On the p-periodicity of arithmetic subgroups of general linear groups.

Comment. Math. Helvetici 55 (1980) 499–509 Birkhäuser Verlag, Basel

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1. Introduction

1.1. For a group G of virtually finite (cohomological) dimension let $\hat{H}^i(G, A)$, $i \in \mathbf{Z}$, be the ith Farrell–Tate cohomology with coefficients in the G-module A (see [4]). The groups $\hat{H}^i(G, A)$ are always torsion-groups; they agree with the usual cohomology $H^i(G, A)$ for dimensions i greater than the virtual (cohomological) dimension of G.

The group G is called *periodic* if $\hat{H}^i(G, A)$ is periodic for all G-modules A (i.e., if there exists $m \in \mathbf{N}$, such that the functors $\hat{H}^i(G, -)$ and $\hat{H}^{i+m}(G, -)$ are naturally equivalent for all $i \in \mathbf{Z}$). The smallest such number m is called the period of G. The group G is called *p-periodic* for a prime p, if the p-primary component of $\hat{H}^i(G, A)$, written $\hat{H}^i(G, A, p)$, is periodic for all A; the p-period is denoted by m_p.

G is periodic if and only if G is p-periodic for all primes p; the period of G is then the greatest common multiple of the p-periods. If G is p-periodic, then every subgroup U of G is p-periodic, and the p-period of U divides the p-period of G.

1.2. In this paper, we consider linear groups $G = GL(n, R)$ where R is a ring of complex numbers containing $\mathbf{Z}$. In several cases $GL(n, R)$ is known to be virtually of finite dimension. We determine then the primes p for which $GL(n, R)$ is p-periodic, and the p-period m_p (Section 2). In Section 3 we consider, in particular, the following situation: Let K be an (algebraic) number field, $\mathbf{O}(K)$ the ring of algebraic integers in K and $\Delta(K)$ the discriminant of K. We then have:

THEOREM 1.1. *Let p be a prime not dividing $\Delta(K)$.*
(*a*) *If $p \leq (n/2)+1$, then $GL(n, \mathbf{O}(K))$ is not p-periodic.*
(*b*) *If $(n/2)+1 < p \leq n+1$, then $GL(n, \mathbf{O}(K))$ is p-periodic with p-period $2(p-1)$.*
(*c*) *If $p > n+1$, then $\hat{H}^i(GL(n, \mathbf{O}(K)), A, p) = 0$ for all $i \in \mathbf{Z}$ and all A.*

This theorem is established in Section 3 in a more general case including the S-arithmetic groups $GL(n, \mathbf{O}_S(K))$; the p-periodicity of these groups depends only on K, but not on the set S of valuations.

A further result (Theorem 2.1) can be used to decide, for primes p dividing $\Delta(K)$, whether $GL(n, \mathbf{O}(K))$ is p-periodic or not.

1.3. We also investigate the special linear groups $G = SL(n, R)$, where R is a ring of real numbers containing $\mathbf{Z}$; their p-periodicity is examined in Section 4. For a real number field K, we get

THEOREM 1.2. *Let p be a prime not dividing $\Delta(K)$ and $n \geq 3$.*

(*a*) *If $p \leq (n/2)+1$, then $SL(n, \mathbf{O}(K))$ is not p-periodic.*

(*b*) *If $(n/2)+1 < p \leq n+1$, then $SL(n, \mathbf{O}(K))$ is p-periodic. For $(n/2)+1 < p < n+1$, the p-period is $2(p-1)$; for $p = n+1$, it is $p-1$.*

(*c*) *If $p > n+1$, then $\hat{H}^i(SL(n, \mathbf{O}(K)), A, p) = 0$ for all $i \in \mathbf{Z}$ and all A.*

If we take $K = \mathbf{Q}$ in Theorem 1.2, we obtain

COROLLARY 1.3. *Let p be a prime and $n \geq 3$.*

(*a*) *If $p \leq (n/2)+1$, then $SL(n, \mathbf{Z})$ is not p-periodic.*

(*b*) *If $(n/2)+1 < p \leq n+1$, then $SL(n, \mathbf{Z})$ is p-periodic. For $(n/2)+1 < p < n+1$, the p-period is $2(p-1)$; for $p = n+1$, it is $p-1$.*

(*c*) *If $p > n+1$, then $\hat{H}^i(SL(n, \mathbf{Z}), A, p) = 0$ for all $i \in \mathbf{Z}$ and all A.*

We note that $SL(2, \mathbf{Z})$ is easily seen to be periodic with period 2.

Further applications and special cases are treated in Section 5.

1.4. This paper grew out of certain parts of the author's Doctoral Thesis, Chapter 3 of [2]. Like there, the method is based on a theorem of Brown–Venkov [1, §14] combined with properties of characteristic classes of linear groups; the results of the present paper are much more general than those of [2, Chapter 3]. The Brown–Venkov theorem states: If $a \in \hat{H}^m(G, \mathbf{Z})$ exists such that $\widehat{\mathrm{Res}}^G_{S_p}(a)$ is a maximal generator in $\hat{H}^m(S_p, \mathbf{Z})$ for every p-Sylow subgroup S_p of G, then G is p-periodic with p-period m_p, where m_p divides m; more precisely, the cup-product with a is an isomorphism $\hat{H}^i(G, A, p) \cong \hat{H}^{i+m}(G, A, p)$ for all i and A. Such an element a is called a maximal p-generator of G. We will exhibit such generators by means of certain characteristic classes.

In [2, Chapter 1], a different method is described to obtain p-periodicity for $SL(n, \mathbf{Z})$ and similar groups of virtually finite dimension; it uses finite factor groups of the groups in question, and their p-periodicity.

I wrote my thesis [2] under the direction of Prof. B. Eckmann. I thank him for his stimulating interest, his criticism and many helpful conversations.

2. *p*-periodicity of $GL(n, R)$

2.1. Throughout this paper we use the following notations: Let n be a natural number, R a ring of complex numbers containing $\mathbf{Z}$ and Q the quotient field of R. For $r \in \mathbf{N}$, C_r denotes the cyclic group of order r; and $\varphi_Q(r) = \dim_Q Q(\xi_r)$, where $\xi_r \in \mathbf{C}$ is a primitive rth root of unity.

THEOREM 2.1. *Let R be integrally closed and $GL(n, R)$ virtually of finite dimension, and let p be a prime.*

(*a*) *If $\varphi_Q(p) \leq (n/2)$, then $GL(n, R)$ is not p-periodic.*

(*b*) *If $(n/2) < \varphi_Q(p) \leq n$, then $GL(n, R)$ is p-periodic, and the p-period divides $2\varphi_Q(p)$.*

(*c*) *If $\varphi_Q(p) > n$, then $\hat{H}^i(GL(n, R), A, p) = 0$ for all $i \in \mathbf{Z}$ and all A.*

Proof. We write φ for $\varphi_Q(p)$.

(a) Let f be the irreducible polynomial of ξ_p over Q and D the $\varphi \times \varphi$-diagonal matrix with the φ complex roots of $f(x)$ in the diagonal. Then D has order p in $GL(\varphi, \mathbf{C})$, and D is similar to the companion matrix B of χ_D(= characteristic polynomial of D). Since $\chi_D(x) = f(x) \in R[x]$ (because R is integrally closed), B is in $GL(\varphi, R)$, and its order is p.

We now consider the subgroup

$$U = \left\{ \begin{pmatrix} B^k & 0 & 0 \\ 0 & B^i & 0 \\ 0 & 0 & E \end{pmatrix} \middle| \; 0 \leq k, i < p \right\}$$

of $GL(n, R)$, where E is the unit matrix; since $\varphi \leq (n/2)$, we can form such matrices. U is isomorphic to $C_p \times C_p$, and therefore $GL(n, R)$ cannot be p-periodic.

(b) Let S_p be any p-Sylow subgroup of $GL(n, R)$. According to Lemma 2.2 below, every abelian subgroup of S_p is cyclic. Hence [3, Chapter XII], S_p is cyclic or a generalized quaternion group. But S_2 cannot be a generalized quaternion group (of order ≥ 8); for $\varphi_Q(2) = 1$ and the assumption implies $n = 1$, and $GL(1, R)$ is abelian. S_p is therefore cyclic of order p^α. Since $\varphi \leq n$ and since R is integrally closed, $\alpha \geq 1$ (see the proof of a) above). Let A be a generator of S_p; A is similar to a complex diagonal matrix D. Since the order of $D \in GL(n, \mathbf{C})$ is p^α, D contains a ξ_{p^α} in the diagonal. Therefore the irreducible polynomial $f(x)$ of ξ_{p^α} over Q divides $\chi_D(x) = \chi_A(x) \in Q[x]$. Because $n < 2\varphi$, f has degree φ. It follows that D contains φ primitive p^α-th roots of unity in the diagonal; and since $n < 2\varphi$ the other elements in the diagonal are 1.

Let η be the canonical complex representation of $GL(n, R)$, and $c_\varphi(\eta) \in H^{2\varphi}(GL(n, R), \mathbf{Z})$ the φth *Chern class* of η. Then $\eta \mid_{S_p} = \bigoplus_{i=1}^n \eta_i$, where the η_i are one-dimensional representations of S_p. The above consideration about A and D shows that φ out of these, say $\eta_1, \ldots, \eta_\varphi$, are just multiplication with some ξ_{p^α}, and $\eta_{\varphi+1}, \ldots, \eta_n$ are trivial. $c_1(\eta_i)$ has therefore order p^α for $1 \leq i \leq \varphi$; and in view of the structure of the cohomology ring $H^*(C_{p^\alpha}, \mathbf{Z})$, the Chern class $c_\varphi(\eta \mid_{S_p}) = \prod_{i=1}^{\varphi} c_1(\eta_i)$ has also order p^α. This means that $\operatorname{Res}_{S_p}^{GL(n,R)}(c_\varphi(\eta))$ is a maximal generator in $H^{2\varphi}(S_p, \mathbf{Z})$.

We now recall that one has a commutative diagram

$$\begin{array}{ccc} H^{2\varphi}(GL(n, R), \mathbf{Z}) & \xrightarrow{\text{Res}} & H^{2\varphi}(S_p, \mathbf{Z}) \\ g^* \downarrow & & \downarrow \cong \\ \hat{H}^{2\varphi}(GL(n, R), \mathbf{Z}) & \xrightarrow[\widehat{\text{Res}}]{} & \hat{H}^{2\varphi}(S_p, \mathbf{Z}) \end{array}$$

where g^* is the canonical map from cohomology to Farrell–Tate cohomology [2, Lemma 6.3]. The theorem of Brown–Venkov quoted in the introduction then tells us that $GL(n, R)$ is p-periodic and that the p-period divides 2φ (the element $g^*(c_\varphi(\eta))$ is a maximal p-generator of $GL(n, R)$).

c) The above argument shows that if there exists $A \in GL(n, R)$ of order p, then $\varphi \leq n$. Hence, if $\varphi > n$, $GL(n, R)$ has no p-torsion; which implies that all Farrell–Tate cohomology groups have no p-torsion [2, Theorem 2.3].

We now formulate and prove the lemma.

LEMMA 2.2. *Let R be any ring of complex numbers containing $\mathbf{Z}$ and p a prime such that $(n/2) < \varphi_Q(p) \leq n$. Then there is no injective homomorphism $C_p \times C_p \to GL(n, R)$.*

Proof. Let $\mu: C_p \times C_p \to GL(n, R)$ be an injective homomorphism; we may regard μ as a faithful complex representation of $C_p \times C_p$. Therefore there is a faithful representation $\theta: C_p \times C_p \to GL(n, \mathbf{C})$ such that θ is equivalent to μ and $\theta(a)$ is a diagonal matrix for all $a \in C_p \times C_p$. If $a \neq 1 \in C_p \times C_p$, then $\theta(a)$ contains some ξ_p in the diagonal. Since the characteristic polynomial $\chi_{\theta(a)}$ lies in $Q[x]$, the irreducible polynomial of ξ_p over Q must divide $\chi_{\theta(a)}$. Taking into account $n/2 < \varphi_Q(p) = \varphi$ one concludes:

(*) $\theta(a)$ contains φ primitive p-th roots of unity, and the other diagonal elements are 1.

Let $M(a)$ be the set of those indices i, $1 \leq i \leq n$, for which the diagonal element $\theta(a)_{ii}$ is equal to 1. For each $a \neq 1$, $M(a)$ consists of $n-\varphi$ elements. If $M(a) \neq M(b)$ for certain $a, b \in C_p \times C_p - \{1\}$, one easily deduces a contradiction to property (*). If $M(a) = M(b)$ for all $a, b \in C_p \times C_p - \{1\}$, then–because of property (*)–θ induces an injective homomorphism $\bar{\theta}: C_p \times C_p \to GL(\varphi, \mathbf{C})$ such that $\bar{\theta}(a)$ is a diagonal matix, which contains for $a \neq 1$ only primitive p-th roots of unity in the diagonal. Let U_i $(1 \leq i \leq \varphi)$ be the subgroup of those diagonal matrices in $GL(\varphi, \mathbf{C})$, which contain a p-th root of unity at the place (i, i) and the other diagonal elements are 1. The group generated by $U_2, \ldots, U_\varphi$, written $\langle U_2, \ldots, U_\varphi \rangle$, has order $p^{\varphi-1}$. But $\bar{\theta}(C_p \times C_p) \cap \langle U_2, \ldots, U_\varphi \rangle = \{E\}$, which yields a contradiction.

Remark 2.3. From the proof it is clear that statements b) and c) of Theorem 2.1 remain true without the assumption "R is integrally closed". It is possible then, however, that the Farrell–Tate cohomology is 0, for all i and A, also in case b). (Take for example $R = \mathbf{Z}[\sqrt{5}]$; then $\varphi_Q(5) = 2$ and $GL(2, R)$ has no element of order 5.)

For $n = 1$, Theorem 2.1.(b) and (c) and Remark 2.3 imply that the group of units of R, written $U(R)$, if it is virtually of finite dimension, is periodic with period 2. If R is the ring of algebraic integers in a number field, then this statement follows also directly from the well-known structure of $U(R)$ (Dirichlet's Unit Theorem).

2.2. We now look closer at the case where $\varphi_Q(p) = p-1$; we then can calculate the precise value of the p-period of $GL(n, R)$ (provided this group is p-periodic).

THEOREM 2.4. *Let R be any ring of complex numbers containing $\mathbf{Z}$, Q the quotient field of R, p a prime such that $\varphi_Q(p) = p-1$, and assume that $GL(n, R)$ is virtually of finite dimension.*

(*a*) *If $p \leq (n/2)+1$, then $GL(n, R)$ is not p-periodic.*

(*b*) *If $(n/2)+1 < p \leq n+1$, then $GL(n, R)$ is p-periodic with p-period $2(p-1)$.*

(*c*) *If $p > n+1$, then $\hat{H}^i(GL(n, R), A, p) = 0$ for all $i \in \mathbf{Z}$ and all A.*

Proof. (a) If $p-1 \leq (n/2)$, the proof of Theorem 2.1.a) shows that there is a subgroup $U \cong C_p \times C_p$ in $GL(n, \mathbf{Z})$. Hence, $GL(n, R)$ is not p-periodic.

(b) According to Theorem 2.1.b) and Remark 2.3, $GL(n, R)$ is p-periodic, the p-period m_p dividing $2(p-1)$.

To get the precise value of m_p, it suffices to find a subgroup U of $GL(n, \mathbf{Z})$ whose p-period is $2(p-1)$. If $(n/2)+1 < p < n+1$, we take for U the image of S_n (the symmetric group of n letters) under the standard faithful permutation representation $S_n \to GL(n, \mathbf{Z})$. An easy calculation using [7, Theorem 3] shows that the p-period of S_n is $2(p-1)$. If $p = n+1$, we let U be the group generated by the following matrices A and B in $GL(p-1, \mathbf{Z})$: A is the companion matrix of the p-th cyclotomic polynomial; B is defined as follows: Let $(\mathbf{Z}/p\mathbf{Z})^*$ be the cyclic multiplicative group of non-zero residue classes modulo p; and let $q \in \mathbf{Z}$ be such that the residue class of q, written $[q]$, is a generator of $(\mathbf{Z}/p\mathbf{Z})^*$. Take for B the matrix which is 1 at the places $([i], [q] \cdot [i])$ and 0 otherwise. A has order p and B order $p-1$ in $GL(p-1, \mathbf{Z})$, and $A \cdot B = B \cdot A^q$. In fact, $\langle A, B\rangle$ is isomorphic to the group $\langle a, b \mid a^p = 1,\ b^{p-1} = 1,\ a \cdot b = b \cdot a^q\rangle$. The p-period of the latter group is $2(p-1)$ (for details see [2, Theorem 5.6.(c)]).

(c) The assertion follows immediately from Theorem 2.1.(c) and Remark 2.3.

3. The number field case

3.1. We assume in this section that R is contained in a number field (i.e., a finite dimensional field extension of $\mathbf{Q}$). Then the quotient field of R is a number field which we denote by K.

We recall the definition of the discriminant of K, written $\Delta(K)$: Let $x_1, \ldots, x_m$ be a base of K over $\mathbf{Q}$; as usual, $\Delta(x_1, \ldots, x_m) = \det\{T(x_i x_j)\}$ (one takes the determinant of that matrix, whose (i, j)-entry is the trace of $x_i x_j$). The ring of algebraic integers in K, written $\mathbf{O}(K)$, is a free $\mathbf{Z}$-module; let $w_1, \ldots, w_m$ be a base. Then $\Delta(K) = \Delta(w_1, \ldots, w_m)$, and this integer is independent of the choice of the base $w_1, \ldots, w_m$. We further recall that $p \cdot \mathbf{O}(K)$ has a ramified prime ideal factor if and only if $p \mid \Delta(K)$.

LEMMA 3.1. *Let p be any prime. If $p \cdot \mathbf{O}(K)$ has an unramified factor, then* $\varphi_K(p) = p-1$.

Proof. There is a prime ideal P in $\mathbf{O}(K)$, such that $P \mid p \cdot \mathbf{O}(K)$ but $P^2 \nmid p \cdot \mathbf{O}(K)$. Let $\Phi_p(x) \in \mathbf{Z}[x]$ be the p-th cyclotomic polynomial. It follows that $\Phi_p(x+1) = \sum_{k=1}^{p} \binom{p}{k} x^{k-1}$ is an Eisenstein polynomial over $\mathbf{O}(K)_p$, hence irreducible over K. Therefore Φ_p is irreducible over K, which means $\varphi_K(p) = p-1$.

3.2. It is convenient to introduce the following definition:

DEFINITION 3.2. We denote by $\delta(K)$ the product of those primes p, for which $p \cdot \mathbf{O}(K)$ has only ramified factors (i.e., $p \cdot \mathbf{O}(K) = \prod_{i=1}^{s} P_i^{\alpha_i}$, P_i different prime ideals in $\mathbf{O}(K)$, $\alpha_i > 1$ for $1 \le i \le s$).

Remark 3.3. $\delta(K) \mid \Delta(K)$.

If K is Galois over $\mathbf{Q}$, then $\delta(K)$ and $\Delta(K)$ contain the same prime factors. Otherwise it is possible that $\delta(K)$ contains less different primes than $\Delta(K)$ (see example 5.3).

Theorem 2.4 and Lemma 3.1 imply:

THEOREM 3.4. *Let R be a ring containing $\mathbf{Z}$ and contained in a number field and K the quotient field of R. Let p be a prime not dividing $\delta(K)$ and assume that $GL(n, R)$ is virtually of finite dimension.*

(*a*) *If $p \leq (n/2)+1$, then $GL(n, R)$ is not p-periodic.*

(*b*) *If $(n/2)+1 < p \leq n+1$, then $GL(n, R)$ is p-periodic with p-period $2(p-1)$.*

(*c*) *If $p > n+1$, then $\hat{H}^i(GL(n, R), A, p) = 0$ for all $i \in \mathbf{Z}$ and all A.*

3.3. According to Serre [6] the group $GL(n, R)$ is virtually of finite dimension for the following type of rings R: Let Σ be the set of equivalence classes $[v]$ of valuations v on a number field K, and Σ^∞ the subset of Σ consisting of all the classes of archimedian valuations; Σ^∞ is finite. Let S be a finite subset of Σ containing Σ^∞. The valuation ring of a nonarchimedian valuation v is denoted by $\mathbf{O}(v)$, and $\mathbf{O}_S(K) = \bigcap_{[v]\in\Sigma-S} \mathbf{O}(v)$. This ring $\mathbf{O}_S(K)$ is a Dedekind ring with quotient field K. Serre proved [6, Theorem 4] that $GL(n, \mathbf{O}_S(K))$ is virtually of finite dimension.

Therefore Theorem 3.4 can be applied to investigate p-periodicity of $GL(n, \mathbf{O}_S(K))$. If $S = \Sigma^\infty$, then $\mathbf{O}_S(K) = \mathbf{O}(K)$; taking Remark 3.3 into account we get Theorem 1.1.

4. *p*-periodicity of *SL*(*n*, *R*)

4.1. In all of Section 4, we assume that R consists of *real* numbers.

THEOREM 4.1. *Let R be integrally closed, p a prime, $n \geq 3$ and assume that $SL(n, R)$ is virtually of finite dimension.*

(*a*) *If $\varphi_Q(p) \leq (n/2)$, then $SL(n, R)$ is not p-periodic.*

(*b*) *If $(n/2) < \varphi_Q(p) \leq n$, then $SL(n, R)$ is p-periodic with p-period m_p; where $m_p \mid 2\varphi_Q(p)$ for $(n/2) < \varphi_Q(p) < n$, $m_p \mid \varphi_Q(p)$ for $\varphi_Q(p) = n$.*

(*c*) *If $\varphi_Q(p) > n$, then $\hat{H}^i(SL(n, R), A, p) = 0$ for all $i \in \mathbf{Z}$ and all A.*

Proof. We write φ for $\varphi_Q(p)$.

(a) We look at the proof of Theorem 2.1.(a): f has now real coefficients. Therefore, if $p > 2$, the product of all the roots of $f(x)$ is 1; hence D is in $SL(\varphi, \mathbf{C})$. Therefore $U \subset SL(n, R)$, which implies that $SL(n, R)$ is not p-periodic.

The case $p=2$: it is easy to see that $SL(3, \mathbf{Z})$ contains a subgroup isomorphic to $C_2 \times C_2$; since $n \geq 3$, $SL(n, R)$ is not 2-periodic.

(b) Like in the proof of Theorem 2.1.(b), it follows from Lemma 2.2 that any p-Sylow subgroup S_p of $SL(n, R)$ is cyclic; say of order p^α, where $\alpha \geq 1$, because R is integrally closed.

Let η be the canonical complex representation of $SL(n, R)$. Like in the proof of Theorem 2.1 we get: $g^*(c_\varphi(\eta))$ is a maximal p-generator of $SL(n, R)$. Therefore $SL(n, R)$ is p-periodic, the p-period dividing 2φ. If $\varphi = n$, we look at the *Euler class* of the canonical real representation $\eta_{\mathbf{R}}$ of $SL(n, R)$, written $\varepsilon(\eta_{\mathbf{R}})$. Since

$$c_n(\eta \mid s_p) = (-1)^{n(n-1)/2} \cdot \varepsilon^2(\eta_{\mathbf{R}} \mid s_p)$$

and since $c_n(\eta \mid s_p)$ has order p^α, the Euler class $\varepsilon(\eta_{\mathbf{R}} \mid s_p)$ has also order p^α. Therefore $g^*(\varepsilon(\eta_{\mathbf{R}})) \in \hat{H}^n(SL(n, R), \mathbf{Z})$ is a maximal p-generator, hence the p-period of $SL(n, R)$ divides $n = \varphi$.

(c) If $\varphi > n$, then there exists no element of order p in $SL(n, R)$, which implies statement c).

The assumption "R integrally closed" is not necessary for statements (b) and (c) of Theorem 4.1 (cf. Remark 2.3).

The method used above can also be applied to $SL(2, R)$; this group turns out to be periodic.

4.2. We again look closer at the case where $\varphi_Q(p) = p-1$; we then can give the precise value of the p-period of $SL(n, R)$ (provided this group is p-periodic).

THEOREM 4.2. *Let R be any ring of real numbers containing $\mathbf{Z}$, Q the quotient field of R, p a prime such that $\varphi_Q(p) = p-1$, $n \geq 3$ and assume that $SL(n, R)$ is virtually of finite dimension.*

(a) If $p \leq (n/2)+1$, then $SL(n, R)$ is not p-periodic.

(b) If $(n/2)+1 < p \leq n+1$, then $SL(n, R)$ is p-periodic with p-period m_p; where $m_p = 2(p-1)$ for $(n/2)+1 < p < n+1$, $m_p = p-1$ for $p = n+1$.

(c) If $p > n+1$, then $\hat{H}^i(SL(n, R), A, p) = 0$ for all $i \in \mathbf{Z}$ and all A.

Proof. We apply Theorem 4.1. The proof is completely analogous to the proof of Theorem 2.4.

4.3. We now assume that R is contained in a number field. Lemma 3.1 and Theorem 4.2 imply:

THEOREM 4.3. *Let R be a ring containing $\mathbf{Z}$ and contained in a real number field and K the quotient field of R. Let p be a prime not dividing $\delta(K)$,$n \geq 3$ and assume that $SL(n, R)$ is virtually of finite dimension.*

(*a*) *If $p \leq (n/2)+1$, then $SL(n, R)$ is not p-periodic.*

(*b*) *If $(n/2)+1 < p \leq n+1$, then $SL(n, R)$ is p-periodic with p-period m_p; where $m_p = 2(p-1)$ for $(n/2)+1 < p < n+1$, $m_p = p-1$ for $p = n+1$.*

(*c*) *If $p > n+1$, then $\hat{H}^i(SL(n, R), A, p) = 0$ for all $i \in \mathbf{Z}$ and all A.*

In particular, R can be taken to be $\mathbf{O}_S(K)$ in this theorem; and we thus get the S-arithmetic version of Theorem 1.2 (cf. Section 3.3).

5. Examples

5.1. $K = \mathbf{Q}(\theta)$, where θ is any root of $x^r - q$ ($r \in \mathbf{N}$, $q \in \mathbf{Z}$). Let m be the dimension of K over $\mathbf{Q}$. Then $\Delta(K)$ divides $\Delta(1, \theta, \ldots, \theta^{m-1})$ [5, Lemma 7.2]. It is well-known that $\Delta(1, \theta, \ldots, \theta^{m-1}) = \pm\prod_{i=1}^{m} f'(\theta_i)$, where f is the irreducible polynomial of θ over $\mathbf{Q}$, and $\theta_1, \ldots, \theta_m$ are all the roots of $f(x)$. Clearly $f(x) \in \mathbf{Z}[x]$. Let be $x^r - q = f(x) \cdot g(x)$, where $g(x) \in \mathbf{Z}[x]$. Let $\theta_1, \ldots, \theta_m$, $\theta_{m+1}, \ldots, \theta_r$ be all the roots of $x^r - q$. We have

$$(*) \quad \prod_{i=1}^{r} r \cdot \theta_i^{r-1} = \prod_{i=1}^{m} f'(\theta_i) \cdot \prod_{i=1}^{m} g(\theta_i) \cdot \prod_{i=m+1}^{r} r \cdot \theta_i^{r-1}.$$

An easy calculation shows that

$$\prod_{i=1}^{r} r \cdot \theta_i^{r-1} = \pm r^r \cdot q^{r-1}.$$

Formula (*) shows that $\prod_{i=1}^{m} g(\theta_i) \cdot \prod_{i=m+1}^{r} r \cdot \theta_i^{r-1}$ is rational, hence an integer (being an algebraic integer). Therefore $\prod_{i=1}^{m} f'(\theta_i) \mid r^r \cdot q^{r-1}$, hence $\Delta(K) \mid r^r \cdot q^{r-1}$. Since $\mathbf{Z}[\theta] \subset \mathbf{O}(K)$, the group $GL(n, \mathbf{Z}[\theta])$ is virtually of finite dimension; and we conclude from Theorem 3.4:

THEOREM 5.1. *Let θ be any root of $x^r - q$ ($r \in \mathbf{N}$, $q \in \mathbf{Z}$) and p a prime not dividing $r \cdot q$.*

(*a*) *If $p \leq (n/2)+1$, then $GL(n, \mathbf{Z}[\theta])$ is not p-periodic.*

(*b*) *If* $(n/2)+1<p\leq n+1$, *then* $GL(n, \mathbf{Z}[\theta])$ *is p-periodic with p-period* $2(p-1)$.
(*c*) *If* $p>n+1$, *then* $\hat{H}^i(GL(n, \mathbf{Z}[\theta]), A, p)=0$ *for all* $i\in\mathbf{Z}$ *and all* A.

From Theorem 4.3 we obtain an analogous theorem for $SL(n, \mathbf{Z}[\sqrt[r]{q}])$, $q\in\mathbf{N}$.

Remark (without proof): If we assume that x^r-q is irreducible and r odd, then statements a)–c) of Theorem 5.1 are true for every prime p.

5.2. $K=\mathbf{Q}(\theta)$, where θ is a root of x^3+ax+b $(a, b\in\mathbf{Z})$. With the technique mentioned above one easily shows that $\Delta(K)\mid 27b^2+4a^3$. From Theorem 4.3 we get

THEOREM 5.2. *Let* θ *be a real root of* x^3+ax+b $(a, b\in\mathbf{Z})$, p *a prime not dividing* $27b^2+4a^3$ *and* $n\geq 3$.
(*a*) *If* $p\leq(n/2)+1$, *then* $SL(n, \mathbf{Z}[\theta])$ *is not p-periodic.*
(*b*) *If* $(n/2)+1<p\leq n+1$, *then* $SL(n, \mathbf{Z}[\theta])$ *is p-periodic with p-period* m_p; *where* $m_p=2(p-1)$ *for* $(n/2)+1<p<n+1$, $m_p=p-1$ *for* $p=n+1$.
(*c*) *If* $p>n+1$, *then* $\hat{H}^i(SL(n, \mathbf{Z}[\theta]), A, p)=0$ *for all* $i\in\mathbf{Z}$ *and all* A.

We add the following remark: For any (real) field extension $K=\mathbf{Q}(\theta)$ of dimension $\leqslant 3$ there is at most one prime p such that $\varphi_K(p)<p-1$. Taking into account Theorem 4.2 we conclude, that there is at most one prime p, for which statements *a*)–*c*) of Theorem 5.2 are not true.

5.3. $K=\mathbf{Q}(\theta)$, where θ is any root of x^5-x-1. It is known, that $\Delta(K)= 19\cdot 151$, and that $19\cdot\mathbf{O}(K)=P_1^2\cdot P_2$ and $151\cdot\mathbf{O}(K)=P_3^2\cdot P_4\cdot P_5$, where all the P_i are prime ideals in $\mathbf{O}(K)$. Therefore $\delta(K)=1$. Hence Theorem 3.4 answers, for every prime p, the question whether $GL(n, \mathbf{Z}[\theta])$ is p-periodic or not.

Remark to 5.1–5.3. Instead of $R=\mathbf{Z}[\theta]$, we could as well take the ring $R=\mathbf{Z}[\theta, (1/m)]$ for any $m\in\mathbf{N}$ (cf. Theorem 3.4 and the remarks following that theorem).

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Received March 3, 1980