A of cardinality μ* that contain C^*.

Proof. If no such group C^* exists, then we can construct by induction a continuous ascending chain $\{C_\nu \mid \nu < \mu^+\}$ of subgroups of A of cardinality μ such that $C_0 = C$ and for all ν, $1 \leq \nu < \mu^+$, C_ν is pure in A and Ext $(C_{\nu+1}/C_\nu, G) \neq 0$. Let $\tilde{C} = \bigcup_{\nu<\mu^+} C_\nu$. Then $|\tilde{C}| = \mu^+ < \kappa$, but Ext $(\tilde{C}, G) \neq 0$ by Lemma 1.4 of [E₂], which contradicts the hypothesis. This completes the proof of the lemma.

Now we can verify (HI) by defining g_i so that for all $(A_1, A_2) \in S'_i \times S_i$, $A_2 \subseteq g_i(A_1, A_2)$ and $C^* = g_i(A_1, A_2)$ satisfies (*) of Lemma 3.4. (Note that if $A_1 = 0$, then $(g_i(A_1, A_2), 0) \in \mathscr{F}$ by the hypothesis of Theorem 3.2(1).)

It remains to verify (HII). Given $A_1 \subsetneqq A_2$ in $S'_{i+1}(i \geq 1)$ such that $(A_2, A_1) \in \mathscr{F}$, there exists by Lemma 3.1(6) a generating element X of $K_{\lambda^+}(A_2)$ such that for all $H \in X$, $A_1 + H$ is a pure subgroup of A_2. Moreover since by inductive hypothesis A_2/A_1 satisfies Theorem 3.2(3) we may assume – using Lemma 3.1(4) – that every element H of X satisfies Ext$(A_2/(A_1+H), G) = 0$. Hence for all $H \in X$, $(A_2, A_1 + H) \in \mathscr{F}$. Now the winning strategy of player II is as follows. Suppose $C_{n-1} \in S'_i$ has been chosen so that $C_{n-1} \cap A_2 \in X$. If player I chooses $B_n \in S'_i$ with $C_{n-1} \subseteq B_n$, then player II chooses $C_n \in S'_i$ such that $B_n \subseteq C_n$ and $C_n \cap A_2 \in X$; this is possible because $\kappa_i \geq \lambda$. Now

$$(A_2 + \bigcup_{n<\omega} C_n)/(A_1 + \bigcup_{n<\omega} C_n) \cong A_2/(A_1 + \bigcup_{n<\omega} (C_n \cap A_2)),$$

and $(A_2, A_1 + \bigcup_{n<\omega} (C_n \cap A_2)) \in \mathscr{F}$ since X is closed under unions of chains. It follows easily that $(A_2 + \bigcup_{n<\omega} C_n, A_1 + \bigcup_{n<\omega} C_n) \in \mathscr{F}$.

Therefore by Theorem 3.3 we have a continuous ascending chain $\{A_\nu \mid \nu < \omega \text{ cf}(\kappa)\}$ of κ-generated subgroups with union A such that $(A_{\nu+1}, A_\nu) \in \mathscr{F}$ for all $\nu < \omega \text{ cf}(\kappa)$. Note that the continuity of the chain implies that for all ν, A_ν is pure in A. By choosing a continuous subchain of length cf(κ) we obtain the chain required for Theorem 3.2(2). This completes the proof of Theorem 3.2.

COROLLARY 3.5 ($V = L$). *Let G be a group of countable torsion-free rank. There is a countable quotient H of G such that for any torsion-free group A,* Ext$(A, G) = 0$ *if and only if* Ext$(A, H) = 0$.

Proof. Let H be the countable quotient of G given by Corollary 2.5. For any group A, Ext$(A, G) = 0$ implies Ext$(A, H) = 0$. We shall prove by induction on $|A|$ that the converse is also true if A is torsion-free. For countable A this is

Corollary 2.5. If A is uncountable and $\operatorname{Ext}(A, H)=0$, then by Theorem 3.2(2), A is the union of a continuous chain $\{A_\nu \mid \nu<\operatorname{cf}(\kappa)\}$ of κ-generated subgroups such that $\operatorname{Ext}(A_0, H)=0$ and for all $\nu<\operatorname{cf}(\kappa)$, $A_{\nu+1}/A_\nu$ is torsion-free and $\operatorname{Ext}(A_{\nu+1}/A_\nu, H)=0$. By induction, $\operatorname{Ext}(A_0, G)=0$ and for all $\nu<\operatorname{cf}(\kappa)$, $\operatorname{Ext}(A_{\nu+1}/A_\nu, G)=0$; hence (by Theorem 1.2 of [E$_2$]) $\operatorname{Ext}(A, G)=0$.

Remark. As an immediate consequence of Corollary 3.5 we obtain Theorem 3.2 for A torsion-free and G any group of countable torsion-free rank (and arbitrary cardinality) with $\lambda=\omega$ in 3.2(3). This generalizes Theorem 3.4 of [E$_3$].

We can also generalize Shelah's solution of Whitehead's problem in L using Corollary 2.3.

COROLLARY 3.6 ($V=L$). *Suppose that G is a countable torsion-free group such that for all primes p, G is not p-divisible. For any group A, if* $\operatorname{Ext}(A, G)=0$ *then A is free.*

Proof. We proceed by induction on the cardinality of A, using Corollary 2.3 and Theorem 3.2(2).

Remark. By [Sh$_1$], Corollary 3.6 is independent of ZFC. Moreover the same holds for Corollary 3.5 and Theorem 3.2(2) and (3) (see [E$_3$]). We do not know, however, if Theorem 3.2(1) is independent of ZFC.

The following result is related to Corollary 2.5 as Hill's theorem ([Hi]) is related to Pontryagin's criterion.

THEOREM 3.7 ($V=L$). *Let A be a torsion-free group and G a group of countable torsion-free rank. Suppose $A=\bigcup_{n<\omega} A_n$, where $\{A_n \mid n<\omega\}$ is a chain of pure subgroups of A such that* $\operatorname{Ext}(A_n, G)=0$ *for all $n<\omega$. Then* $\operatorname{Ext}(A, G)=0$.

Proof. By Corollary 3.5 we may assume that G is countable. the proof of the theorem will be by induction on $|A|$. If A is countable the result follows easily from Corollary 2.7. Suppose now that $|A|=\kappa>\aleph_0$. Theorem 3.2(3) and Lemma 3.1(2) and (3) imply that there is a generating element X of $K_{\omega_1}(A)$ consisting of subgroups H such that for all $n<\omega$, $\operatorname{Ext}(A_n/(H\cap A_n), G)=0$. Moreover by Lemma 3.1(6) we may assume that for all $H\in X$ and all $n<\omega$, A_n+H is pure in A. Now using the properties of a generating element we can define by transfinite induction a continuous ascending chain $\{H_\nu \mid \nu<\kappa\}$ of elements of X such that $H_0=0$, $A=\bigcup_{\nu<\kappa} H_\nu$, and for all $\nu<\kappa$, $|H_\nu|<\kappa$. For all $\nu<\kappa$

$$H_{\nu+1}/H_\nu=\bigcup_{n<\omega}((H_{\nu+1}\cap A_n)+H_\nu)/H_\nu \quad \text{and} \quad ((H_{\nu+1}\cap A_n)+H_\nu)/H_\nu \cong (H_{\nu+1}\cap A_n)/(H_\nu\cap A_n).$$

Hence Ext $(((H_{\nu+1}\cap A_n)+H_\nu)/H_\nu, G)=0$ since by choice of X, Ext $(A_n/(H_\nu \cap A_n), G)=0$. Moreover $((H_{\nu+1}\cap A_n)+H_\nu)/H_\nu$ is pure in $H_{\nu+1}/H_\nu$ since by choice of X, A_n+H_ν is pure in A. Therefore by inductive hypothesis, Ext $(H_{\nu+1}/H_\nu, G)=0$. Since this is true for all $\nu<\kappa$, it follows that Ext $(A, G)=0$. This completes the proof of the theorem.

4. The uncountable case: structure of Ext (A, G)

The aim of this final section is to determine the torsion-free rank of Ext (A, G) in the case where A is uncountable torsion-free and G satisfies suitable cardinality conditions. The results of this section extend those of [H–H–S], when Shelah's solution of Whitehead's problem in L is taken into account. We do not know, however, whether our results remain valid without additional axioms of set theory. We start with

THEOREM 4.1 ($V=L$). *Let A be a torsion-free group of uncountable cardinality κ and let G be any group of cardinality $<\kappa$. Suppose that for every κ-generated pure subgroup B of A,* Ext $(A/B, G)\neq 0$. *Then the torsion-free rank of* Ext (A, G) *is* 2^κ.

To prove this we follow the pattern of the proof of the corresponding result (Theorem 1) of [H–H–S]. The regular case is an easy consequence of the subsequent proposition. Recall that Ext (A, G) can be defined as the quotient group Fact (A, G)/Trans (A, G), where Fact (A, G) is the abelian group of all factor sets on A to G and Trans (A, G) is the subgroup of transformation sets (see e.g. [F_1], pp. 209–211).

PROPOSITION 4.2 ($V=L$). *Let A be a torsion-free group of regular uncountable cardinality κ and let G be any group of cardinality $\leqslant\kappa$. Suppose that for every κ-generated pure subgroup B of A,* Ext $(A/B, G)\neq 0$. *If A_0 is any κ-generated pure subgroup of A, then for every $f_0\in$* Fact (A_0, G) *there exists a subset $\{f^\alpha \mid \alpha<2^\kappa\}$ of* Fact (A, G) *such that*

(i) *for all $\alpha<2^\kappa$, f^α extends f_0;*
(ii) *for each pair $\alpha\neq\beta$, $f^\alpha-f^\beta$ represents an element of infinite order of* Ext (A, G).

The proof of this proposition is almost identical with that of Proposition 1 in [H–H–S]. We only have to replace the statement "A is free" by "Ext $(A, G)=0$". Instead of [E_1, Theorem 2.6] we make use of Theorem 1.2 of [E_2]. Note that the cardinality hypothesis on G is needed in order that Lemma 3 of [H–H–S] can be applied.

The proof of Theorem 4.1 in case κ is singular relies on the above proposition and

THEOREM 4.3 ($V=L$). *Let A be any group of singular cardinality κ. Let G be a group of cardinality $\lambda<\kappa$ and let γ be any infinite cardinal $<\kappa$. Suppose that every κ-generated subgroup B of A contains a γ^+-generated subgroup C such that C is pure in B and* $\operatorname{Ext}(B/C, G)=0$. *Then A contains a γ^+-generated pure subgroup C such that* $\operatorname{Ext}(A/C, G)=0$.

Proof. Let $\mathcal{F}_1$ be the class of pairs $(B, 0)$ where B is a subgroup of A that contains a γ^+-generated subgroup C such that C is pure in B and $\operatorname{Ext}(B/C, G)=0$. Let $\mathcal{F}_2$ be the class of pairs (B, C) of subgroups of A where C is a non-trivial pure subgroup of B such that $\operatorname{Ext}(B/C, G)=0$. We show that the class $\mathcal{F}=\mathcal{F}_1\cup\mathcal{F}_2$ satisfies (HI) and (HII) of Theorem 3.3, assuming that the sequence $\{\kappa_i \mid i<\operatorname{cf}(\kappa)\}$ is chosen such that $\kappa_1\geq\max\{\operatorname{cf}(\kappa), \lambda, \gamma\}$. Condition (HI) is easily verified by means of the following analogue of Lemma 3.4.

LEMMA 4.4 ($V=L$). *Suppose that A and G satisfy the hypotheses of Theorem 4.3. Let B be a subgroup of A of cardinality μ, where $\max\{\lambda, \gamma\}\leq\mu<\kappa$. Then there is a pure subgroup B^* of A of cardinality μ such that B^* contains B and* $\operatorname{Ext}(B'/B^*, G)=0$ *for all subgroups B' of A of cardinality μ that contain B^*.*

Proof. We proceed as in the proof of Lemma 3.4. Supposing that no such B^* exists, we obtain a subgroup $\tilde{B}$ of A of cardinality μ^+ which is the union of a continuous ascending chain $\{B_\nu \mid \nu<\mu^+\}$ of subgroups of cardinality μ such that $B_0=B$ and for all ν, $1\leq\nu<\mu^+$, B_ν is pure in A and $\operatorname{Ext}(B_{\nu+1}/B_\nu, G)\neq 0$. By hypothesis $\tilde{B}$ contains a γ^+-generated pure subgroup C such that $\operatorname{Ext}(\tilde{B}/C, G)=0$. As $|\tilde{B}|$ is regular, we may assume that C is contained in B_0; so we have $\tilde{B}/C=\bigcup_{\nu<\mu^+}B_\nu/C$. But then Lemma 1.4 of [E_2] yields a contradiction, and our lemma is proved.

It remains to check (HII). Given $A_1\subsetneq A_2$ in S'_{i+1} such that $(A_2, A_1)\in\mathcal{F}$, we distinguish the following two cases. First if $A_1\neq 0$, we proceed exactly as in the proof of Theorem 3.2. In the second case, suppose that $A_1=0$. So $(A_2, A_1)\in\mathcal{F}$ means that A_2 contains a γ^+-generated pure subgroup C such that $\operatorname{Ext}(A_2/C, G)=0$. Then by Theorem 3.2(3) and Lemma 3.1 there exists a generating element X of $K_{\lambda^+}(A_2)$ consisting of subgroups H of A_2 such that $(A_2, C+H)\in\mathcal{F}$. Now the winning strategy of player II is to choose $C_n\in S'_i$ such that C_0 contains C and $C_n\cap A_2\in X$. This is possible by the assumption on κ_i.

Then

$$(A_2+\bigcup_{n<\omega} C_n)/(A_1+\bigcup_{n<\omega} C_n)\cong A_2/\bigcup_{n<\omega}(C_n\cap A_2)$$

$$\cong A_2/(C+\bigcup_{n<\omega}(C_n\cap A_2)).$$

But $(A_2, C+\bigcup_{n<\omega}(C_n\cap A_2))$ is in $\mathcal{F}$ since X is closed under unions of chains; so indeed $(A_2+\bigcup_{n<\omega} C_n, A_1+\bigcup_{n<\omega} C_n)\in\mathcal{F}$.

Therefore by Theorem 3.3 we obtain a continuous chain $\{A_\nu \mid \nu<\omega\,\mathrm{cf}(\kappa)\}$ of κ-generated pure subgroups of A such that $A=\bigcup_{\nu<\omega\mathrm{cf}(\kappa)} A_\nu$, A_1 contains a γ^+-generated pure subgroup C with $\mathrm{Ext}(A_1/C, G)=0$ and for all ν, $1\leq\nu<\omega\,\mathrm{cf}(\kappa)$, $\mathrm{Ext}(A_{\nu+1}/A_\nu, G)=0$. Hence we have $\mathrm{Ext}(A/C, G)=0$ by Theorem 3.2(2). This completes the proof of Theorem 4.3.

Proof of Theorem 4.1. Clearly 2^κ is an upper bound for $r_0(\mathrm{Ext}(A, G))$. If κ is regular, Proposition 4.2 implies that the quotient group of $\mathrm{Ext}(A, G)$ modulo torsion is of cardinality 2^κ. Hence in this case 2^κ is also a lower bound for $r_0(\mathrm{Ext}(A, G))$. Note that for κ regular the theorem still holds if the cardinality of G is κ.

Now assume that κ is singular. In this case we define by induction a chain of pure subgroups $\{A_\nu \mid \nu<\mathrm{cf}(\kappa)\}$ of A such that

(i) $A=\bigcup_{\nu<\mathrm{cf}(\kappa)} A_\nu$;

(ii) $|A_\nu|$ is a regular cardinal $>\max\{|G|, |\bigcup_{\mu<\nu} A_\mu|\}$;

(iii) if C is a $|A_\nu|$-generated pure subgroup of A_ν, then $\mathrm{Ext}(A_\nu/C, G)\neq 0$.

The definition of the chain is similar to the one in the proof of Theorem 1 of [H-H-S]. Instead of Theorem 2 of [H-H-S] we apply our Theorem 4.3, while condition (iii) is checked by making use of Theorem 3.2(3).

Let $\tilde{A}_\nu=\bigcup_{\mu<\nu} A_\mu$. As in [H-H-S], we deduce from Proposition 4.2 that to each sequence η of ordinals of length ν with $\eta(\mu)\in 2^{|A_\nu|}$, $\mu<\nu$, a factor set $f^\eta\in\mathrm{Fact}(\tilde{A}_\nu, G)$ can be assigned such that

(iv) if ξ is an initial segment of η, then f^η extends f^ξ;

(v) if $\xi\neq\eta$ are of the same length ν, then $f^\xi-f^\eta$ represents an infinite order element of $\mathrm{Ext}(\tilde{A}_\nu, G)$.

We conclude that there are $\prod_{\nu<\mathrm{cf}(\kappa)} 2^{|A_\nu|}=2^\kappa$ factor sets on A to G which represent pairwise different elements of $\mathrm{Ext}(A, G)$ modulo torsion. This completes the proof of Theorem 4.1.

THEOREM 4.5 ($V=L$). *Let A be a torsion-free group and let G be any group of countable torsion-free rank such that* $\mathrm{Ext}(A, G)\neq 0$. *Suppose that B is a pure* *subgroup of A such that* $\mathrm{Ext}(A/B, G)=0$. *If B is of minimal cardinality, then* $r_0(\mathrm{Ext}(A, G)) \geqslant 2^{|B|}$.

Proof. By hypothesis we have $\mathrm{Ext}(A, G) \cong \mathrm{Ext}(B, G)$. The case where B is countable is therefore settled by Theorem 2.8. For uncountable B the result follows from Corollary 3.5 and Theorem 4.1.

Remark. The following special case of the above theorem is implicit in [N_2]: If A is a torsion-free group and T a torsion group such that $\mathrm{Ext}(A, T) \neq 0$, then $r_0(\mathrm{Ext}(A, T)) \geqslant 2^{\aleph_0}$. Note that this does not require $V=L$.

The next two results are immediate consequences of Theorem 4.5.

COROLLARY 4.6. ($V=L$). *Let A be torsion-free and let G be countable such that* $\mathrm{Ext}(A, G) \neq 0$. *Then*

(a) $r_0(\mathrm{Ext}(A, G)) = 2^{\mu}$ *for some infinite cardinal* μ;

(b) $r_0(\mathrm{Ext}(A, G)) = |\mathrm{Ext}(A, G)|$.

COROLLARY 4.7. ($V=L$). *Let A be κ-free for some infinite cardinal κ and let G be countable. If* $\mathrm{Ext}(A, G) \neq 0$, *then* $r_0(\mathrm{Ext}(A, G)) = 2^{\mu}$ *for some* $\mu \geqslant \kappa$.

Recall that a group A is called *κ-free* if every κ-generated subgroup of A is free. We already mentioned that the results of this section extend those of [H–H–S]. Corollaries 4.7 and 4.8 generalize, moreover, Théorème 1 and Corollaire 2 of [Hu], respectively.

COROLLARY 4.8 ($V=L$). *Let A be any group and let G be countable. If* $\mathrm{Ext}(A, G)$ *is nonzero and divisible, then* $r_0(\mathrm{Ext}(A, G)) = 2^{\mu}$ *for some infinite* μ.

Proof. Clearly we may assume that G is reduced. We consider the exact sequence

$$\mathrm{Hom}(tA, G) \to \mathrm{Ext}(A/tA, G) \xrightarrow{\varphi} \mathrm{Ext}(A, G) \to \mathrm{Ext}(tA, G) \to 0.$$

From Lemma 55.3 of [F_1] we know that $\mathrm{Ext}(tA, G)$ is reduced. On the other hand, the hypothesis implies that $\mathrm{Ext}(tA, G)$ is divisible; hence $\mathrm{Ext}(tA, G)=0$. Using Lemma 1.1 we conclude that G is p-divisible for every prime p for which $t_pA \neq 0$. Therefore we have $t_pG=0$ whenever $t_pA \neq 0$. It follows that $\mathrm{Hom}(TA, G) \cong \prod_p \mathrm{Hom}(t_pA, t_pG) = 0$. Hence by exactness of the above sequence φ is an isomorphism. Thus it sufficies to consider the case where A is torsion-free. But this case has already been settled by Corollary 4.6(a).

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