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§1. Introduction

Let $f: V^n \to W^{n+r}$ be a smooth immersion, where V^n and W^{n+r} are smooth manifolds of dimension n and $n+r$ respectively; we denote by $V^{(k)}$ the k-fold product of V, $\Delta_V(k)=\{(x_1,\ldots,x_k)\in V^{(k)} \mid \exists i\neq j \text{ with } x_i=x_j\}$, $\delta_W(k)=\{(y,\ldots,y)\in W^{(k)}\}$. We shall say that f is regular if $f^k: V^{(k)}\to W^{(k)}$ is transversal to $\delta_W(k)$ outside $\Delta_V(k)$. This means that if $f(x_1)=\cdots=f(x_k)=y$, $x_i\neq x_j$, the vector spaces $\mathrm{Im}\,(df_{x_1}),\ldots,\mathrm{Im}\,(df_{x_k})$ are in general position in TW_y.

The following theorem has been proved by Ralph J. Herbert in his thesis [3]:

1.1 THEOREM. *Let $f: V^n\to W^{n+r}$ be a regular proper immersion and set $N_k=\{y\in W \mid \#(f^{-1}(y))=k\}$, $M_k=f^{-1}(N_k)$. Then $\bar{M}_k$ and $\bar{N}_k$ carry fundamental classes over the integers modulo two; denoting by m_k and n_k their Poincaré duals in V and W respectively and by $e=e(N_f)$ the Euler class of the normal bundle N_f of f, we have:*

$$m_k=f^*(n_{k-1})-e\cdot m_{k-1} \qquad (*)$$

If r is even and V and W oriented, $\bar{M}_k$ and $\bar{N}_k$ carry fundamental classes over the integers, and the above formula is valid in integral cohomology.

The fundamental classes are meant as in [2], §2.2.

Remarks.

(i) If r is even and N_f only is oriented, we still have integral dual classes, for which (*) stays valid.

(ii) In proving (*) we will exhibit minimal desingularisations of $\bar{M}_k$ and $\bar{N}_k$ which provide fundamental classes in bordism theory (oriented bordism if r is even and N_f oriented, complex bordism if N_f has a stable complex structure, unoriented bordism otherwise). In the corresponding cobordism theories (*) still holds

(iii) From (*) we deduce:

$$m_k = \sum_{j=0,\ldots,k-1} (-1)^j e^j f^*(n_{k-1-j})$$

In particular if $W=\mathbf{R}^{n+r}$, $m_k=(-1)^{k-1}e^{k-1}$. This recovers the formula for triple points of immersed surfaces in $\mathbf{R}^3$ given in [1].

(iv) When r is even and N_f oriented, the orientations we shall give for the dual classes to $\bar{M}_k$ and $\bar{N}_k$ are such that $f_!(m_k)=k\cdot n_k$, where $f_!:H^*(V)\to H^*(W)$ is the Gysin homomorphism associated to f. Defining $\varphi_h:H^*(V)\to H^*(V)$ by $\varphi_h(a)=f^*f_!(a)-h(e\cdot a)$, we deduce from (*):

$$(k-1)!m_k=\varphi_{k-1}\cdot\varphi_{k-2}\cdot\cdots\cdot\varphi_1(1)$$

Herbert's theorem corrects a formula given in [4]. The purpose of this note is to give a simple proof of (*). My contribution is the idea of proving (*) using Proposition 2.2 below, which is a generalization of a proposition of D. Quillen ([5], prop. 3.3).

Particular cases of (*) were known before Herbert's thesis. In [7], p. 131, H. Whitney shows that $m_2=f^*f_!(1)-e$; Herbert's method for proving (*) appears to be a generalisation of Whitney's method, which also inspired our approach. By different methods, the case of triple points of surfaces in $\mathbf{R}^3$ is treated in [1] and [6] deals with the number of triple points of an immersion $V^{4n}\to\mathbf{R}^{6n}$.

§2. Proofs

We adopt the following notations: a smooth map $\alpha:A\to X$ means a C^∞ map between C^∞ manifolds. TA denotes the tangent bundle of A, $N_\alpha=\alpha^*(TX)-TA$ the virtual normal bundle of α; if α is an immersion, N_α denotes the genuine normal bundle of α, namely $\alpha^*(TX)/d\alpha(TA)$, where $d\alpha:TA\to\alpha^*TX$ denotes the derivative of α.

Let $f:V^n\to W^{n+r}$ be a smooth regular proper immersion. We set:

– $N_k(f)=\{y\in W \mid \#(f^{-1}(y))=k\}$, $\quad M_k(f)=f^{-1}(N_k)$

– $\hat{M}_k(f)=\{(x_1,\ldots,x_k)\in V^{(k)}-\Delta_V(k) \mid f(x_i)=f(x_j)\}$

The group of permutations of k objects S_k acts fixed-point free on $\hat{M}_k(f)$ in the obvious way.

$$- \tilde{N}_k(f) = \hat{M}_k/S_k, \qquad \tilde{M}_k(f) = \hat{M}_k/S_{k-1},$$

where S_{k-1} acts on the last $k-1$ coordinates.

We write $[x_1, \ldots, x_k]$, resp. $(x_1, [x_2, \ldots, x_k])$ for the class of $(x_1, \ldots, x_k) \in \hat{M}_k$ in $\tilde{N}_k$, resp. $\tilde{M}_k$. We define $f_k : \tilde{M}_k \to V$, $f_k(x_1, [x_2, \ldots, x_k]) = x_1$ and $g_k : \tilde{N}_k \to W$, $g_k([x_1, \ldots, x_k]) = f(x_1)$ $(= f(x_2) = \cdots = f(x_k))$. We set $\tilde{M}_k^0 = f_k^{-1}(M_k)$, $\tilde{N}_k^0 = g_k^{-1}(N_k)$. Recall that $N_f^{(k)}$ denotes the k-fold product of N_f.

2.1 LEMMA.

(i) *f_k and g_k are proper immersions with normal bundles* $N_{g_k} = (N_f^{(k)} \mid \hat{M}_k)/S_k$ *and* $N_{f_k} = (0 \times N_f^{(k-1)} \mid \hat{M}_k)/S_{k-1}$.

(ii) *$\tilde{M}_k^0$ and $\tilde{N}_k^0$ are open dense in $\tilde{M}_k$ and $\tilde{N}_k$ respectively, $f_k \mid \tilde{M}_k^0 : \tilde{M}_k^0 \to M_k$ and $g_k \mid \tilde{N}_k^0 : \tilde{N}_k^0 \to N_k$ are diffeomorphisms.*

(iii) $f_k(\tilde{M}_k) = \bar{M}_k = \bigcup_{h \geq k} M_h$, $g_k(\tilde{N}_k) = \bar{N}_k = \bigcup_{h \geq k} N_h$.

Proof. Since $\hat{M}_k = (f^k)^{-1}(\delta_W(k)) - \Delta_V(k)$, we deduce from the transversality of f^k to $\delta_W(k)$ outside $\Delta_V(k)$ that $T(\hat{M}_k)_{(x_1, \ldots, x_k)} = \{(v_1, \ldots, v_k) \in T(V)^{(k)}_{(x_1, \ldots, x_k)} \mid df_{x_i}(v_i) = df_{x_j}(v_j)\}$. So, $v_1 = 0$ implies $v_2 = \cdots = v_k = 0$. Hence f_k and g_k are immersions; it is easily seen that their normal bundles are as stated.

Let us check that $\hat{M}_k$ is closed in $V^{(k)}$: if not, there are sequences $\{x_1^h\}$, $\{x_2^h\} \subset V$, $f(x_1^h) = f(x_2^h)$, $x_1^h \neq x_2^h$, with $\lim_{h \to \infty}(x_1^h) = \lim_{h \to \infty}(x_2^h) = x$. We write f in local coordinates as a map $f : \mathbf{R}^n \to \mathbf{R}^{n+r}$; we can assume that $x_1^h - x_2^h/\|x_1^h - x_2^h\|$ tends to $v \in \mathbf{R}^n$, $\|v\| = 1$. But then $df_x(v) = 0$ and f is no longer an immersion. Hence $\tilde{M}_k$ and $\tilde{N}_k$ are closed in $V^{(k)}/S_{k-1}$ and $V^{(k)}/S_k$ respectively and since f is proper we deduce that f_k and g_k are proper. This proves (i). The assertions (ii) and (iii) follow from the fact that f_k and g_k are proper and, using the implicit function theorem, by writing f locally as a linear map.

We digress now to sub-cartesian diagrams; they generalize the notion of clean intersection of Quillen ([5], §3), which concerns the case when α and β below are embeddings.

DEFINITION. The diagram of smooth proper immersions:

$$\begin{array}{ccc} Z & \xrightarrow{f_B} & B \\ {\scriptstyle f_A}\downarrow & & \downarrow{\scriptstyle \beta} \\ A & \xrightarrow{\alpha} & X \end{array}$$

is said to be sub-cartesian if:

(i) $f_A \times f_B : Z \to A \times B$ is an embedding onto $A \times_X B = \{(a, b) \in A \times B \mid \alpha(a) = \beta(b)\}$.

(ii) the following sequence is exact:

$$0 \to TZ \xrightarrow{d(f_A \times f_B)} f_A^* TA \times f_B^* TB \xrightarrow{(d\alpha, -d\beta)} f_A^* \alpha^* TX$$

where $(d\alpha, -d\beta)$ is meant to send $(v, w) \in (f_A^* TA \times f_B^* TB)_z$ to $d\alpha(v) - d\beta(w)$. The vector bundle $E = f_A^* \alpha^* TX / \mathrm{Im}\,(d(f_A \times f_B))$ over Z is called the excess vector bundle.

Remarks.

(i) The above diagram is cartesian if and only if E is the zero bundle.

(ii) We have not assumed Z to have constant dimension, hence E won't have constant rank in general.

(iii) The above condition (ii) is equivalent to say that if for $a \in A$ and $b \in B$ we choose open neighbourhoods A' and B' respectively such that $\alpha \mid A'$ and $\beta \mid B'$ are embeddings, then $\alpha(A') \cap \beta(B')$ is a sub-manifold of X and $T(\alpha(A') \cap \beta(B')) = T(\alpha(A')) \cap T(\beta(B'))$. This is to say that $\alpha(A')$ and $\beta(B')$ intersect cleanly in X in the terminology of [5].

2.2 PROPOSITION. *For $c \in H^*(B)$ we have:*

$$\alpha^* \beta_!(c) = f_{A!}(e(E) \cdot f_B^*(c))$$

where e denotes the Euler class, $\beta_!$ and $f_{A!}$ are the Gysin homomorphisms associated to those maps. The cohomology is taken over the integers whenever N_β and E are oriented, the integers modulo two otherwise. (The proposition and its proof remain valid in any generalized cohomology theory in which N_β and E have orientations.)

Proof. We replace Z by its image in $A \times B$, still denoted by Z. We provide TX with a metric and identify E with the orthogonal to $\mathrm{Im}\,(d(f_A \times f_B|_Z))$ in $f_A^* \alpha^* TX$. Let $e: TX \to X$ be the exponential mapping associated to the metric; for $x \in X$ there is an open neighbourhood U_x of $0 \in TX_x$ such that $e_x = e \mid U_x$ is a diffeomorphism onto an open neighbourhood of x in X. Let Ω be a closed tubular neighbourhood of Z in $A \times B$; it is a manifold with boundary $\partial\Omega$. If Ω is small enough, for $(a, b) \in \Omega$ we have $b \in e_{\alpha(a)}(U_{\alpha(a)})$. Let $v: Z \to E$ be a section transversal to the zero section and denote by $\bar{E}$ and $\bar{v}$ extensions of E and v to Ω, with

$\bar{E}$ still a sub-bundle of $TX' = p_A^*\alpha^*(TX) \mid \Omega$, where $p_A : \Omega \to A$ denotes the obvious projection. Define the section $w : \Omega \to TX'$ by $w(a, b) = e^{-1}_{\alpha(a)}(\beta(b))$, and the section $\bar{w} : \Omega \to TX'$ by $\bar{w} = w + \bar{v}$. If Ω is small enough, $w(a, b) \notin \bar{E}_{(a,b)}$ for $(a, b) \notin Z$ and hence, setting $Z'' = \{(a, b) \in \Omega \mid \bar{w}(a, b) = 0\}$, we have $z'' = \{(a, b) \in Z \mid v(a, b) = 0\}$.

It follows from the exact sequence (ii) of the definition of a sub-cartesian diagram that $\bar{w}$ is transversal to the zero section in TX'. Hence the map $F : \Omega \to X \times X$, $F(a, b) = (e_{\alpha(a)}(\bar{w}(a, b)), \beta(b))$ is transversal to Δ_X and $F^{-1}(\Delta_X) = Z''$ if v has been chosen near enough the zero section in E.

Let $\alpha' : A \to X$ be near α such that $\alpha' \times \beta : A \times B \to X \times X$ is transversal to Δ_X and set $Z' = (\alpha' \times \beta)^{-1}(\Delta_X)$. The following diagram is cartesian:

$$\begin{array}{ccc} Z' & \xrightarrow{f_B} & B \\ {\scriptstyle f'_A}\downarrow & & \downarrow{\scriptstyle \beta} \\ A & \xrightarrow{\alpha'} & X \end{array}$$

where f'_A and f'_B are the obvious projections; hence $\alpha'^*\beta_!(c) = f'_{A!}f'^*_B(c)$. If α' is near enough to α, $F' = (\alpha' \times \beta) \mid \Omega$ and F are homotopic through maps transversal to Δ_X and sending $\partial\Omega$ into $X \times X - \Delta_X$. Hence there is an isotopy of Ω leaving $\partial\Omega$ fixed and sending Z' onto Z''.

Consider the inclusions $i : Z \subset \Omega$, $i' : Z' \subset \Omega$, $i'' : Z'' \subset \Omega$, $j : Z'' \subset Z$, the projection $p_A : \Omega \to A$ and the associated Gysin homomorphisms $i_! : H^*(Z) \to H^*(\Omega, \partial\Omega)$, similarly for $i'_!$ and $i''_!$, and $p_{A!} : H^*(\Omega, \partial\Omega) \to H^*(A)$. Since Z' and Z'' are isotopic in Ω rel. $\partial\Omega$, $i'_!f'^*_B = i''_!(f_Bj)^*$. Also, since Z'' is the set of zeroes of $v : Z \to E$ which is transversal to the zero section, $j_!(1) = e(E)$. Hence, using that $f'_A = p_Ai'$, $i'' = ij$, $f_A = p_Ai$ and $j_!(j^*(x)) = j_!(1) \cdot x : \alpha^*\beta_!(c) = \alpha'^*\beta_!(c) = f'_{A!}f'^*_B(c) = p_{A!}i'_!f'^*_B(c) = p_{A!}i''_!j^*f^*_B(c) = p_{A!}i_!j_!j^*f^*_B(c) = (p_Ai)_!(j_!(1) \cdot f^*_B(c)) = f_{A!}(e(E) \cdot f^*_B(c))$.

Proof of 1.1. Consider the diagram:

$$\begin{array}{ccc} \tilde{M}_k \cup \tilde{M}_{k-1} & \xrightarrow{p} & \tilde{N}_{k-1} \\ {\scriptstyle f_k \cup f_{k-1}}\downarrow & & \downarrow{\scriptstyle g_{k-1}} \\ V & \xrightarrow{f} & W \end{array}$$

where $p(x_1, [x_2, \ldots, x_k]) = [x_2, \ldots, x_k]$, $p(x_1, [x_2, \ldots, x_{k-1}]) = [x_1, \ldots, x_{k-1}]$. It follows from the transversality of $f^k : V^{(k)} - \Delta_V(k) \to W^{(k)}$ to $\delta_W(k)$ that the above diagram is sub-cartesian, the excess bundle being zero on $\tilde{M}_k$ and $f_{k-1}^*(N_f)$ on $\tilde{M}_{k-1}$. From 2.1 we deduce that $f_{k*}([\tilde{M}_k])$ and $g_{k*}([\tilde{N}_k])$, where [ ] denotes the fundamental class, are fundamental classes for $\bar{M}_k$ and $\bar{N}_k$ respectively, for which $m_k = f_{k!}(1)$, $n_k = g_{k!}(1)$. Applying 2.2 to the above diagram with $c = 1$ we get:

$$f^*(n_{k-1}) = f^*(g_{k-1!}(1)) = f_{k!}(1) + f_{k-1!}(f_{k-1}^*(e(N_f))) = m_k + e \cdot m_{k-1}.$$

If r is even and N_f oriented, the induced orientation on $N_f^{(k)} \mid \hat{M}_k^0$ is invariant by the action of S_k and 2.1 (i) shows that $N_{f_k \cup f_{k-1}}$ and $N_{g_{k-1}}$ are oriented. The above calculations hold in integral cohomology. If W is not orientable, m_k and n_k can be interpreted as follows. Let θ_W denote the sheaf of orientations of W; then $f^*(\theta_W) = \theta_V$ since N_f is oriented, and also $f_k^*(\theta_V) = \theta_{\tilde{M}_k}$, $g_k^*(\theta_W) = \theta_{\tilde{N}_k}$. Letting [ ] denote the fundamental class with twisted coefficients, we have that $f_{k*}([M_k])$ and $g_{k*}([N_k])$ are fundamental classes for $\bar{M}_k$ and $\bar{N}_k$ respectively with twisted coefficients, whose Poincaré duals are $m_k = f_{k!}(1)$ and $n_k = g_{K!}(1)$.

In the terminology of [7], the above considerations amount roughly to say that the homological intersection of $f(V)$ and $\bar{N}_{k-1}$ in W consists of the "far intersection" (that is $\bar{M}_k$) plus the "near intersection" (that is the set of zeroes of a section of the non-zero part of the excess bundle).

§3. Divisibility conditions

3.1 PROPOSITION. *If the compact oriented manifold* V^{4pr} *immerses in* $\mathbf{R}^{4pr+2r}$, $\bar{P}_r(V)^p$ *is divisible by* $2p+1$, *where* $\bar{P}_r(V)$ *denotes the* r-th *Pontriagin class of the stable normal bundle of* V.

Proof. Let $f: V^{4pr} \to \mathbf{R}^{4pr+2r}$ be an immersion; after perturbing it slightly we can assume it to be regular. Then M_{2p+1} consists of isolated points whose number equals m_{2p+1} evaluated on $[V]$; since $e(N_f)^2 = \bar{P}_r$, by 1.1 $m_{2p+1} = (-1)^{2p+1} \cdot \bar{P}_r(V)$. If $x_1, \ldots, x_{2p+1} \in V$ are distinct and $f(x_1) = \cdots = f(x_{2p+1}) = y$, the orientation we have given to $N(f_{2p+1})$ shows that they are all counted with the same sign, say ε_y. Hence $(-1)^{2p+1} \cdot \bar{P}_r(V)$ evaluated on $[V]$ equals $(\sum_{y \in N_{2p+1}} \varepsilon_y) \cdot (2p+1)$.

For example, if V^{4n} immerses in $\mathbf{R}^{4n+2}$, P_1^n is divisible by $2n+1$. (The case $n = 1$ was considered by J. H. White in [6]). If V^{12} immerses in $\mathbf{R}^{18}$, $\bar{P}_3 = P_1^3 - 2P_1P_2 + P_3$ is divisible by 3. If V^{16} immerses in $\mathbf{R}^{20}$, $(P_1^2 - P_2)^2$ is divisible by 5.

In fact 3.1 is probably a consequence of the integrality of the L-genus, taking inaccount that $\bar{P}_i = 0$ for $i > r$.

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