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Quasiconformal circles and Lipschitz classes

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1. Introduction

A Jordan curve J in the extended complex plane $\bar{\mathbf{C}}$ is termed a *quasiconformal circle* if there exists a quasiconformal mapping of $\bar{\mathbf{C}}$ onto itself which carries J onto the unit circle in $\mathbf{C}$. A purely geometric characterization of the class of quasiconformal circles was provided by Ahlfors in [1]. In order to formulate his result in the manner most convenient for our purposes we associate to an arbitrary Jordan curve J in $\bar{\mathbf{C}}$ a number $k(J)$ in the interval $[0,1]$ as follows:

$$k(J)=\inf\frac{|z_1-z_3|\,|z_2-z_4|}{|z_1-z_2|\,|z_3-z_4|+|z_1-z_4|\,|z_2-z_3|},\tag{1}$$

where the infimum is extended over the set of ordered quadruples z_1, z_2, z_3, z_4 of finite points on J with the property that z_1 and z_3 separate z_2 and z_4 on J. Ahlfors proved that J is a quasiconformal circle if and only if $k(J)>0$. Conforming to the usage in [2], [3] and [5], we refer to a Jordan curve J as a k-*circle* if $k(J)\geq k>0$.

The invariance of cross-ratios under Möbius transformations implies that the image of a k-circle under a Möbius transformation is again a k-circle. It is not difficult to verify that a 1-circle is either a euclidean circle or a straight line. An arbitrary k-circle, on the other hand, can be quite an exotic curve. For example, a k-circle in the finite plane $\mathbf{C}$ may fail to be rectifiable or even to contain a rectifiable subarc, although it must be of 2-dimensional Lebesgue measure zero. For k in $(0, 1]$ the canonical example of a k-circle is supplied by the Jordan curve J,

$$J=\{z\in\bar{\mathbf{C}}\colon\ z=0,\ z=\infty,\quad \text{or}\quad |\mathrm{Arg}\, z|=\arcsin k\}.$$

Indeed, an elementary calculation reveals that $k(J)=k$ for this curve.

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A complex-valued function f on a set A in $\mathbf{C}$ is said to belong to the class $\mathrm{Lip}_\alpha(A)$, where $0<\alpha\leq 1$, if there is a number M such that

$$|f(z)-f(w)|\leq M|z-w|^\alpha$$

for all z and w in A. Suppose that D is a bounded Jordan domain in $\mathbf{C}$ and that f is a K-quasiconformal mapping of D onto the open unit disk B in $\mathbf{C}$. It has long been known that f belongs to $\mathrm{Lip}_\alpha(A)$ for each compact subset A of D, with $\alpha=1/K$. In general, however, f need not belong to $\mathrm{Lip}_\alpha(D)$ for any α. If, on the other hand, ∂D is a quasiconformal circle, it is possible to extend f to a quasiconformal mapping of $\bar{\mathbf{C}}$ onto itself and this fact implies that f belongs to $\mathrm{Lip}_\alpha(D)$ for some α in $(0, 1/K]$. How large can one expect this α to be? In this paper we determine the largest Hölder exponent α valid for all D in the class of domains bounded by k-circles. We also consider the analogue of this problem for K-quasiconformal mappings of B onto domains belonging to this class. We prove the following theorems.

THEOREM 1. *Let D be a bounded domain in $\mathbf{C}$ such that ∂D is a k-circle and let f be a K-quasiconformal mapping of D onto B. Then f belongs to* $\mathrm{Lip}_\alpha(D)$ *for*

$$\alpha=\frac{\pi}{2K(\pi\text{-}\arcsin k)}. \tag{2}$$

This Hölder exponent is the best possible.

THEOREM 2. *Let D be a bounded domain in $\mathbf{C}$ such that ∂D is a k-circle and let f be a K-quasiconformal mapping of B onto D. Then f belongs to* $\mathrm{Lip}_\beta(B)$ *for*

$$\beta=\frac{2\arcsin^2 k}{\pi K(\pi\text{-}\arcsin k)}. \tag{3}$$

We are uncertain whether the Hölder exponent β in Theorem 2 is the best possible or whether it is subject to improvement. Theorem 1 has an interesting corollary which is worth stating as a separate theorem. This result was obtained independently by Lesley [7] using different methods.

THEOREM 3. *Let D be a bounded domain in $\mathbf{C}$ such that ∂D is a quasiconformal circle and let f map D conformally onto B. Then f belongs to* $\mathrm{Lip}_\alpha(D)$ *for some α greater than* $1/2$.

The proofs of Theorems 1 and 2 are presented in Sections 2 and 3, respectively, of this paper.

The genesis of the present paper is to be found in work of Warschawski [11, 12] concerning conformal mappings and boundary Hölder continuity. (See also [4], [7] and [13].) He established results reminiscent of Theorems 1 and 2 in the conformal case for a class of domains included in, but less general than, the class of domains which are bounded by quasiconformal circles. For example, Warschawski considered, exclusively, domains with rectifiable boundaries. The techniques he utilized are quite different from those which we employ.

A number of authors have studied k-circles and the properties of conformal and quasiconformal mappings of domains bounded by such curves. In addition to the seminal paper of Ahlfors referred to earlier, we would like to cite, in particular, the work of Blevins [2, 3] and of Rickman [9]. We wish also to acknowledge the help afforded us by conversations with D. Blevins and with F. W. Gehring.

2. Proof of Theorem 1

In matters regarding notation and terminology we will conform to the usage in the book of Lehto and Virtanen [6], unless some explicit stipulation to the contrary is made. We denote by B the open unit disk in $\mathbf{C}$ and by $S(r)$ the circle in $\mathbf{C}$ of radius $r>0$ centered at the origin. The diameter of a set A in $\mathbf{C}$ will be denoted diam A.

The following simple observation permits us to deal exclusively with the conformal case in carrying out the proofs of Theorems 1 and 2.

LEMMA 1. *Let D be a bounded simply connected domain in $\mathbf{C}$. If some conformal mapping of D onto B belongs to* $\mathrm{Lip}_\alpha(D)$, *then each K-quasiconformal mapping of D onto B belongs to* $\mathrm{Lip}_{\alpha/K}(D)$; *if some conformal mapping of B onto D belongs to* $\mathrm{Lip}_\beta(B)$, *then each K-quasiconformal mapping of B onto D belongs to* $\mathrm{Lip}_{\beta/K}(B)$.

Proof. We provide the details of the proof for the first assertion. The latter assertion can be treated in a similar manner. Suppose that f is a conformal mapping of D onto B which belongs to $\mathrm{Lip}_\alpha(D)$ and let g be an arbitrary K-quasiconformal mapping of D onto B. Then $h=g\circ f^{-1}$ is a K-quasiconformal self-mapping of B and, as such, h belongs to $\mathrm{Lip}_{1/K}(B)$. (See [6, 8].) Consequently, $g=h\circ f$ belongs to $\mathrm{Lip}_{\alpha/K}(D)$.

We record for use in the proof of Theorem 1 the following geometric property of k-circles. Its elementary verification is included for the sake of completeness.

LEMMA 2. *Let J be a k-circle in $\mathbf{C}$. For any pair of points z and w on J the arcs A and A' into which J is divided by z and w satisfy*

$$\min\{\operatorname{diam} A, \operatorname{diam} A'\} \leq \frac{2}{k}|z-w|. \tag{4}$$

Proof. Fix a pair of points z and w on J and set $r=|z-w|$. Suppose that the corresponding arcs A and A' contain points y and y', respectively, such that $|z-y|=|z-y'|=s>0$. From (1) we infer

$$k \leq \frac{|z-w|\,|y-y'|}{|z-y|\,|w-y'|+|z-y'|\,|y-w|} = \frac{r|y-y'|}{s|w-y'|+s|y-w|} \leq \frac{r}{s},$$

whence $s \leq r/k$. This implies that either A or A' is contained in the closed disk of radius r/k centered at z and (4) follows.

Proof of Theorem 1. Let f be a conformal mapping of D onto B. We verify that f belongs to $\mathrm{Lip}_\alpha(D)$, with

$$\alpha = \frac{\pi}{2(\pi\text{-}\arcsin k)}. \tag{5}$$

Theorem 1 then follows from Lemma 1. The mapping f admits an extension to a homeomorphism of $\bar{D}$ onto $\bar{B}$. We retain the notation f for the extended mapping.

Write $d=\operatorname{dist}(f^{-1}(0), \partial D)$. Consider a pair of points z and w on ∂D, dividing ∂D into two arcs, of which A will denote the one of minimal diameter. We estimate the harmonic measures $\omega=\omega(f^{-1}(0), A, D)$ and $\omega'=\omega(0, f(A), B)$. First, using a result of Blevins [2], we obtain

$$\omega \leq \frac{4}{\pi}\arctan\left[\left(\frac{\operatorname{diam} A}{d}\right)^\alpha\right] \leq \frac{4}{\pi}\left(\frac{\operatorname{diam} A}{d}\right)^\alpha,$$

with α given by (5). In view of (4), we infer that

$$\omega \leq \frac{4}{\pi}\left(\frac{2}{dk}\right)^\alpha |z-w|^\alpha. \tag{6}$$

On the other hand, it is apparent that

$$\omega' \geq \frac{|f(z)-f(w)|}{2\pi}. \tag{7}$$

Since f is conformal in D, $\omega = \omega'$. Combining this fact with (6) and (7), we arrive at the conclusion that

$$|f(z)-f(w)| \leq M\,|z-w|^{\alpha} \tag{8}$$

for all z and w on ∂D, where $M = 8(2/dk)^{\alpha}$. Theorem 10.1 in [10] allows us to conclude that (8) is valid for all z and w in D. Therefore f belongs to $\mathrm{Lip}_{\alpha}\,(D)$, as asserted.

The following example demonstrates that the Hölder exponent in Theorem 1 is the best possible.

EXAMPLE 1. Let $k \in (0, 1]$, let $K \in [1, \infty)$ and let α be given by (2). There exist a bounded domain D in $\mathbf{C}$ such that ∂D is a k-circle and a K-quasiconformal mapping of D onto B which does not belong to $\mathrm{Lip}_{\alpha'}\,(D)$ for any α' greater than α.

Proof. Consider the domains

$$G = \{z \in \mathbf{C}\colon\ |\mathrm{Arg}\ z| < \pi\text{-arcsin}\ k\}$$

and

$$G_1 = \left\{z \in \mathbf{C}\colon\ |\mathrm{Arg}\ z| < \frac{\pi}{2}\right\}.$$

The Möbius transformation ϕ which satisfies $\phi(1) = 0$, $\phi(0) = 1$ and $\phi(-1) = \infty$ maps G_1 onto B. Moreover, $D = \phi(G)$ is a bounded domain in $\mathbf{C}$ and, since ∂G is a k-circle, ∂D is a k-circle as well. There exists a constant C such that

$$C^{-1}\,|z| \leq |\phi(z)-1| \leq C\,|z| \tag{9}$$

for all z in some neighborhood of the origin. Let g and h designate the homeomorphisms defined on $\bar{G}_1$ and $\bar{G}$, respectively, by

$$g(z) = z\,|z|^{(1/K)-1}, \quad h(z) = z^{\pi/2(\pi\text{-arcsin}\ k)}$$

for $z \neq 0, \infty$, while $g(0)=h(0)=0$ and $g(\infty)=h(\infty)=\infty$. The mapping f defined on $\bar{D}$ by

$$f(z)=\phi \circ g \circ h \circ \phi^{-1}(z)$$

is a homeomorphism which maps D K-quasiconformally onto B. Obviously $f(1)=1$. It is a consequence of (9) that, for z in D sufficiently close to 1,

$$C^{-2}|z-1|^{\alpha} \leq |f(z)-1| \leq C^{2}|z-1|^{\alpha}.$$

This implies that f cannot belong to $\mathrm{Lip}_{\alpha'}(D)$ when α' exceeds α.

3. Proof of Theorem 2

The proof of Theorem 2 is a good deal more complicated than that of Theorem 1 and will require several preparatory results. In these results G will denote a domain whose boundary is a k-circle passing through 0 and ∞. We use the terminology cross-cut of G to indicate an open arc in G with two endpoints in ∂G. For $r>0$ each component of $G \cap S(r)$ is a cross-cut of G. Furthermore, it follows from elementary plane topology that $G \cap S(r)$ has a component whose endpoints separate 0 and ∞ on ∂G.

Our first result will be needed to establish a subsequent modulus estimate.

LEMMA 3. *Let γ be a rectifiable cross-cut of G whose endpoints separate* 0 *and ∞ on ∂G. Then*

$$\int_{\gamma} \frac{ds}{|z|} \geq 2 \arcsin k. \tag{10}$$

Proof. Replacing, if necessary, G by $\phi(G)$ and γ by $\phi(\gamma)$, where ϕ is a mapping either of the form $\phi(z)=az$ or of the form $\phi(z)=a\bar{z}$, we are free to assume that γ has endpoints $z_1=1$ and $z_2=re^{i\theta}$, where $r \geq 1$ and where $0 \leq \theta \leq \pi$. Choose a cross-cut γ^* of the domain G^* complementary to $\bar{G}$ so that γ^* has endpoints 0 and ∞. Let L designate the branch of the logarithm in the domain $\mathbf{C} \setminus \bar{\gamma}^*$ which satisfies $L(1)=0$. Parametrizing the arc $\bar{\gamma}$ by arc length with initial point z_1, we find that

$$\int_{\gamma} \frac{ds}{|z|} \geq \left| \int_{\gamma} \frac{dz}{z} \right| = |L(z_2)-L(z_1)| = (\log^2 r + [\operatorname{Im} L(z_2)]^2)^{1/2}.$$

If $|\mathrm{Im}\, L(z_2)| \geq 2 \arcsin k$, (10) is clearly valid. This is certainly the case for $k=1$, when G is simply an open half-plane in $\mathbf{C}$. In what follows, therefore, we assume that $0<k<1$ and that $|\mathrm{Im}\, L(z_2)|<2 \arcsin k$. Since the complex number z_2 has a unique argument in the interval $(-\pi, \pi]$, we conclude that $\mathrm{Im}\, L(z_2)=\theta$. Consequently, $0 \leq \theta < 2 \arcsin k$ and

$$\int_\gamma \frac{ds}{|z|} \geq (\log^2 r + \theta^2)^{1/2}. \tag{11}$$

It follows from (1) that

$$\frac{|z-w|}{|z|+|w|} \geq k, \tag{12}$$

whenever the points z and w separate 0 and ∞ on ∂G. This fact, applied to the endpoints of γ, yields

$$|re^{i\theta}-1| \geq k(1+r).$$

To complete the proof of Lemma 3 it is sufficient to verify that for k in $(0,1)$ the function Φ,

$$\Phi(r, \theta) = \log^2 r + \theta^2,$$

satisfies

$$\Phi(r, \theta) \geq 4 \arcsin^2 k, \tag{13}$$

when (r, θ) is constrained to lie in the set E described by the conditions:

$$r \geq 1, \quad 0 \leq \theta \leq 2 \arcsin k, \quad |re^{i\theta}-1| \geq k(1+r).$$

Elementary considerations reveal that the minimum of Φ on E is attained at some point of the arc A,

$$A = \left\{ (r, \theta) \colon 1 \leq r \leq \frac{1+k}{1-k}, \quad \theta = \theta(r) \right\}.$$

Here

$$\theta(r) = \arccos \left[\frac{1}{2}(1-k^2)\left(r+\frac{1}{r}\right) - k^2 \right].$$

The minimum of Φ on A can be computed, albeit with some effort, using standard calculus techniques. We spare the reader the details: the minimum of Φ on A is $4\arcsin^2 k$ and is attained when $r=1$ and $\theta=2\arcsin k$. Consequently, (13) holds and (10) follows from (11) and (13).

We remind the reader that the modulus $M(\Gamma)$ of a family Γ of arcs in $\mathbf{C}$ is defined by

$$M(\Gamma)=\inf\int_{\mathbf{C}}\rho^2\,dm,$$

where m is 2-dimensional Lebesgue measure and where the infimum is taken over the collection $F(\Gamma)$ of Borel functions $\rho\colon \mathbf{C}\to[0,\infty]$ such that

$$\int_\gamma \rho\,ds\geq 1$$

for each rectifiable arc γ in Γ. If f is a conformal mapping, then

$$M[f(\Gamma)]=M(\Gamma) \tag{14}$$

for each family Γ of arcs in the domain of f.

In the next lemma the notation $A(a,b)$ designates the annulus $\{z\in\mathbf{C}\colon a<|z|<b\}$, where $0<a<b<\infty$.

LEMMA 4. *Let Γ be a family of cross-cuts of G which lie in $A(a,b)$ and which have endpoints separating* 0 *and* ∞ *on* ∂G. *Then*

$$M(\Gamma)\leq\frac{\pi\text{-}\arcsin k}{2\arcsin^2 k}\log\frac{b}{a}. \tag{15}$$

Proof. Define a Borel function ρ on $\mathbf{C}$ by

$$\rho(z)=\frac{1}{2|z|\arcsin k}$$

for z in $G\cap A(a,b)$ and $\rho(z)=0$ otherwise. Lemma 3 implies that

$$\int_\gamma \rho\,ds\geq 1$$

for each rectifiable γ in Γ. We conclude that ρ belongs to $F(\Gamma)$. Hence

$$M(\Gamma) \leq \int_{\mathbf{C}} \rho^2 \, dm = \frac{1}{4 \arcsin^2 k} \int_a^b \left(\int_{E_r} d\theta \right) \frac{dr}{r}, \tag{16}$$

where $E_r = \{\theta \in [0, 2\pi]: re^{i\theta} \in G\}$.

Let G^* denote the domain complementary to $\bar{G}$. For each $r > 0$, $G^* \cap S(r)$ has a component γ^* whose endpoints separate 0 and ∞ on $\partial G^* = \partial G$. The application of Lemma 3 to γ^* shows that the length of γ^* is at least $2r \arcsin k$. We can thus assert that

$$\int_{E_r} d\theta \leq 2\pi - 2 \arcsin k$$

for each $r > 0$. In conjunction with (16) this implies (15).

REMARK. Lemmas 3 and 4 make only weak use of the hypothesis that ∂G is a k-circle. Indeed, it is condition (12) which is crucial for their proofs, to wit, that

$$\frac{|z-w|}{|z|+|w|} \geq k,$$

whenever z and w separate 0 and ∞ on ∂G. This condition does not even imply that ∂G is a quasiconformal circle, much less a k-circle. Examples suggest that, in fact, the conclusion of Lemma 4 might be strengthened to

$$M(\Gamma) \leq \frac{1}{2 \arcsin k} \log \frac{b}{a}, \tag{17}$$

if full use were to be made of the assumption that ∂G is a k-circle. When $G = \{z \in \mathbf{C}: |\operatorname{Arg} z| < \arcsin k\}$ and when Γ is the family of cross-cuts of G in $A(a, b)$ which join the components of $\partial G \cap A(a, b)$, the modulus estimate (17) holds with equality. It can be argued on purely heuristic grounds that this configuration is extremal in estimating $M(\Gamma)$. At this juncture, however, we are unable to fashion a rigorous proof for such an assertion.

The next result contains the heart of the proof of Theorem 2. In it, H denotes the open upper half-plane in $\mathbf{C}$.

LEMMA 5. *There exists a constant $c \geq 1$ depending only on k such that, if g is* *a homeomorphism of $\bar{H}$ onto $\bar{G}$ which is conformal in H and which is normalized by* $g(0)=0$ *and* $g(\infty)=\infty$, *then*

$$|g(z)| \leq c\,|g(1)|\,|z|^{\beta} \tag{18}$$

for all z in $\bar{B} \cap \bar{H}$, *where*

$$\beta = \frac{2 \arcsin^2 k}{\pi(\pi - \arcsin k)}\,. \tag{19}$$

Proof. Theorem 1 in [1] implies that the k-circle ∂G admits a K^*-quasiconformal reflection, where K^* depends only on k. It then follows that g has a K^*-quasiconformal extension g^* to $\bar{\mathbf{C}}$. For $r>0$ set

$$L(r) = \max_{|z|=r} |g^*(z)|$$

and

$$l(r) = \min_{|z|=r} |g^*(z)|.$$

Because $g^*(\infty)=\infty$, we are assured that

$$L(r) \leq b\,l(r) \tag{20}$$

for all $r>0$, with $b=\exp(\pi K^*)$. (See [6, p. 111].)

Now fix z in $B \cap \bar{H}$, $z \neq 0$, and set $r=|z|$. Consider the family Γ of cross-cuts of H which lie in $A(r, 1)$ and which join the components of $\partial H \cap A(r, 1)$. A simple calculation gives

$$M(\Gamma) = \frac{1}{\pi} \log \frac{1}{r}\,. \tag{21}$$

It is apparent that

$$g[\bar{H} \cap A(r, 1)] \subset A(l, L),$$

where $l=l(r)$ and $L=L(1)$. The family $g(\Gamma)$ consists of cross-cuts of G in

$G \cap A(l, L)$ whose endpoints separate 0 and ∞ on ∂G. We apply Lemma 4 to $g(\Gamma)$ and conclude that

$$M[g(\Gamma)] \leq \frac{\pi\text{-}\arcsin k}{2 \arcsin^2 k} \log \frac{L}{l} = \frac{1}{\pi\beta} \log \frac{L}{l}, \tag{22}$$

with β given by (19). Since g is conformal in H, we can invoke (14), together with (21) and (22), and infer that

$$\beta \log \frac{1}{r} \leq \log \frac{L}{l}.$$

In combination with (20) this implies

$$|g(z)| \leq L(r) \leq bl \leq bLr^\beta \leq b^2 l(1) r^\beta \leq b^2 |g(1)| \, |z|^\beta,$$

establishing (18) for z in $B \cap \bar{H}$ with $c = \exp(2\pi K^*)$. By continuity, (18) is valid throughout $\bar{B} \cap \bar{H}$.

Having completed all preparations, we turn to the proof of Theorem 2.

Proof of Theorem 2. Let f be a conformal mapping of B onto D. We show that f belongs to $\mathrm{Lip}_\beta(B)$, where β is given by (19). Theorem 2 then follows from Lemma 1. It will be assumed that f has been extended to a homeomorphism of $\bar{B}$ onto $\bar{D}$, which we continue to denote by f.

Write $\delta = \operatorname{diam} D$. For points w and w' in $\bar{B}$ satisfying $|w - w'| \geq 1$ we can apply (8) to f^{-1} and observe that

$$1 \leq |w - w'| \leq M \, |f(w) - f(w')|^\alpha,$$

where α is given by (5) and where M is a constant which depends only on k and $d = \operatorname{dist}(f(0), \partial D)$. It follows, for such w and w', that

$$M^{-(1/\alpha)} \leq |f(w) - f(w')| \leq \delta. \tag{23}$$

We next fix a point z of ∂B and introduce auxiliary Möbius transformations ϕ_1 and ϕ_2,

$$\phi_1(w) = -i \frac{w - z}{w + z}, \qquad \phi_2(w) = \frac{w - f(z)}{w - f(-z)}. \tag{24}$$

Then ϕ_1 maps B onto the open upper half-plane H with

$$\phi_1(z)=0, \qquad \phi_1(-z)=\infty, \qquad \phi_1(iz)=1. \tag{25}$$

Furthermore, ϕ_1 maps the closed semi-circle $A=\{zw\colon w\in\partial B,\ \mathrm{Re}\, w\geq 0\}$ onto $\bar{B}\cap\partial H$ and satisfies

$$\frac{1}{2}|w-w'|\leq|\phi_1(w)-\phi_1(w')|\leq 2|w-w'| \tag{26}$$

for all w and w' in A. A straightforward calculation using (23) reveals that

$$\frac{1}{\delta}|w-w'|\leq|\phi_2(w)-\phi_2(w')|\leq \delta M^{(2/\alpha)}|w-w'|, \tag{27}$$

whenever w and w' belong to $f(A)$. The domain $G=\phi_2(D)$ is bounded by a k-circle which passes through 0 and ∞. In view of (24) and (25), Lemma 5 can be applied to the mapping g defined on $\bar{H}$ by

$$g(w)=\phi_2\circ f\circ\phi_1^{-1}(w).$$

With the aid of (18), (25), (26) and (27) we obtain

$$\begin{aligned}|f(z)-f(w)|&\leq\delta\,|\phi_2\circ f(w)|=\delta\,|g\circ\phi_1(w)|\\&\leq c\delta\,|g(1)|\,|\phi_1(w)|^{\beta}\leq 2c\delta^3M^{(2/\alpha)}\,|z-w|^{\beta}\end{aligned}$$

for all w in A, where $c\geq 1$ depends only on k and where β is given by (19). For w in $\partial B\backslash A$ we have $|z-w|>1$ and, as a consequence,

$$|f(z)-f(w)|\leq\delta\leq\delta\,|z-w|^{\beta}\leq 2c\delta^3M^{(2/\alpha)}\,|z-w|^{\beta}.$$

We have succeeded, therefore, in demonstrating that

$$|f(z)-f(w)|\leq M_1\,|z-w|^{\beta} \tag{28}$$

for all points z and w in ∂B, with $M_1=2c\delta^3M^{(2/\alpha)}$. Invoking Theorem 10.1 in [10], we can assert that (28) holds for all z and w in B. This establishes that f belongs to $\mathrm{Lip}_{\beta}\,(B)$, as maintained. The proof of Theorem 2 is complete.

Examples indicate that the Hölder exponent in Theorem 2 might be subject to improvement. Indeed, should the modulus estimate (17) be established, an improvement in Theorem 2 would result, without change in the proof: the exponent β could be replaced by the larger exponent β_0,

$$\beta_0 = \frac{2 \arcsin k}{\pi K}. \tag{29}$$

The next example shows that no further improvement of the exponent in Theorem 2 would then be possible.

EXAMPLE 2. Let $k \in (0, 1)$, let $K \in [1, \infty)$ and let β_0 be given by (29). There exist a bounded domain D in $\mathbf{C}$ such that ∂D is a k-circle and a K-quasiconformal mapping of B onto D which does not belong to $\mathrm{Lip}_{\beta'}(B)$ for any β' greater than β_0.

Proof. The present example is a simple variation on the theme of Example 1. Rather than the domain G used there, we consider its complementary domain G^*,

$$G^* = \{z \in \mathbf{C}\colon |\mathrm{Arg}\, z| < \arcsin k\},$$

along with the domain G_1,

$$G_1 = \left\{z \in \mathbf{C}\colon |\mathrm{Arg}\, z| < \frac{\pi}{2}\right\}.$$

Again we let ϕ be the Möbius transformation which satisfies $\phi(1) = 0$, $\phi(0) = 1$ and $\phi(-1) = \infty$. The domain $D = \phi(G^*)$ is contained in B and is bounded by a k-circle. Homeomorphisms g and h are defined on $\bar{G}_1$ by

$$g(z) = z|z|^{(1/K)-1}, \qquad h(z) = z^{(2 \arcsin k)/\pi}$$

for $z \neq 0, \infty$, while $g(0) = h(0) = 0$ and $g(\infty) = h(\infty) = \infty$. The mapping f defined on $\bar{B}$ by

$$f(z) = \phi \circ g \circ h \circ \phi^{-1}(z)$$

is a homeomorphism which maps B K-quasiconformally onto D and which satisfies $f(1) = 1$. It follows from (9) without difficulty that

$$C^{-2}\,|z-1|^{\beta_0} \le |f(z) - 1| \le C^2\,|z-1|^{\beta_0}$$

for z in B sufficiently close to 1. From this we infer that f does not belong to $\mathrm{Lip}_{\beta'}(B)$, if β' is larger than β_0.

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