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The secondary characteristic classes of parabolic foliations

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When a manifold M has a foliation of codimension n, its de Rham cohomology carries secondary characteristic classes associated to the normal bundle ν in degrees $\geq 2n+1$: such classes exist even if ν is trivial or flat, although some of them depend on the choice of trivialization (flat reduction) (see Bott and Haefliger [2]). A natural class of foliations with trivialized or flat normal bundles is obtained from pairs (G, H) where G is a lie-group and $H \subseteq G$ a closed subgroup: in the first instance, one has the foliation of G by the cosets of H, and from this we can construct two further classes of foliations – one on $\Gamma \backslash G$, $\Gamma \subseteq G$ a discrete subgroup, and one on G^c (the corresponding complex group) by cosets of H^c. The secondary classes for all these foliations can be computed "simultaneously" in the left-invariant complex $\Lambda^*(\mathfrak{g}^{*c})$, $\mathfrak{g}$ = lie algebra of G, and so we shall by abuse of notation refer to all these examples collectively by the notation (G, H).

In this paper we shall prove Fuks' conjecture [4] for several cases of the type (G, H): roughly speaking, this says that the examples (G, H) detect nothing more than what is already detected on $(\mathrm{SL}(n+1), P_1)$ where P_1 is the parabolic subgroup which is the isotropy group of a point in the projective space P_n. The example $(\mathrm{SL}(n+1), P_1)$ is well-understood, and its secondary classes have been computed independently by several people. Our main result essentially verifies this for all pairs (G, P) where G = noncompact, semi-simple lie-group, and P a parabolic subgroup: we call this a parabolic pair. Our methods also allow us to deal with pairs (G, S) where G is *arbitrary*, and S is nilpotent or solvable, and here one gets more vanishing.

THEOREM 1. *Let G be an arbitrary lie-group: the secondary classes for* (G, S) *are given thus*

a. *If S is nilpotent, all secondary classes are zero* .

b. *If S is solvable, then all secondary classes are of the form* $\{h_i\tau^a \mid i+|a| = n+1\}$ (notation in Section 1).

Theorem 1b does not quite fall under Fuks' conjecture: but this case is a side

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issue here which we shall pursue elsewhere. The point of the theorem is that one does not get very many more secondary classes by looking at nilpotent or solvable groups.

THEOREM 2. *Let* (G, P) *be a parabolic pair: then every rigid class is zero, and every movable class is equal to* $ah_1h_{i_1}\cdots h_{i_k}\tau_1^n$, *where* a *is some scalar.*

By combining Theorem 1b and 2, one obtains

COROLLARY. *If* (G, P) *is a parabolic pair with* P *minimal* (i.e. Borel) *then all nonzero classes are scalar multiples of the Godbillon–Vey class* $h_1\tau_1^n$.

We should mention that the first half of Theorem 2, that the rigid classes for (G, P) vanish has been recently announced by Bott and Haefliger (in Bott [1]), and they certainly had this result earlier. The other half of Theorem 2 is not really as strong as what Fuks conjectured, but for all practical purposes it comes to the same thing. The difference can be explained thus. Call the subspace of $H^*(W_n)$ generated by the classes $\{h_1h_{i_1}\cdots h_{i_k}\tau_1^n\}$ the G-V space: our result says that the image $H^*(W_n)\to H^*(\mathfrak{g})$ is contained in the image of the G-V space. Fuks stated his conjecture in the stronger form that any class in $H^*(W_n)$ which is zero for the "standard" example $(sl(n+1), \mathfrak{p}_1)$ is zero on all (G, H). This is false, as one can show by direct computation (Section 3). Quite recently D. Baker [12] has given systematic computations which also refute the strong Fuks' conjecture. For example, Fuks' version implies that if $\{h_1\tau_n\}=a\{h_1\tau_1^n\}$ on the standard example, then this equation is true for all (G, P): our calculations show that the constant a depends on the pair (G, P). In principle these constants can be computed in terms of the root-system and if this could be done in some explicit way, one would get more independence for the movable classes than one gets from the G-V space. In practice the calculation of these constants becomes very penurious. Finally, our proof of Theorem 2 yields an analogue of the "proportionality principle" of Bott and Haefliger ([1], again), and we prove this in Section 3.

To end, here is a brief description of the paper by sections. In Section 1 we recall the basic facts about homogeneous foliations, and we construct a particular left-invariant Bott connection which we will use throughout the paper: after recalling the facts about the associated secondary classes which come from $H^*(W_n)$, we prove Theorem 1 – which essentially boils down to invoking the theorems of Lie and Engel. In Section 2 we obtain further symmetries on the connection and curvature matrix arising from the semi-simplicity of G and the parabolicity of P: one main result here is that the first Chern form of the curvature, $\tau_1=\operatorname{trace}(\Omega)$ satisfies $\tau_1^n\neq 0$, and indeed is a Kähler class for the complex variety G^c/P^c. Finally in Section 3 we prove Theorem 2: the proof uses nothing beyond the standard structure theory of pairs (G^c, P^c) (from Section 2) together with elementary homological properties of Kähler manifolds. After giving the proof we make some miscellaneous remarks to round out the discussion.

It is a pleasure to thank Herb Shulman for conversations in the early stages of this work, and Conner Lazarov for bringing Fuks' paper to my notice.

1. Homogeneous foliations

Let G be a connected lie-group with S a closed, connected subgroup, and let $\mathfrak{g}$ and $\mathfrak{s}$ be their lie algebras. G is foliated by the cosets $\{gS\}$ of S. Denoting the quotient projection by $p: G \to G/S$, the normal-bundle ν of this foliation is $p^*(T(G/S))$ where $T(\cdot)$ is the tangent-bundle functor. The pull-back $p^*(T(G/S))$ is canonically isomorphic to the trivial bundle $G \times (\mathfrak{g}/\mathfrak{s})$, and since TG is also canonically trivial, left-invariant connections on ν are in 1-1 correspondence with R-linear maps

$$\nabla : \mathfrak{g} \to \mathfrak{gl}(V)$$

where $V = \mathfrak{g}/\mathfrak{s}$ (we shall use this convention for the rest of the section). Among these we can single out a set of Bott connections, which we call "homogeneous connections" for short, by requiring that the composite map

$$\mathfrak{s} \xrightarrow{i} \mathfrak{g} \xrightarrow{\nabla} \mathfrak{gl}(V)$$

coincide with the "adjoint" action of $\mathfrak{s}$ on V: this is the action of $\mathfrak{s}$ on V obtained (by passage to the quotient) from the adjoint action of $\mathfrak{s}$ on $\mathfrak{g}$, since $\mathfrak{s}$ is stable under ad $(\mathfrak{s})$. It is easily verified that the curvature of this connection has entries in the ideal $I_V \subseteq \Lambda^*(\mathfrak{g}^*)$ generated by $V^* \subseteq \mathfrak{g}^*$, which is the salient property of a Bott connection. It is equally easy to check that any two homogeneous connections are homotopic through a family of homogeneous connections, so the secondary classes can be computed by using any one of these.

We shall now show how to construct homogeneous connections. The obvious and traditional choice is to choose a linear splitting $\sigma : V \to \mathfrak{g}$ for the exact sequence

$$0 \longrightarrow \mathfrak{s} \longrightarrow \mathfrak{g} \xrightarrow{\pi} V \longrightarrow 0$$

and then to set

$$\nabla^0_X(v) = \pi[X, \sigma(v)].$$

If σ^1 is another splitting, the line segment joining σ to σ^1 gives an (integrable) homotopy of the corresponding ∇^0's regarded as G-invariant connections for the foliation of $G \times R$ by $S \times R$. However, we shall use another homogeneous connection, which is better adapted to the lie algebra set-up.

$$\nabla^1_X(v) = \pi[X - \sigma\pi(X), \ \sigma(v)]$$

Thus ∇^0, ∇^1 coincide on $\mathfrak{s}$, but ∇^1 totally ignores the bracket-relations in $\sigma(V)$: to see that these two are homotopic, put

$$\nabla^t_X(v) = \pi[X - t\sigma\pi(X), \ \sigma(v)]$$

The dependence on σ is therefore harmless, and in any case for parabolic pairs there will be a "natural" splitting.

We now point out some features of ∇^1 which will be important in the sequel: and these features allow us to prove Theorem 1 of the introduction simply by invoking the theorems of Lie and Engel (see the end of this section). Fix a basis $v_1, \ldots, v_n$ for V, and let Θ^1 be the connection form for this ∇^1 matrix: by construction the entries (Θ_{ij}) of Θ^1 lie in $\mathfrak{s}^*$, and in fact Θ^1 may be called the "Maurer–Cartan" matrix for the representation of $\mathfrak{s}$ on V. We could give a basis-free description of Θ^1, but the reader can do that for himself. Now consider the associated curvature form: $\Omega^1 = d\Theta^1 - \Theta^1 \wedge \Theta^1$. Since the second term lies in $\Lambda^2(\mathfrak{s}^*)$, it is natural to decompose $d = d' + d''$ where

$$d' : \mathfrak{s}^* \longrightarrow \Lambda^2(\mathfrak{g}^*) \xrightarrow{i^*} \Lambda^2(\mathfrak{s}^*)$$

and

$$d'' : \mathfrak{s}^* \longrightarrow \Lambda^2(\mathfrak{g}^*) \longrightarrow (\mathfrak{s}^* \wedge V^*) + \Lambda^2(V^*)$$

Since ∇^1 is a Bott-connection, we must have $d'\Theta^1 = \Theta^1 \wedge \Theta^1$ and so our curvature formula becomes $\Omega^1 = d''\Theta^1$. This is of course just part of the general nonsense about integrable structures, but it simplifies calculations considerably: moreover for parabolic pairs this formula will acquire further symmetries.

The foliation described above – with the left invariant trivialization of ν – gives several other foliations which can all be treated simultaneously. We describe two such classes now, and we shall refer to them as "the foliation" associated to the pair $(\mathfrak{g}, \mathfrak{s})$.

a. One can consider a complex group G^c corresponding to $\mathfrak{g}^c = \mathfrak{g} \otimes C$ and the subgroup $S^c \subseteq G^c$ corresponding to $\mathfrak{s}^c = \mathfrak{s} \otimes C$. This foliation is holomorphic of course.

b. For any discrete subgroup $\Gamma \subseteq G$, Γ/G inherits a foliation "induced" by S. This can be seen by noting that the differential ideal generated by V^* in $\Lambda^*(\mathfrak{g}^*)$ gives a (locally-free) differential ideal in the deRham algebra $A^*(\Gamma \backslash G)$.

Our discussion of connections above adapts at once to both these cases: and with reference to case (b) in practice all the constructions and calculations are made on G in the complex $\Lambda^*(\mathfrak{g}^*)$ and then pushed down to $\Gamma \backslash G$. Thus from now on we shall confine our attention almost exclusively to lie-algebra pairs $(\mathfrak{g}, \mathfrak{s})$.

Since our constructions involve a particular choice of trivialization of ν, the appropriate truncated Weil-algebra is $W_n (n = \dim_{\mathbf{R}} V = \operatorname{codim}_G S)$:

$$W_n = \Lambda^*_{\mathbf{R}}(h_1, h_2, \ldots, h_n) \otimes R[\tau_1, \ldots, \tau_n]/(\text{ideal deg} > n)$$

where $\deg h_i = 2i - 1$, $dh_i = \tau_i$ and the τ_i are the trace-forms associated to the universal $\mathfrak{gl}(n)$-curvature $\mathcal{K}$ – i.e., $\tau_j = \text{trace}\,(\mathcal{K}^j)$. Recall that there is a given recipe for calculating the h_j by a formal integration over the fiber of $\mathcal{K}$: see Haefliger [7]. Since our connections are left-invariant the Bott–Haefliger map is

$$(\beta; \chi): W_n \to \Lambda^*(\mathfrak{g}^*)$$

inducing a map on cohomology

$$(\beta; \chi): H^*(W_n) \to H^*(\mathfrak{g}).$$

In this framework there is clearly no harm in working with complexifications – and this has considerable technical advantages. Thus we start with $\mathfrak{g}^c$, $\mathfrak{s}^c$, $V^c = \mathfrak{g}^c/\mathfrak{s}^c$ and construct C linear maps $\nabla: \mathfrak{g}^c \to \mathfrak{gl}(V^c)$: the Weil algebra is then $W^c = W_n \otimes C$, and the Bott–Haefliger map is $(\beta; \chi) \otimes 1$. Unlike the passage from WO_n to WU_n, W_n^c does *not* acquire the ring generated by the $\bar{\tau}_j$, because we are comparing the Bott connection with a flat, affine one (rather than a Riemannian one). Moreover, by working with the complexifications from the start, we get secondary classes for both (G^c, S^c) (case (a) above) and the foliations on $\Gamma \backslash G$

(case b):[2] of course in the second case we get complex cohomology classes for a *real* foliation, but this is of no importance. Note finally that when $\mathfrak{g}$ is semi-simple, then

a. $H^*(\mathfrak{g}:C) \simeq H^*_{\mathrm{DR}}(G^c:C)$ by Weyl's unitarian trick
b. if $\Gamma \subseteq G$ is co-compact, then

$H^*(\mathfrak{g}:C) \hookrightarrow H^*_{\mathrm{DR}}(\Gamma \backslash G:C)$ by Poincaré duality

so we can work entirely in the lie-algebra without losing information.

The cohomology of W_n has been computed by J. Vey [5]: a linear basis for $H^*(W_n)$ is given by $\{h_{i_1} \cdots h_{i_k}\tau^a \mid i_1 < i_2 \cdots < i_k\}$ where $a = (a_1, \ldots, a_n)$ is a multi-index

1. $i_1 + |a| \geq n+1 \qquad \left(|a| = \sum_1^n ja_j\right)$

2. $i_1 \leq a_1$.

Heitsch's formulae [9] for the rigid and movable classes adapt easily to this case and we have

rigid classes: $\quad i_1 + |a| \geq n+2$

movable classes: $\quad i_1 + |a| = n+1$ (for the monomials given above).

We end this section by giving the proof of Theorem 1.

PROPOSITION 1.1. *Suppose* $\mathfrak{s}$ *is nilpotent,* $\mathfrak{g}$ *arbitrary: then all the* h_j *and* τ_j (re. ∇^1) *are zero.*

Proof. By Engel's theorem, we can choose a basis for V in which the action of $\mathfrak{s}$ on V is given by *strictly* upper triangular matrices: we call this $\mathfrak{n}(V)$. Since ∇^1 only records the information given by this action, it is trivial to see that in the chosen basis, Θ^1 lies in $\mathfrak{n}(V) \otimes \mathfrak{s}^*$. It follows that the whole algebra generated by Θ^1 and Ω^1 in $\mathrm{End}(V) \otimes \Lambda^*(\mathfrak{g}^*)$ is comprised of strictly upper-triangular matrices; since the h_j and τ_j are traces of certain elements in this algebra, they are identically zero.

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[2] To quote a well-known logic text, we get the full sweep of classes for $(\mathfrak{g}, \mathfrak{s})$ at one fell-swoop. (With nothing swapped).

It follows that any secondary form $h_{i_1}\cdots h_{i_k}\tau^a$ for $(\mathfrak{g}, \mathfrak{s})$ as above, and with respect to any homogeneous connection, is exact, which proves Theorem 1a.

PROPOSITION 1.2. *Suppose $\mathfrak{s}$ is solvable $\mathfrak{g}$ arbitrary: then with respect to ∇^1 every closed form in the Vey-basis is zero, except possibly for $h_j\tau^a$, $j+|a|=n+1$.*

Proof. From the discussion above, we may first complexify $\mathfrak{g}$, $\mathfrak{s}$ and V without loss of generality. Then we find as above (by invoking Lie's theorem) that Θ^1 is upper-triangular. Let $\theta_1, \ldots, \theta_n$ be the diagonal entries. It follows that Ω^1 is upper-triangular with diagonal entries $d\theta_1, \ldots, d\theta_n$: moreover that $[\Omega^1, \Theta^1]= 0 \bmod \mathfrak{n}(V)$ and $\Theta^1\wedge\Theta^1=0 \bmod \mathfrak{n}(V)$.

Armed with these facts, the result follows quite simply: to calculate h_j and τ_j we may work modulo the ideal $\mathfrak{n}(V)\otimes\Lambda^*(\mathfrak{g}^*)$: then

$$\tau_k=\sum_1^n (d\theta_i)^k$$

and

$$h_k=\text{trace}\,(\Omega^{k-1}\Theta)=\sum_1^n \theta_j(d\theta_j)^{k-1}.$$

Now consider a closed form $h_{i_1}\cdots h_{1_k}\tau^a$. Since $i_1+|a|\geq n+1$, no more than $1 h_i$ can occur, by Bott's vanishing theorem: else one would have a polynomial in the $d\theta_j$ of weight $>n$. For the same reason, $i_1+|a|=n+1$, concluding the proof.

2. Parabolic foliations

Let $\mathfrak{g}$ be a real semi-simple algebra of non-compact type, and $\mathfrak{p}$ a parabolic subalgebra: in this section we shall work out the special constraints on the connection and curvature matrices associated to our ad-connection ∇^1 introduced above, which arise from the parabolicity of $\mathfrak{p}$. As explained above, we may pass to complexifications $\mathfrak{g}^c$, $\mathfrak{p}^c$ first. Our key result here is that if $\tau_1=\text{trace}\,(\Omega)$, the first Chern form for ∇^1, then $\tau_1^n\neq 0$, $n=\dim_c(\mathfrak{g}^c/\mathfrak{p}^c)$: indeed, that τ_1 represents a Kähler class for the complex variety G^c/P^c.

Let $\mathfrak{h}$ be a Cartan subalgebra. One can find a simple system of roots $\{a_1, \ldots, \alpha_l\}\subseteq\mathfrak{h}^*$ which is adapted to $\mathfrak{p}^c$ in the following sense. Let Δ be the set of all positive roots *re* $\cdot\{\alpha_1, \ldots, \alpha_l\}$ and decompose $\mathfrak{g}^c$ as an ad$\,(\mathfrak{h})$-module in the usual way:

$$\mathfrak{g}^c=\mathfrak{h}+\sum_\Delta(\mathfrak{g}_\alpha+\mathfrak{g}_{-\alpha}). \tag{1}$$

For any $\beta \in \Delta$, define $\mu_j(\beta)$ = coefficient of α_j in the expansion $\beta = \sum_1^l m_k\alpha_k$ – so that $\mu_j(\beta) = m_j$. Then there is a subset $\{i_1, \ldots, i_k\} \subseteq \{1, 2, \ldots, \}$ such that if we put

$$\Delta' = \{\gamma \in \Delta \mid \mu_{i_j}(\gamma) \neq 0 \text{ for at least one } i_j \text{ in the subset}\}$$
$$\Delta'' = \Delta - \Delta'$$

then we can write the above decomposition (1) as

$$\mathfrak{g}^c = \Big(\mathfrak{h} + \sum_{\Delta''} \mathfrak{g}_\beta + \mathfrak{g}_{-\beta}\Big) + \Big(\sum_{\Delta'} \mathfrak{g}_\gamma\Big) + \Big(\sum_{\Delta'} \mathfrak{g}_{-\gamma}\Big) \tag{2}$$

and $\mathfrak{p}^c$ = direct sum of the first two summands. We shall say that $\mathfrak{p}^c$ is defined by $\{\alpha_{i_1}, \ldots, \alpha_{i_k}\}$.

Label the three summands in the decomposition (2) by $\mathfrak{r}$, $\mathfrak{u}$, $\mathfrak{v}$:

$$\mathfrak{r} = \Big(\mathfrak{h} + \sum_{\Delta''} \mathfrak{g}_\beta + \mathfrak{g}_{-\beta}\Big), \qquad \mathfrak{u} = \sum_{\Delta'} \mathfrak{g}_\gamma, \qquad \mathfrak{v} = \sum_{\Delta'} \mathfrak{g}_{-\gamma}.$$

In these terms, $\mathfrak{p}^c \simeq \mathfrak{r} \oplus \mathfrak{u}$, and $\mathfrak{v}$ gives the canonical splitting $\sigma : \mathfrak{g}^c/\mathfrak{p}^c \to \mathfrak{v}$ alluded to in Section 1.

Choosing a Weyl basis for $\mathfrak{g}^c$ (decomposed as in (1)) it follows from the standard theory that $\mathfrak{r}$ is a reductive subalgebra – the so-called Levi factor of $\mathfrak{p}^c$: that $\mathfrak{u}$, $\mathfrak{v}$ are isomorphic nilpotent subalgebras, which are both $\mathfrak{r}$-modules under ad, and dual as $\mathfrak{r}$-modules. Finally $\mathfrak{v} \simeq \mathfrak{g}^c/\mathfrak{p}^c$ as $\mathfrak{r}$-modules, so our splitting σ is $\mathfrak{r}$-equivariant. All this is standard structure-theory and can be found in Bourbaki [3] or Helgason [8].

At this point we can already make some reductions in our general formula for the curvature Ω of ∇^1 which was given by $d''\Theta$ (Section 1). By construction the entries θ_{ij} of Θ lie in $\mathfrak{p}^*$ [3] and d'' on $\mathfrak{p}^*$ is given by

$$d'' : \mathfrak{p}^* \xrightarrow{d} (\mathfrak{p}^* \wedge \mathfrak{p}^*) \oplus (\mathfrak{p}^* \oplus \mathfrak{v}^*) \wedge \mathfrak{v}^* \xrightarrow{\pi} (p^* \oplus \mathfrak{v}^*) \wedge \mathfrak{v}^*$$

the second map π being projection on the second summand.

PROPOSITION 2.1. $\omega_{ij} = d''\theta_{ij} \in \mathfrak{u}^* \wedge \mathfrak{v}^*$.

Proof. Writing $\mathfrak{p} = \mathfrak{r} \oplus \mathfrak{u}$, we have to show that d'' has no components in $\mathfrak{r}^* \wedge \mathfrak{v}^*$

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[3] Henceforth we will be dealing with $\mathfrak{g}^c$, $\mathfrak{p}^c$, and so we drop the Superscript $c : \mathfrak{p}^* = \mathrm{Hom}_C(\mathfrak{p}^c, C)$.

and $\mathfrak{v}^* \wedge \mathfrak{v}^*$. Using the duality of d and the Lie-bracket, it is equivalent to show that $[\mathfrak{r}, \mathfrak{v}]$ and $[\mathfrak{v}, \mathfrak{v}]$ have no component in $\mathfrak{p}$: but $\mathfrak{v}$ is an $\mathfrak{r}$-module, and subalgebra of $\mathfrak{g}$.

This proposition establishes two basic properties of the trace-forms $\tau_j = \operatorname{trace}(\Omega^j)$. From 2.1 we see that $\tau_j \in \Lambda^{2j}(\mathfrak{u}^* \oplus \mathfrak{v}^*)$ and even more, that τ_j lies in $\Lambda^j(\mathfrak{u}^*) \otimes \Lambda^j(\mathfrak{v}^*)$: in the natural double-grading of $\Lambda^*(\mathfrak{u}^* \oplus \mathfrak{v}^*)$, τ_j is of type (j, j). Moreover, each τ_j is $\mathfrak{r}$-invariant: for any $R \in \mathfrak{r}$, the adjoint action of R on $\Lambda^*(\mathfrak{u}^* \oplus \mathfrak{v}^*)$ is given by the lie-derivative $\mathscr{L}_R = i_R d + d i_R$; now τ_j is closed by construction, and by 2.1, $i_R \cdot \tau_j = 0$. We summarize these facts in

PROPOSITION 2.2. *The trace forms* $\tau_j = \operatorname{trace}(\Omega^j)$ *lie in the relative complex* $\Lambda^*(\mathfrak{g}^*/\mathfrak{r}^*)^{\mathfrak{r}} = \Lambda^*(\mathfrak{u}^* \oplus \mathfrak{v}^*)^{\mathfrak{r}}$.

Curiously enough, the fact that the τ_j are of type (j, j) is of no importance for us.

We now come to analyzing τ_1 in more detail, and for this we need to introduce the whole notational paraphernalia associated to a Weyl basis for $\mathfrak{g}^c$. Let $(\,,\,)$ denote the Killing form: for every $\alpha \in \Delta$ there is a "co-root" $H_\alpha \in \mathfrak{h}$ which is defined by

$$\alpha'(H_\alpha) = \frac{2(\alpha', \alpha)}{\|\alpha\|^2}, \quad \text{for all} \quad \alpha' \in \Delta.$$

Also, for $\alpha \in \Delta$ we can find generators $X_\alpha \in \mathfrak{g}_\alpha$, $Y_\alpha \in \mathfrak{g}_{-\alpha}$ such that $[X_\alpha, Y_{-\alpha}] = H_\alpha$. It will be convenient to distinguish the roots in Δ', Δ'' by using different Greek letters for them: we adopt the convention that α denotes a *general* root in Δ, β a root in Δ'' and γ a root in Δ'. We number the roots in Δ' in some fashion: $\Delta' = \{\gamma_1, \ldots, \gamma_n\}$ and then we write X_j, Y_j for X_{γ_j}, Y_{γ_j} and correspondingly ξ_j, η_j for the dual basis elements in $\mathfrak{g}^*$. With this understood, we have our basic

THEOREM 2.3. $\theta_{ii} = \gamma_i : \tau_1 = \sum_1^n d''\gamma_i = -\sum_1^n b_j \xi_j \wedge \eta_j$ *where each* $b_j > 0$.

Proof. By construction, the θ_{ij} are the *unique* forms in $\mathfrak{p}^*$ which satisfy $d\eta_i = \sum \theta_{ij} \wedge \eta_j$. Evaluating this on a pair (X, Y_k) where $X \in \mathfrak{p}$ we get $\theta_{ik}(X) = -\eta_i[X, Y_k]$. In particular, $\theta_{ii} \in \mathfrak{h}^*$ and $\theta_{ii}(H) = -\eta_i[H, Y_1] = +\gamma_i(H)$, so $\theta_{ii} = \gamma_i$.

Now consider $d''\gamma_i$, evaluated on a pair of Weyl elements $X_j \in \mathfrak{g}_{\gamma_j}$, $Y_k \in \mathfrak{g}_{-\gamma_k} : d''\gamma_i(X_j, Y_k) = -\gamma_i[X_j, Y_k]$. This is nonzero only when $j = k$, and then it is

$$-\gamma_i(H_j) = -\frac{2(\gamma_i, \gamma_i)}{\|\gamma_j\|^2}.$$

Thus

$$d''\gamma_i = -\sum_1^n \frac{(\gamma_i, \gamma_j)}{\|\gamma_j\|^2}\, \xi_j \wedge \eta_j$$

and summing over i, we get $\tau_1 = -\sum_1^n b_j \xi_j \wedge \eta_j$ where

$$b_j = \frac{(\gamma, \gamma_j)}{\|\gamma_j\|^2}$$

with $\gamma = \sum_1^n \gamma_i$.

It remains to prove that $(\gamma, \gamma_j) > 0$ for every γ_j. But $\rho = \sum_\Delta \alpha$ and $\sigma = \sum_{\Delta''} \beta$ so that $\gamma = \rho - \alpha$; and recall that for any $\alpha \in \Delta$, $(\rho, \alpha) > 0$ because $\rho = 2\sum$ (fund. weights). We claim that $(\gamma, \beta) = 0$ for all $\beta \in \Delta''$: it suffices to prove this for the simple roots α_j which lie in Δ'', and here it follows from the fact that $(\rho, \alpha_j) = \|\alpha_j\|^2 = (\sigma, \alpha_j)$. Now suppose $(\gamma, \gamma_i) \leq 0$: without loss of generality we may suppose that γ_i is "minimal" in the sense that $\gamma_i \neq \gamma_j + \beta$. Then

$$0 \geq (\gamma, \gamma_i) = (\rho, \gamma_i) = (\sigma, \gamma_i)$$

so $(\sigma, \gamma_i) > 0$: hence $(\beta, \gamma_i) > 0$ for at least one $\beta \in \Delta''$ which implies that $\gamma_i - \beta$ is a root. If $\{\alpha_{i_1}, \ldots, \alpha_{i_k}\}$ are the roots defining p, at least one of them occurs in γ_i, and none of them occur in β: hence $\gamma_i - \beta \in \Delta'$, contradicting the minimality of γ_i.

As a first consequence of the theorem, $\tau_1^n \neq 0$. Actually τ_1 is a Kähler class for a suitable left-invariant form on G^c/P^c as we now proceed to explain. Let $\mathfrak{g}_0$ be a compact real-form of $\mathfrak{g}^c$: and let $\mathfrak{k} = \mathfrak{g}_0 \cap \mathfrak{r}$. Then $\mathfrak{g}_0 = \mathfrak{k} \oplus \mathfrak{m}$, as vector spaces and even real ad $(\mathfrak{k})$-modules: and $\mathfrak{k} \otimes \mathbf{C} = \mathfrak{r}$, $\mathfrak{m} \otimes \mathbf{C} = \mathfrak{u} + \mathfrak{v}$. Indeed, if G_0, K denote the compact connected groups corresponding to $\mathfrak{g}_0, \mathfrak{k}$, then (Helgason [8])

1. $G_0/K \simeq G^c/P^c$.
2. the complex-structure on G_0/K induced from #1 is realized on the tangent space $T_e(G_0/K) \simeq \mathfrak{m}$ by an element $Z \in$ center of K such that $\mathrm{Ad}(Z)^2|_{\mathfrak{m}} = -1_{\mathfrak{m}}$; and $\mathfrak{u}$, $\mathfrak{v}$ are the $\pm i$-eigenspaces of $\mathrm{Ad}(Z)$.

Thus we may identify the (1, 0) and the (0, 1) cotangent spaces of G^c/P^c at e with $\mathfrak{u}^*$, $\mathfrak{v}^*$ respectively. Then τ_1 is a left-invariant 2-form of type (1, 1) on G^c/P^c (pulled back to G^c): since $\{\xi_1, \ldots, \xi_n\}$ is a basis for $\mathfrak{u}^*$, $\{\eta_1, \ldots, \eta_n\}$ for $\mathfrak{v}^*$, and each $b_j > 0$ (Theorem 2.3) τ_1 is the Kähler class[(4)] associated to a suitable left

[4] Unless my minus signs are in error, $\tau_1 = -$(Kähler-class) but this is of no moment.

invariant metric ds^2 on G^c/P^c. This observation is indispensable for the next section.

While we are on the subject, let us point out that the relative complex $\Lambda^*(\mathfrak{g}, \mathfrak{r}) = \Lambda^*(\mathfrak{g}^*/\mathfrak{r}^*)^{\mathfrak{r}}$ introduced in 2.2 computes the cohomology $H^*_{DR}(G^c/P^c : C)$. To see this, it is enough to note $\Lambda^*(\mathfrak{g}, \mathfrak{r}) = \Delta^*_{\mathbf{R}}(\mathfrak{g}_0, \mathfrak{k}) \otimes C$, and to apply the classical Chevalley–Eilenberg theorem (see for example my notes [10], Appendix II, Section 2). Finally, by using the Laplace operator associated to ds^2, τ_1 is harmonic: and we shall identify $H^*_{DR}(G^c/P^c : C)$ with the harmonic part of the complex $\Lambda^*(\mathfrak{g}, \mathfrak{r})$.

3. The vanishing theorem

We shall now prove our main theorem for the secondary classes associated to $(\mathfrak{g}, \mathfrak{p})$: this is theorem 2 of the introduction.

THEOREM 3.1. *Let* $\phi = h_{i_1} \cdots h_{i_k} \tau^a (i_1 < i_2 \cdots)$ *be a closed form in* $\Lambda^*(\mathfrak{g}^*)$ *representing an element of the Vey basis for* $H^*(W)_n$ *associated to the connection* ∇^1.

(a) *If* $i_1 + |a| > n+1$, ϕ *is exact*

(b) *If* $i_1 + |a| = n+1$, *then* ϕ *is cohomologous to a class* $c \cdot h_1 h_{i_2} \cdots h_{i_k} \tau_1^n$, *for some scalar* c.

The first assertion 3.1(a) says precisely that the rigid classes are zero: the second assertion verifies Fuks' conjecture [4] that the only nonzero secondary classes one gets are already detected in the "standard" foliation $(\mathfrak{sl}(n+1), \mathfrak{p}_1)$ coming from the linear action of $SL(n+1)$ on the projective space P_n. (see Section 1). This does *not* preclude the possibility that there might be more independence among these classes than the 2^{n-1} independent classes in $H^*(\mathfrak{sl}(n+1))$: for it may well happen that two given Vey forms ϕ_1, ϕ_2 as in 3.1(b) evaluated on two examples $(\mathfrak{g}, \mathfrak{p})_1$ and $(\mathfrak{g}, \mathfrak{p})_2$ give different "ratios": for this purpose, one would have to compute the scalars c (in 3.1(b)) in some explicit way. This can in principle be done via the structure-constants associated to the root system: we shall return to this question in a subsequent paper when we discuss variation.

The proof of the theorem is a fairly straightforward consequence of a certain divisibility result which we now explain. In Section 2 we explained that τ_1 was the Kähler class associated to a certain left invariant Kähler metric on G^c/P^c: and using this metric we identified the de Rham cohomology of G^c/P^c with the graded *subspace* of harmonic forms in $\Lambda^*(\mathfrak{g}, \mathfrak{r})$. We now define τ^a_H = harmonic-component of τ^a: note that in general this is not multiplicative. We set

$$\tau^a = \tau^a_H + d\sigma \qquad \sigma \in \Lambda^*(\mathfrak{g}, \mathfrak{r})$$

LEMMA 3.2. *Given ϕ as in the theorem, ϕ is cohomologous to $h_{i_1}\cdots h_{i_k}\tau_H^a$.*

Proof. We need only show that $h_{i_1}\cdots h_{i_k}\,d\sigma$ is exact. Since ϕ is a closed form in the Vey basis, $i_1+|a|\geq n+1$, or $2i_1+\deg(\tau^a)\geq 2n+2$. Since $\deg(\sigma)=\deg(\tau^a)-1$, we find that $\tau_{i_j}\sigma=0$: it is sufficient to check this for $\tau_{i_1}\sigma$, and this follows from the inequalities above, since $\Lambda^*(\mathfrak{g},\mathfrak{r})$ is concentrated in degrees $\leq 2n$. Hence, if we put $\psi=h_{i_1}\cdots h_{i_k}\sigma$, $d\psi=\pm h_{i_1}\cdots h_{i_k}\,d\sigma$.

What the lemma 3.2 really says is that we can replace τ^a by any closed form in $\Lambda^*(\mathfrak{g},\mathfrak{r})$ which defines the same cohomology class in $H^*(G^c/P^c:C)$, without altering the cohomology class of ϕ. We now state the divisibility result proper. First recall that since ϕ is a form in the Vey basis, $i_1\leq a_1\leq|a|$, so that $|a|\geq(n+1)/2$, or $\deg(\tau^a)\geq n+1$.

PROPOSITION 3.3. *If $\deg(\tau^a)\geq n+1$, then $\tau_H^a=\tau_1\rho$, for some closed form ρ in $\Lambda^*(\mathfrak{g},\mathfrak{r})$.*

Proof. G^c/P^c is a Kähler variety with Kähler class τ_1, so the "Hard Lefschetz Theorem" (Griffiths and Schmid [6], p. 41 or Weil [11], p. 28) says that the Lefschetz operator L (multiplication by τ_1) induces isomorphisms

$$L^k:H^{n-k}(G^c/P^c)\xrightarrow{\sim}H^{n+k}(G^c/P^c)$$

and so every element in $H^{n+l}(G^c/P^c)$, $l\geq 1$ is divisible by τ_1.

Now the proof of the theorem is deduced thus.

Proof of 3.1. Using the two lemmas, we may suppose without loss of generality that

$$\phi=h_{i_1}\cdots h_{i_k}\tau_1\rho$$

where $\deg\rho=2|a|-2$.

a: If $i_1+|a|\geq n+2$, then $i_1>1$, else $\tau^a=0$. So define $\zeta=h_1h_{i_1}\cdots h_{i_k}\rho$. As above, $\tau_{i_j}\,\rho=0$ for each i_j, (because $2i_j+\deg\rho\geq 2n+4$) so $d\zeta=\pm\phi$.

b: If $i_1=1$, then $i_1+|a|=n+1$ implies that $|a|=n$. The complex $\Lambda^*(\mathfrak{g},\mathfrak{r})$ is 1-dimensional in degree $2n$, so $\tau^a=c\cdot\tau_1^n$. If $i_1>1$, use the same form ζ as above: now $\deg(\tau_{i_1}\rho)=2n$ and $\tau_{i_j}\cdot\rho=0$ for $j>1$. Hence $d\zeta=\pm\phi\pm h_1h_{i_2}\cdots h_{i_k}\tau_{i_1}\cdot\rho$ and $\tau_{i_1}\cdot\rho=c\tau_1^n$ for some constant c.

Using the methods above we can prove a "proportionality principle" analogous to the one in Bott [1]. In some ways it is stronger because we can treat all the movable secondary classes, rather than just $h_1\tau^a$: on the other hand, we do not have a "concrete" description of the τ^a in terms of the characteristic numbers of G^c/P^c.

PROPOSITION 3.4. *Let $\phi = h_{i_1} \cdots h_{i_k}\tau^a$ and $\psi = h_{i_1} \cdots h_{i_k}\tau^b$ be two closed movable Vey forms with the same string of h_i's and $|a| = |b| \geq (n+1)/2$. Then there are constants A, B depending on $(\tau_{i_1}, \tau^a, \tau^b)$ such that $B\phi = A\psi$ in cohomology.*

Proof. The conditions on $|a|$, $|b|$ imply that $\tau^a = \tau_1\rho$, $\tau^b = \tau_1\sigma$ and $\tau_{i_1}\rho = A\tau_1^n$, $\tau_{i_1}\sigma = B\tau_1^n$. Then by the proof of 3.1 above,

$$\phi \sim Ah_1h_{i_2} \cdots h_{i_k}\tau_1^n \quad \text{(cohomologous)}$$

$$\psi \sim Bh_1h_{i_2} \cdots h_{i_k}\tau_1^n$$

so the result follows. Of course, just as in [1], we do not make any claim about when $h_1h_{i_2} \cdots h_{i_k}\tau_1^n$ is nonzero.

Finally, here is a computation to show that the strong form of Fuks' conjecture is not true. We will compare the classes $h_1\tau_1^7$ and $h_1\tau_7$ for the standard example $(\mathfrak{sl}(8), \mathfrak{p}_1)$ of codimension 7, with another codimension 7 pair $(\mathfrak{g}, \mathfrak{p})$ to be described below. Both these classes lie in H^{15}.

(i) In the standard example, it is known that $h_1\tau_1^7 = 8^6 \cdot h_1\tau_7$ (see my notes [10], page 35) and by Fuks [4] or the work of Kamber Tondeur, that $h_1\tau_1^7$ is not zero in $H^{15}(\mathfrak{sl}(8))$.

(ii) For the example $(\mathfrak{g}, \mathfrak{p})$ we take $\mathfrak{g} = \mathfrak{sl}(5)$ and $\mathfrak{p}$ to be the non-maximal parabolic defined by the first two roots α_1, α_2: in terms of matrices

$$\mathfrak{p} = \{(a_{ij}),\ 0 \leq i,\ j \leq 4 \mid \textstyle\sum a_{jj} = 0,\ a_{i0} = 0,\ i \geq 1,\ a_{j1} = 0,\ j \geq 2\}$$

This is a codimension 7 foliation whose associated complex variety is the natural P_3-bundle over P_4. An explicit calculation of $h_1 \in H^1(\mathfrak{r})$ together with an elementary spectral sequence argument shows that $h_1\tau_1^7 \neq 0$ in $H^{15}(\mathfrak{sl}(5))$. For a suitable generator x of this group one gets

$$h_1\tau_1^7 = (2^{12} \cdot 3^3 \cdot 5 \cdot 7)x$$

rather easily: and with considerably more work,

$$h_1\tau_7 = -(2^4 \cdot 5 \cdot 7)x$$

so that the ratios are quite different in the two cases.

As mentioned in the introduction, D. Baker [12] has systematic independence results for some classes of parabolic foliations. These give counterexamples to the strong Fuks' conjecture, and they also go some way towards the computation of the scalars alluded to in the beginning of this section. However, for the most part his explicit computations (Section 5 of [12]) deal only with maximal parabolic subgroups: it is for this reason alone that we have sketched the example above.

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