Poincaré duality groups of dimension two.

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In this paper we prove that 2-dimensional Poincaré duality groups with positive first Betti number β_1 are surface groups. As a corollary it follows that a connected Poincaré 2-complex with $\beta_1 > 0$ is homotopy equivalent to a closed surface, and so is any finite connected Poincaré 2-complex.

1. Statement of algebraic results

1.1. A Poincaré duality group of dimension n, in short PD^n-group, is a group G acting on $\tilde{\mathbf{Z}}$ such that there are natural duality isomorphisms

$$H^k(G; A) \cong H_{n-k}(G; \tilde{\mathbf{Z}} \otimes A) \tag{1}$$

for all integers k and all G-modules A (where G acts diagonally on $\tilde{\mathbf{Z}} \otimes A$); the isomorphisms (1) can be given by the cap-product $e \cap -$ with an element $e \in H_n(G; \tilde{\mathbf{Z}})$ called fundamental class. If (1) holds, the "formal dimension" n (= cohomology dimension of G) and the G-module $\tilde{\mathbf{Z}}$ ($\cong H^n(G; \mathbf{Z}G)$) are determined by G. A PD^n-group G is called orientable or non-orientable according to whether $\tilde{\mathbf{Z}}$ is a trivial G-module or not.

The fundamental group $\pi_1(M^n)$ of a closed connected aspherical n-dimensional manifold is a PD^n-group. In particular, if M^2 is a closed surface of genus $\geqslant 1$, then $\pi_1(M^2)$ is a PD^2-group. We will call such a group $\pi_1(M^2)$ a "surface group"; it admits a finite presentation of well-known canonical type. It has been conjectured that these surface groups are the only PD^2-groups. We will show that this is so except in a very special case which remains open.

1.2. From general arguments [5], [2] it is known that PD^n-groups are of type (*FP*); this means that there exists a $\mathbf{Z}G$-projective resolution of the trivial G-module $\mathbf{Z}$, of finite length and finitely generated over $\mathbf{Z}G$. In particular, a PD^n-group G is finitely generated, and its Betti numbers $\beta_i(G)$ and the Euler characteristic $\chi(G) = \sum_{i=0}^{n} (-1)^i \beta_i$ are defined. Our main result is

THEOREM 1. *Let G be a* PD^2*-group with $\beta_1(G) > 0$. Then G is a surface group.*

The condition $\beta_1(G)>0$ means, in the orientable case, that $\beta_1(G)$ is an even integer $\geqslant 2$; in the non-orientable case, an integer $\geqslant 1$. Thus $\beta_1(G)>0$ is equivalent to $\chi(G)\leqslant 0$ (since $\chi(G)=2-\beta_1(G)$ in the orientable, $1-\beta_1(G)$ in the non-orientable case). If G is non-orientable, it contains an orientable PD^2-group G_1 as subgroup of index 2. By the multiplicative property of the Euler characteristic (which holds for groups of type (*FP*), cf. [6]) one has $\chi(G_1)=2\chi(G)$; hence $\beta_1(G)>0$ if and only if $\beta_1(G_1)>0$.

1.3. A group G is said to be of type (*FF*) if it admits a $\mathbf{Z}G$*-free* resolution of finite length and finitely generated over $\mathbf{Z}G$. Obviously surface groups are of type (*FF*). It is not known whether there exist groups of type (*FP*) which are not of type (*FF*).

COROLLARY 1. *A* PD^2*-group G of type (FF) is a surface group.*

Proof. We first assume G orientable. Then the method of proof used by J. Cohen [7] is valid for any (*FF*)-resolution and shows that the assumption $\beta_1(G)=0$ (i.e. $H_1(G;\mathbf{Z})=0$) leads to a contradiction. Hence $\beta_1(G)>0$, and the assertion follows from Theorem 1.

If G is non-orientable, let G_1 be the orientable subgroup of index 2; it is also of type (*FF*), and thus $\beta_1(G_1)>0$. The Euler characteristic argument above then shows that $\beta_1(G)>0$.

1.4. We thus see that the case $\beta_1(G)=0$ not covered by Theorem 1 is equivalent to the existence of a PD^2-group G *not* of type (*FF*), but of course of type (*FP*). We further note that, by Theorem 1, the condition $\beta_1(G)>0$ not only implies type (*FF*) but also finite presentability.

1.5. A further corollary concerns the "Nielsen conjecture" for surface groups.

COROLLARY 2. *Let G be a torsion-free group containing a surface group* G_1 *as a subgroup of finite index. Then G itself is a surface group.*

Proof. Any torsion-free group G containing a PD^2-group G_1 as subgroup of finite index is itself a PD^2-group (cf. [1], [2]). Since $\beta_1(G_1)>0$, i.e., $\chi(G_1)\leqslant 0$, the multiplicative property of the Euler characteristic, $\chi(G_1)=|G:G_1|\chi(G)$, yields $\chi(G)\leqslant 0$. Hence $\beta_1(G)>0$, and the assertion follows from Theorem 1.

1.6. The relative analogue of a PD^n-group is a PD^n-pair, cf. Bieri-Eckmann [3]. A group pair $(G; S_0, S_1, \ldots, S_m)$ consists of a group G and a family of subgroups $\underset{\sim}{S}=(S_0, S_1, \ldots, S_m)$, $m\geqslant 0$; it is called a PD^n-pair if for some G-action on $\mathbf{Z}$ there are duality isomorphisms between the cohomology of G and the relative homology of $(G; \underset{\sim}{S})$, analogous to (1) and also given by the cap product

$e \cap -$ with a fundamental class $e \in H_n(G, \underset{\sim}{S}; \mathbf{Z})$. The duality is, of course, of exactly the same form as that of compact manifolds-with-boundary. Examples of PD^2-pairs are obtained by taking for G the fundamental group of a closed surface with $m+1$ discs removed ($m \geq 0$, and $m \geq 1$ if the surface is the sphere) together with the family of infinite cyclic subgroups generated by the circles bounding the discs. These PD^2-pairs of groups are called "geometric".

THEOREM 2. *All* PD^2*-pairs of groups are geometric.*

This result is actually a consequence of Corollary 1. Indeed it is shown in [3] that it is implied by the assertion that one-relator PD^2-groups are surface groups. Since one-relator PD^2-groups are of type (*FF*), Corollary 1 tells that this is the case.

However, Theorem 2 will be used in the proof of Theorem 1 and therefore requires a direct proof.

1.7. The proof of Theorem 2 will be given in Section 4, of Theorem 1 in Section 5. In Section 3 we describe the procedure of proof and list some auxiliary results, in particular the "decomposition theorems for group pairs" (H. Müller [10]). Section 2 deals with the topological aspect.

2. Topological application: Poincaré 2-complexes

2.1. A Poincaré n-complex is a CW-complex X dominated by a finite complex and fulfilling Poincaré duality for arbitrary local coefficients, with respect to a dualizing $\pi_1(X)$-module $\mathbf{Z}$ and a formal dimension n. We will always assume here that it is *connected.*

The study of Poincaré complexes was initiated by Wall in the 60-s. In [15] Wall proved, in particular, that if X is a Poincaré 2-complex with $\pi_1(X)$ finite, then X is homotopy equivalent to S^2 or $\mathbf{R}P^2$; if $\pi_1(X)$ is infinite, then X is aspherical, i.e., it is an Eilenberg-Mac Lane complex $K(G, 1)$ for $G = \pi_1(X)$. In the latter case the investigation is thus reduced to the study of finitely presented PD^2-groups. Later J. Cohen [7] showed that if X is a *finite* Poincaré 2-complex with $\beta_1(X) = 0$ then the conclusion is the same as for $\pi_1(X)$ finite; and that a Poincaré 2-complex X with $\beta_1(X) = 1$ or 2 is homotopy equivalent to the appropriate closed surface.

2.2. As a consequence of Theorem 1 we obtain

COROLLARY 3. *Let X be a Poincaré 2-complex with $\beta_1(X) > 0$. Then X is homotopy equivalent to a closed surface* (*of genus* ≥ 1).

Indeed, since $\beta_1(X)>0$ implies that $\pi_1(X)$ is infinite, $G=\pi_1(X)$ is a PD^2-group with $\beta_1(G)>0$ and thus isomorphic to $\pi_1(Y)$, where Y is a closed surface of genus $\geqslant 1$. The isomorphism provides a homotopy equivalence between $X=K(G,1)$ and Y.

COROLLARY 4. *A finite Poincaré 2-complex X is homotopy equivalent to a closed surface.*

Proof. If $\pi_1(X)$ is finite, one applies Wall's result mentioned above. If $\pi_1(X)=G$ is infinite, then G is a PD^2-group of type (*FF*), hence isomorphic to a surface group by Corollary 1. Thus $X=K(G,1)$ is homotopy equivalent to a closed surface.

2.3. Thus all Poincaré 2-complexes X are homotopy equivalent to closed surfaces, except possibly if (a) $\pi_1(X)$ is infinite and $\beta_1(X)=0$, and (b) X is not homotopy equivalent to a finite complex. Note that each of properties (a) and (b) implies the other. Except for finite presentability of $G=\pi_1(X)$ this exceptional possibility is exactly the same as the case not covered by Theorem 1, cf. 1.4.

3. Splitting of groups and group pairs

3.1. A group G is said to *split over a subgroup H* if it is either (α) an amalgamated free product $G=G_1 *_H G_2$, $G_1 \neq H \neq G_2$ or (β) an HNN-extension $G=G_1 *_{H,p}$. Cases where H is finitely generated or even finite will be of special importance.

If G is a PD^2-group with $\beta_1(G)>0$ then G admits an infinite cyclic factor group (infinite cyclic groups will be denoted by C in the following, or by $C(g)$ if we want to emphasize a generator g). Since G is of type (*FP*), it is "almost finitely presented". By a theorem of Bieri–Strebel [4], any almost finitely presented group admitting a factor group C splits over a *finitely generated* group L (by a splitting (β)). Thus Theorem 1 is a consequence of

THEOREM 1′. *Let G be a* PD^2*-group which splits over a finitely generated subgroup L. Then G is a surface group.*

If one confines attention to *finitely presented* PD^2-groups only (e.g., in the context of Poincaré 2-complexes or of the Nielsen conjecture), the Bieri–Strebel argument can be replaced by a somewhat simpler one which is just a modification of Moldavanskii's method [9]; cf. Eckmann–Müller [8].

3.2. The proof of Theorem 1′ will proceed as follows. By Strebel's theorem [13] the subgroup L, being of infinite index in G, is free. If the rank of L is >1, the splitting can be changed so as to become a splitting of G over a subgroup of smaller rank. One is thus reduced to the case where $L=C$ is infinite cyclic. Then the group pairs $(G_1; C)$ and $(G_2; C)$ in case (α), or $(G_1; C, p^{-1}Cp)$ in case (β), are PD^2-pairs; this follows from general results on PD^n-groups and -pairs (Bieri–Eckmann [3]). By our Theorem 2 these PD^2-pairs are geometric, which easily implies that $G=G_1 *_C G_2$, or $G=G_1 *_{C,p}$ respectively, is a surface group.

3.3. Both the reduction process above and the proof of Theorem 2 are based on "decomposition theorems for group pairs" (H. Müller [10]). For the convenience of the reader we summarize the appropriate definitions and those results which are needed.

In this context, a splitting of G is understood to be over a *finite* subgroup K. A group pair $(G; S_1, S_2, \ldots, S_m)$, $m \geq 0$, and a splitting (α) $G=G_1 *_K G_2$ or (β) $G=G_1 *_{K,p}$ are said to be *adapted to each other* if each S_j, $j=1,\ldots,m$ is conjugate to a subgroup of G_1 or G_2. If for $(G; S_1, S_2, \ldots, S_m)$ such a splitting of *G exists* we simply say that the pair is adapted. If G is finitely generated, the pair $(G; S_1, \ldots, S_m)$ is adapted if and only if $\bigcap_{j=1}^m N_j \neq 0$, where N_j is the kernel of the restriction map $\mathrm{res}_j : H^1(G; \mathbf{Z}G) \to H^1(S_j; \mathbf{Z}G)$. This is just a restatement of Swarup's relative version of Stallings' structure theorem for finitely generated groups with more than one end.

In the following we assume that $(G; S_1, \ldots, S_m)$ is an adapted pair and that G is finitely generated. With respect to the pair $(G; S_1, \ldots, S_m)$ a number $n(T)$, called *weight* of T, is associated with every subgroup T of G. The definition uses the restriction map

$$\mathrm{res} : H^1(G; \mathbf{Z}G) \to H^1(T; \mathbf{Z}G).$$

For simplicity we only consider the case where T is finitely generated. We regard $H^1(T; \mathbf{Z}T)$ as T-submodule of the (right) G-module $H^1(T; \mathbf{Z}G)$ (the embedding is induced by the inclusion $\mathbf{Z}T \to \mathbf{Z}G$). Since T is finitely generated, we have a decomposition (as abelian group)

$$H^1(T; \mathbf{Z}G) = \bigoplus_{x_i \in G/T} H^1(T; \mathbf{Z}T)x_i$$

(see, e.g., [2] Proposition 5.3).

DEFINITION. The weight $n(T)$ is the minimal number of non-trivial components of $\mathrm{res}(c) \in \bigoplus_{x_i \in G/T} H^1(T; \mathbf{Z}T)x_i$ for all $c \in \bigcap_{j=1}^m N_j$, $c \neq 0$.

3.4. For different values of $n(T)$ various types of a *simultaneous* splitting of G and a graph-decomposition of T are obtained. We describe here only two special cases (Corollaire 2 and Corollaire 5 of [11]). In the statements the splitting $G=G_1 * G_2$ or $G=G_1 *_{e,p}$ written $G*\langle p\rangle$, is always meant to be adapted to the pair $(G; S_1, \ldots, S_m)$.

THEOREM A. *Assume that T is torsion-free and $n(T)=1$. Then we have one of the following cases*

1) $G=G_1 * G_2$, $\quad T=T_1 * T_2$, $\quad T_1 \subset G_1,\ T_2 \subset G_2$;

2) $G=G_1 * \langle p\rangle$, $\quad T=T_1 * pT_2p^{-1}$, $\quad T_1,\ T_2 \subset G_1$;

3) $G=\langle p\rangle$, $\quad T=C(p)$, $\quad S_1=\cdots=S_m=e$ *or* $m=0$.

THEOREM B. *Assume that G is torsion-free, T infinite cyclic and $n(T)=2$. Then we have one of the following cases*

1) $G=G_1 * G_2$, $\quad T=C(g_1g_2)$, $\quad e\neq g_i \in G_i$, $\quad i=1,2$;

2) $G=G_1 * \langle p\rangle$, $\quad T=C(pg_1p^{-1}g_2)$, $\quad e\neq g_1,\ g_2 \in G_1$;

3) $G=\langle p\rangle$, $\quad T=C(p^2)$, $\quad S_1=\cdots=S_m=e$ *or* $m=0$.

4. Proof of Theorem 2

4.1. Let $(G; S_0, S_1, \ldots, S_m)$, $m\geq 0$, in short $(G; \underset{\sim}{S})$, be a PD^2-pair. G acts on $\mathbf{Z}$, and there is a fundamental class $e\in H_2(G, \underset{\sim}{S}; \mathbf{Z})$ such that

$$e\cap - : H^k(G; A) \to H_{2-k}(G, \underset{\sim}{S}; \mathbf{Z}\otimes A) \tag{2}$$

is an isomorphism for all k and A. The *geometric* PD^2-pairs (cf. 1.6) are as follows:

Orientable case

(3) G is freely generated by $t_1, \ldots, t_m, x_1, y_1, \ldots, x_g, y_g$, $\quad m+g>0$, $S_1, \ldots, S_m$ are generated by conjugates to $t_1, \ldots, t_m$ and S_0 is generated by $t_1 \cdots t_m \cdot \prod_{i=1}^{g} [x_i, y_i]$.

Non-orientable case

(4) G is freely generated by $t_1, \ldots, t_m, z_0, \ldots, z_g$, $m\geq 0$, $g\geq 0$, $S_1, \ldots, S_m$ are generated by conjugates to $t_1, \ldots, t_m$ and S_0 is generated by $t_1 \cdots t_m \cdot \prod_{i=0}^{g} z_i^2$.

4.2. By Theorem 4.2 and 9.3 of [3] we know that a PD^2-pair $(G; S_0, S_1, \ldots, S_m)$ consists of a finitely generated *free* group G and a family $\underset{\sim}{S} = (S_0, S_1, \ldots, S_m)$ of *cyclic* subgroups. Moreover, the fundamental class $e \in H_2(G; \underset{\sim}{S}; \mathbf{Z})$ determines fundamental classes e_i for the PD^1-groups $S_0, \ldots, S_m$, namely the components of $\partial e \in H_1(\underset{\sim}{S}; \mathbf{Z}) = \bigoplus_{i=0}^m H_1(S_i; \mathbf{Z})$, where ∂ is the connecting homomorphism in the exact homology sequence of G modulo $\underset{\sim}{S}$. By [3], Theorem 2.1 one has the following commutative diagram

$$\begin{array}{ccccc}
0 \to H^1(G; \mathbf{Z}G) & \xrightarrow{\{\mathrm{res}_i\}} & \bigoplus_{i=0}^m H^1(S_i; \mathbf{Z}G) & \xrightarrow{\delta} & H^2(G, \underset{\sim}{S}; \mathbf{Z}G) \to 0 \\
 & & \cong \Big\downarrow \{e_i \cap -\} & & \cong \Big\downarrow (e \cap -) \\
 & & \bigoplus_{i=0}^m H_0(S_i; \mathbf{Z} \otimes \mathbf{Z}G) & \xrightarrow{\mathrm{cor}} & H_0(G; \mathbf{Z} \otimes \mathbf{Z}G) \\
 & & \cong \Big\downarrow j & & \cong \Big\downarrow \\
 & & \bigoplus_{i=0}^m (\mathbf{Z} \otimes_{S_i} \mathbf{Z}G) & \xrightarrow{p} & \mathbf{Z}
\end{array} \tag{5}$$

where the top row is exact and $p(1 \otimes_{S_i} y) = 1 \cdot y$ for $y \in G$.

4.3. We now prove, by induction on the rank $rk(G)$, that $(G; \underset{\sim}{S})$ has a presentation (3) or (4) and thus is geometric.

If $rk(G) = 1$ then $\bigoplus_{i=0}^m (\mathbf{Z} \otimes_{S_i} \mathbf{Z}G)$ is free Abelian of rank 2, by (5). This is possible only if either $m = 1$ and $S_0 = S_1 = G$; or if $m = 0$ and $S_0 = C(a^2)$ where $G = \langle a \rangle$. Thus we either have a presentation (3) with $m = 1$, $g = 0$, or a presentation (4) with $m = 0$, $g = 0$.

If $rk(G) \geq 2$ we put $T = S_0$ and determine the weight $n(T)$ with respect to the pair $(G; S_1, \ldots, S_m)$, which is adapted by (5). We consider elements $\mathrm{res}_0(c)$, $0 \neq c \in \bigcap_{j=1}^m N_j$ (i.e., elements $(d, 0, \ldots, 0) \in \mathrm{im}\{\mathrm{res}_i\}$, $d \neq 0$) and count the number of components of d in $H^1(T; \mathbf{Z}G) = \bigoplus_{x_\nu \in G/T} H^1(T; \mathbf{Z}T)x_\nu$. From (5) we see that $\mathrm{im}\{\mathrm{res}_i\} = \ker \delta = \ker pj\{e_i \cap -\}$, and $pj\{e_i \cap -\}$ restricted to any $H^1(T; \mathbf{Z}T)x_\nu$ is bijective. Thus the minimal number of components of elements $d \neq 0$ is two, i.e., the weight of $T = S_0$ is 2. By Theorem B we therefore have one of the two following cases:

1) $G = G_1 * G_2$; $S_0 = C(g_1 g_2)$, $e \neq g_i \in G_i$, $i = 1,2$, and the subgroups $S_1, \ldots, S_k$ are conjugate to subgroups of G_1, while $S_{k+1}, \ldots, S_m$ are conjugate to subgroups of G_2, for some k, $0 \leq k \leq m$.

2) $G = G_1 * \langle p \rangle$; $S_0 = C(p g_1 p^{-1} g_2)$, $e \neq g_1, g_2 \in G_1$, and $S_1, \ldots, S_m$ are conjugate to subgroups of G_1.

Since hypothesis and assertion are invariant under conjugation we may assume that $S_1, \ldots, S_m$ are actually subgroups of G_1 or G_2 respectively.

Case 1). We can write G as $G=(G_1 * C(g_2)) *_{C(g_2)} G_2$. The subgroups $S_0 = C(g_1 g_2)$ and $S_1, \ldots, S_k$ are in $G_1 * C(g_2)$, and the $S_{k+1}, \ldots, S_m$ in G_2. If $G_2 \neq C(g_2)$, Theorem 8.1 of [3] tells that $(G_2; C(g_2), S_{k+1}, \ldots, S_m)$ is a PD^2-pair. We claim that this is also true if $G_2 = C(g_2)$; namely, that pair is then $(C(g_2); C(g_2), C(g_2))$.

To prove this we note that quite generally, in Case 1), diagram (5) implies that $\operatorname{res}: H^1(G; \mathbf{Z}G) \to \bigoplus_{i=k+1}^{m} H^1(S_i; \mathbf{Z}G)$ is surjective, and so is $\operatorname{res}: H^1(G_2; \mathbf{Z}G_2) \to \bigoplus_{i=k+1}^{m} (S_i; \mathbf{Z}G_2)$. If $G_2 = C(g_2)$, then $H^1(G_2; \mathbf{Z}G_2) = \mathbf{Z}$, so this is possible only if $k = m$, or $k = m-1$ and $S_m = G_2 = C(g_2)$. Assume $k = m$; then all subgroups $S_1, \ldots, S_m$ are in G_1, hence $H^1(G, \underset{\sim}{S}; \mathbf{Z}) \neq 0$, since $G = G_1 * C(g_2) = G_1 * C(g_1 g_2) = G_1 * S_0$. However, for a PD^2-pair $H^1(G, \underset{\sim}{S}; \mathbf{Z}G) = 0$, so $k = m$ is not possible and we are left with $k = m-1$ and $(G_2; C(g_2), S_{k+1}, \ldots, S_m) = (C(g_2); C(g_2), C(g_2))$, which is a PD^2-pair.

Thus $(G_2; C(g_2), S_{k+1}, \ldots, S_m)$ is a PD^2-pair, and so is $(G_1; C(g_1), S_1, \ldots, S_k)$. By induction hypothesis they have presentations of the type (3) or (4). It follows immediately that $(G; \underset{\sim}{S})$ has a presentation (3) or (4): This is obvious if both above pairs have a presentation (3), or both a presentation (4). Otherwise one gets a presentation (4), i.e. non-orientable, by using transformations of the form

$$a^2[b, c] = \bar{a}^2 \bar{b}^2 \bar{c}^2; \quad \bar{a} = a^2 bca^{-1}, \quad \bar{b} = ac^{-1}b^{-1}a^{-1}ca^{-1}, \quad \bar{c} = ac^{-1} \tag{6}$$

Case 2). Write G as $G = (G_1 * C(a)) *_{C(ag_2^{-1}), p}$ with $p^{-1}(ag_2^{-1})p = g_1$. The subgroups $S_0 = C(a)$ and $S_1, \ldots, S_m$ are in $G_1 * C(a)$. By [3], Theorem 8.3, $(G_1 * C(a); C(a), S_1, \ldots, S_m, C(ag_2^{-1}), C(g_1))$ is a PD^2-pair. By the method used in Case 1) it follows that $(G_1; S_1, \ldots, S_m, C(g_1), C(g_2))$ is a PD^2-pair; the induction hypothesis tells that it has a presentation of the type (3) or (4). We may assume that this presentation is as follows.

G_1 is freely generated by $t_0, t_1, \ldots, t_m$ and some x_i, y_i (orientable case (3)) or some z_i (non-orientable case (4)); and S_i is conjugate to $C(t_i)$, $i = 1, \ldots, m$, $C(g_1)$ to $C(t_0)$, i.e., g_1 is conjugate to t_0 or t_0^{-1}; and $g_2 = t_0 \cdots t_m r$ where $r = \prod [x_j, y_j]$ or $\prod z_j^2$ respectively. S_0 is generated by $pg_1p^{-1}t_0 \ldots t_m r$. By changing p if necessary we may assume $g_1 = t_0^{\pm 1}$. Using transformations of the form

$$ptp^{-1}t = \bar{p}^2 \bar{t}^2; \, \bar{p} = ptp^{-1}t^{-1}p^{-1}, \, \bar{t} = pt \tag{7}$$

and of the form (6), we get a presentation (3) or (4) for the pair $(G; S_0, S_1, \ldots, S_m)$.

The passage from the two geometric pairs $(G_1; \ldots)$ and $(G_2; \ldots)$ to $(G; \underline{S})$ in Case 1), or from $(G_1; \ldots)$ to $(G; \underline{S})$ in Case 2) can, of course, be replaced by a geometric procedure on the corresponding surfaces-with-boundary.

5. Proof of Theorem 1′

5.1. We recall that surface groups have canonical presentations

$$G = \left\langle x_1, y_1, \ldots, x_g, y_g \;\middle|\; \prod_{j=1}^{g} [x_j, y_j] = 1 \right\rangle, \qquad g \geq 1 \tag{8}$$

in the orientable, and

$$G = \left\langle z_0, \ldots, z_g \;\middle|\; \prod_{j=0}^{g} z_j^2 = 1 \right\rangle, \qquad g \geq 1 \tag{9}$$

in the non-orientable case.

Let G be a PD^2-group which splits over a finitely generated group L as (α) $G = G_1 *_L G_2$, $G_1 \neq L \neq G_2$ or (β) $G = G_1 *_{L,p}$. Since L has infinite index in G it is free [13].

If $rk(L) = 1$, $L = C$, we consider the pairs $(G_1; C)$ and $(G_2; C)$ in case (α), or $(G_1; C, p^{-1}Cp)$ in case (β). By [3], Theorem 8.1 and 8.3 these pairs are PD^2-pairs and hence geometric; they have presentations (3) or (4), and by amalgamation or HNN-extension these yield presentations of the form (8) or (9) (by using, if necessary, transformations (6) and (7)). Thus G is a surface group.

Of course, the appropriate surface can also be obtained geometrically from the surfaces-with-boundary corresponding to the group pairs.

5.2. If $rk(L) \geq 2$, we will obtain from Theorem A a new splitting of G over a subgroup M with $rk(M) < rk(L)$. This reduces the problem to the case $rk(L) = 1$ above.

(α) Assume first that $G = G_1 *_L G_2$. We consider the Mayer-Vietoris sequence

$$\cdots \to 0 \to H^1(G_1; \mathbf{Z}G) \oplus H^1(G_2; \mathbf{Z}G) \xrightarrow{(\mathrm{res}_1, -\mathrm{res}_2)}$$
$$H^1(L; \mathbf{Z}G) \xrightarrow{\delta} H^2(G; \mathbf{Z}G) \to \cdots$$

and show the following:

(10) If the weight of L with respect to both $(G_1; \emptyset)$ and $(G_2; \emptyset)$ is greater

than one, then $H^1(L;\mathbf{Z}L)\cap \mathrm{im}\,(\mathrm{res}_1, -\mathrm{res}_2)=0$. (Here we consider $H^1(L;\mathbf{Z}L)$ as submodule of $H^1(L;\mathbf{Z}G)$.)

Proof. Let C_L denote $H^1(L;\mathbf{Z}L)$ and $C_i=H^1(G_i;\mathbf{Z}G_i)$, $i=1,2$. Choose sets $\{x_i;\, i\in I\}$ and $\{y_j;\, j\in J\}$ of representatives of the (right) cosets $\in G_1/L$ and G_2/L (both sets containing e). We then have the following sets of representatives:

$$\Sigma_1=\{e\}\cup\{y_{j_1}x_{i_2}\cdots;\, y_{j_1}\neq e\neq x_{i_l}\}\quad\text{for}\quad G/G_1;$$

$$\Sigma_2=\{e\}\cup\{x_{i_1}y_{j_2}\cdots;\, y_{j_l}\neq e\neq x_{i_1}\}\quad\text{for}\quad G/G_2;$$

$$\Sigma_L=\Sigma_1\cup\Sigma_2\quad\text{for}\quad G/L.$$

Hence we get decompositions

$$H^1(G_i;\mathbf{Z}G)=\bigoplus_{z\in\Sigma_i} C_i z,\qquad i=1,2;$$

$$H^1(L;\mathbf{Z}G)=\bigoplus_{z\in\Sigma_L} C_L z.$$

The "length" of a summand $C_i z$ or $C_L z$ is defined as the number of representatives $x_i, y_i\neq e$ occurring in z. Consider now $0\neq(c_1,c_2)\in H^1(G_1;\mathbf{Z}G)\oplus H^1(G_2;\mathbf{Z}G)$. We want to show that $\mathrm{res}_1(c_1)-\mathrm{res}_2(c_2)\notin C_L$. For this we consider a non-trivial component d of (c_1,c_2) lying in a summand (of the above decompositions) of maximal length; say $d=cz_1$ in C_1z_1 of length l. Let $\mathrm{res}_1(c)$ be $\sum_{i\in I} b_i x_i$, $b_i\in C_L$. Because the weight of L with respect to $(G_1;\varnothing)$ is greater than one, there is at least one i_0 with $x_{i_0}\neq e$, $b_{i_0}\neq 0$. So $\mathrm{res}_1(cz_1)$ contains the summand $b_{i_0}x_{i_0}z_1$ in $C_Lx_{i_0}z_1$ of length $l+1$, and because of the maximality of l there is no other contribution in $\mathrm{res}_1(c_1)-\mathrm{res}_2(c_2)$ to the component $C_Lx_{i_0}z_1$. So indeed $\mathrm{res}_1(c_1)-\mathrm{res}_2(c_2)\notin C_L$, which proves (10).

By assumption, $H^2(G;\mathbf{Z}G)$ is free abelian of rank one and L has infinitely many ends. Therefore the restriction of δ to $H^1(L;\mathbf{Z}L)$ cannot be injective. Because of the exactness of the Mayer—Vietoris sequence, $H^1(L;\mathbf{Z}L)\cap \mathrm{im}\,(\mathrm{res}_1,-\mathrm{res}_2)\neq 0$. By (10), L has weight *one* with respect to $(G_1;\varnothing)$ or $(G_2;\varnothing)$, say $(G_1;\varnothing)$. (Note that L cannot have weight 0, since res_1 and res_2 are injective.) By Theorem A, we have one of the following two cases:

1) $G_1=H_1*H_2,\qquad L=L_1*L_2,\qquad e\neq L_i\subset H_i,\qquad i=1,2;$

2) $G_1=H_1*\langle t\rangle,\qquad L=L_1*tL_2t^{-1},\qquad e\neq L_1, L_2\subset H_1.$

In Case 1), we have $G=H_1 *_{L_1}(H_2 *_{L_2} G_2)$. If $L_1 \neq H_1$, G splits over L_1; if $L_1=H_1$, then $L_2 \neq H_2$ and $G=H_2 *_{L_2} G_2$ splits over L_2.

In Case 2), $G=(H_1 *_{L_1} G_2) *_{L_2,t^{-1}}$ splits over L_2.

So in both cases we have a splitting of G over a group M with $rk(M)<rk(L)$.

(β) The case $G=G_1 *_{L,p}$ is treated similarly. If L is not cyclic, one can show that (by changing the notation if necessary) $n(L)=1$ with respect to $(G_1; p^{-1}Lp)$; to prove that the pair is adapted and to compute the weight one proceeds by methods analogous to those in the proof of (10). By Theorem A we have again the cases 1) or 2) above, where moreover $p^{-1}Lp$ is conjugate to a subgroup of H_1. By changing the stable letter p we can get $p^{-1}Lp \subset H_1$.

In Case 1), $G=(H_1 *_{L_{1,p}}) *_{L_2} H_2$ splits over L_2 if $L_2 \neq H_2$; or else over L_1.

In Case 2), $G=(H_1 *_{L_{1,p}}) *_{L_2,t^{-1}}$ splits over L_2. This completes the proof of Theorem 1′.

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