The Ahlfors-Schwarz lemma in several complex variables.

Comment. Math. Helvetici **55** (1980) 547–558 Birkhäuser Verlag, Basel

The Ahlfors-Schwarz lemma in several complex variables

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Ahlfors [1] established a beautiful connection between curvature and distortion for holomorphic maps by showing that any holomorphic map of the disk into a Riemann surface furnished with a conformal metric whose Gaussian curvature is everywhere less than or equal to -4 must be distance decreasing from the Poincaré metric of the disk.

This suggests the following question: Given two Hermitian manifolds M and N and a holomorphic map $f: M \to N$, under what conditions can we give a bound for the differential df of the map in terms of bounds on suitable curvatures of M and N? The Ahlfors result immediately generalizes to the case when M is the disk with Poincaré metric and N has holomorphic sectional curvature everywhere less than or equal to $K<0$. Then $\|df\|^2 \leq -K^{-1}$. Chern [3] and Lu [6] obtained results for other special cases of M, including general compact M.

Yau [9] established a general form of this result under the assumption that M is a complete Kähler manifold. He shows in this case that if the Ricci curvature of M is bounded from below by $k \leq 0$ and the holomorphic bisectional curvature of N is bounded from above by $K<0$, *then*

$$\|df\|^2 \leq A\frac{k}{K}.$$

Recently, Chen, Cheng, and Look [2] obtained a different version, assuming that M is a complete Kähler manifold with holomorphic sectional curvature bounded from below by $k \leq 0$ and Riemann sectional curvature bounded from below by some constant and that N is a Hermitian manifold with holomorphic sectional curvature bounded from above by $K<0$. Then

$$\|df\|^2 \leq \frac{k}{K}.$$

[1] This work was supported by the Forschungsinstitut für Mathematik, ETH, Zürich, and the U.S. National Science Foundation.

In this paper we prove variants of these theorems. Theorem 1 differs from Yau's result in that we assume N is also Kähler, but only assume that the holomorphic sectional curvature is bounded from above by $K<0$. Theorem 2 differs from that of Chen, Cheng, and Look in that we do not need to assume that M is Kähler. The proofs given here have some similarity to the original proof of Ahlfors.

1. We shall need to use certain comparison functions on M for our proofs, and we formulate their existence as a series of propositions. A non-negative real-valued continuous function on M is said to be proper if the sets $\{p: u(p) \leqslant c\}$ are compact for each real constant c. We say that a function v defined in a neighborhood V of p is an upper supporting function for u at p if $v(p)=u(p)$ and $v(q) \geqslant u(q)$ for $q \in U$.

PROPOSITION 1. *Let M be a complete Riemannian manifold with Ricci curvature bounded from below. Then there is a continuous proper non-negative function u on M with the property that at each point p it has a smooth upper supporting function v with $\|\nabla v\| \leqslant 1$ and $\Delta v \leqslant 1$ at p.*

Since we may always divide u (and v) by a given positive constant, it suffices to show that there is a constant C and a non-negative proper function u on M which has a supporting function v at each point with $\|\nabla v\| \leqslant C$, $\Delta v \leqslant C$. Let $o \in M$ be a fixed point. Then the function $r=d(p, o)$ is proper by the Hopf-Rinow Theorem, provided M is complete. Let $2a$ be the distance from o to the nearest cut-point of o (i.e., the radius of injectivity of o), and B_a the ball of radius a about o. If the Ricci curvature of M is not less than $-c^2(m-1)$, where $m=\dim M$, then

$$\Delta r \leqslant \frac{1}{r}+c$$

at points p which are not cut-points of o (see [4], [7], or [8]). Let u be a smooth non-negative function inside the ball B_{2a} and equal to r outside B_a. Take C' to be greater than the maxima of $\|\nabla u\|$ and Δu in B_a, and set

$$C=\max\left(C', \frac{1}{a}+c\right).$$

Then $\|\nabla u\| \leqslant C$ and $\Delta u \leqslant C$ at p, provided p is not a cut-point of o. If p is a cut-point of o, take $o' \in B_a$ on the shortest geodesic joining o and p, and set $v=d(o, o')+d(o', p)$. Then v is smooth in a neighborhood of p, $u(p)=v(p)$,

$u(q) \leqslant v(q)$, $\|\nabla v\| \leqslant 1$ and at p we have

$$\Delta v \leqslant \frac{1}{a} + c \leqslant C.$$

In a similar manner one establishes the following propositions (cf. [4], [7], and [8]), where we use the expression $A \leqslant B$ for Hermitian matrices to mean that $B - A$ is positive semi-definite.

PROPOSITION 2. *Let M be a complete Riemannian manifold with Riemann sectional curvature bounded from below. Then there is a continuous non-negative proper function u on M with the property that at each point $p \in M$ it has a smooth upper supporting function v with $\|\nabla v\| \leqslant 1$ and $v_{i,j} \leqslant g_{ij}$ at p. If M is a Hermitian manifold, this implies $v_{\alpha\bar\beta} \leqslant g_{\alpha\bar\beta}$.*

PROPOSITION 3. *Let M be a complete Kähler manifold with holomorphic bisectional curvature bounded from below. Then there is a continuous non-negative proper function u on M with the property that at each point p of M it has a smooth upper supporting function v with $\|\nabla v\| \leqslant 1$ and $v_{\alpha\bar\beta} \leqslant g_{\alpha\bar\beta}$ at p.*

2. Let (M, g) be a Kähler manifold. Then we may introduce normal coordinates $z^1, \ldots, z^n$ in a neighborhood of any point p so that

$$g_{\alpha\bar\beta} = \delta_{\alpha\bar\beta} - \tfrac{1}{2} R_{\alpha\bar\beta\gamma\bar\delta} z^\gamma \bar z^\delta + O(z^3),$$

where $R_{\alpha\bar\beta\gamma\bar\delta}$ is the Riemann curvature tensor of the metric at p. It has the symmetry property $R_{\alpha\bar\beta\gamma\bar\delta} = R_{\gamma\bar\beta\alpha\bar\delta} = R_{\alpha\bar\delta\gamma\bar\beta} = R_{\gamma\bar\delta\alpha\bar\beta}$. The Ricci curvature tensor $R_{\alpha\bar\beta}$ is given by

$$R_{\alpha\bar\beta} = g^{\gamma\bar\delta} R_{\alpha\bar\beta\gamma\bar\delta} = g^{\gamma\bar\delta} R_{\gamma\bar\delta\alpha\bar\beta}.$$

The Ricci curvature in the direction of a tangent vector ξ at p is $R_{\alpha\bar\beta}\xi^\alpha\bar\xi^\beta/(g_{\alpha\bar\beta}\xi^\alpha\bar\xi^\beta)$.

Suppose N, h is another Kähler manifold and $f: M \to N$ is a non-constant holomorphic map. We may introduce normal coordinates $w^1, \ldots, w^n$ in a neighborhood of $f(p)$ so that

$$h_{i\bar g} = \delta_{i\bar g} - \tfrac{1}{2} S_{i\bar j k\bar l} w^k \bar w^l + O(w^3),$$

where S is the Riemann curvature tensor of N. If ξ and η are tangent vectors at $f(p)$, the holomorphic bisectional curvature of N in the directions ξ and η is

$$S_{i\bar{g}k\bar{l}}\xi^i\bar{\xi}^j\eta^k\bar{\eta}^l/(g_{i\bar{j}}\xi^i\bar{\xi}^j)(g_{k\bar{l}}\eta^k\bar{\eta}^l).$$

The holomorphic sectional curvature in the direction ξ is

$$S_{i\bar{j}k\bar{l}}\xi^i\bar{\xi}^j\xi^k\bar{\xi}^l/(g_{i\bar{j}}\xi^i\bar{\xi}^j)^2.$$

In terms of the coordinates z^α and w^i the mapping may be expressed as

$$w^i = f^i(z).$$

The differential df is then given by the matrix $[f^i_\alpha]$, where $f^i_\alpha = \partial f^i/\partial z^\alpha$. By taking unitary changes of coordinates at p and $f(p)$, we may bring $[f^i_\alpha]$ into the canonical form

$$f^i_\alpha = \lambda_\alpha \delta^i_\alpha,$$

with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\nu > \lambda_{\nu+1} = \cdots = 0$, where ν is the rank of $[f^i_\alpha]$. The norm $\|df\|$ of df is just $\lambda = \lambda_1$. In this section we shall be concerned with the behavior of the quantity $\Lambda = \Sigma\lambda^2_\alpha$. We have

$$\|df\|^2 = \lambda^2 \leq \Lambda.$$

Since the λ^2_α are just the eigenvalue of the matrix $\Sigma_i f^i_\alpha \bar{f}^i_\beta$, we see that Λ is just the trace of this matrix, and so $\Lambda = \Sigma_{i,\alpha} f^i_\alpha \bar{f}^i_\alpha$ in our special coordinates. Since $g^{\alpha\bar{\beta}} f^i_\alpha \bar{f}^j_\beta h_{i\bar{j}}$ is invariant under arbitrary changes of coordinates, we have in general

$$\Lambda = g^{\alpha\beta} f^i_\alpha \bar{f}^j_\beta h_{i\bar{j}}.$$

The next proposition gives a useful inequality for the Laplacian $\Delta\Lambda$ (See also Lu [6]).

PROPOSITION 4. *Let f be a non-constant holomorphic map from a Kähler manifold M to a Kähler manifold N. Suppose that the Ricci curvature of M (at a point p) is greater than or equal to k and that the holomorphic sectional curvature of N (at $f(p)$) are all less than or equal to $K \leq 0$. Then at p we have*

$$\Delta \log \Lambda \geq 2k - \frac{\nu+1}{\nu} K\Lambda,$$

where ν is the rank of df at p.

Proof. We use the special coordinates introduced above. Since M is Kähler, we have $\Delta u = 4g^{\gamma\bar\delta}u_{\gamma\bar\delta}$, and so

$$\Delta \log \Lambda = \frac{4g^{\gamma\bar\delta}\Lambda_{\gamma\bar\delta}}{\Lambda} - \frac{4g^{\gamma\bar\delta}\Lambda_\gamma \bar\Lambda_\delta}{\Lambda^2}.$$

In normal coordinates we have $g^{\alpha\bar\beta} = \delta^{\alpha\beta} + \frac{1}{2}R_{\alpha\bar\beta\gamma\bar\delta}z^\gamma \bar z^\delta + O(z^3)$. Since f^i_α is holomorphic and the first derivatives of $g^{\alpha\bar\beta}$ and $h_{i\bar j}$ vanish at p and $f(p)$, we have at p

$$\Lambda_\gamma = g^{\alpha\bar\beta} f^i_{\alpha\gamma} \bar f^j_\beta h_{i\bar j} = \sum_{\alpha,i} f^i_{\alpha\gamma}\bar f^i_\beta$$

and

$$4g^{\gamma\bar\delta}\Lambda_{\gamma\bar\delta} = 2\sum_{\alpha,\beta,i} R_\alpha \beta f^i_\alpha \bar f^i_\beta - 2\sum_{\alpha,\gamma} S_{i\bar j k\bar l} f^i_\alpha \bar f^j_\alpha f^k_\gamma \bar f^l_\gamma + 4\sum f^i_{\alpha\gamma}\bar f^i_{\alpha\gamma}.$$

By the Schwarz inequality

$$\sum_l f^i_\beta \bar f^i_\beta \sum f^i_{\alpha\gamma}\bar f^i_{\alpha\gamma} \geq \left|\sum_l f^i_{\alpha\gamma}\bar f^i_\alpha\right|^2.$$

Since

$$R_{\alpha\bar\beta} f^i_\alpha \bar f^i_\beta \geq k\lambda_i^2,$$

we have

$$\Delta \log \Lambda \geq 2k - 2\Lambda \sum_{\alpha,\gamma} S_{i\bar j k\bar l} f^i_\alpha \bar f^j_\alpha f^k_\gamma \bar f^l_\gamma.$$

The following lemma states that

$$\sum_{\alpha,\gamma} S_{i\bar j k\bar l} f^i_\alpha \bar f^j_\alpha f^k_\gamma \bar f^l_\gamma \leq \frac{1}{2}\frac{\nu+1}{\nu} K\Lambda^2.$$

Hence

$$\Delta \log \Lambda \geq 2k - \frac{\nu+1}{\nu} K\Lambda.$$

LEMMA. *Let* $\xi_1, \ldots, \xi_\nu$ *be* ν *orthogonal tangent vectors. If* $S(\xi, \bar{\eta}, \zeta, \bar{\omega})$ *is a symmetric bihermitian form* [i.e., $S(\xi, \bar{\eta}, \zeta, \bar{\omega}) = S(\zeta, \bar{\eta}, \xi, \bar{\omega})$ *and* $S(\eta, \bar{\xi}, \omega, \bar{\zeta}) = \bar{S}(\xi, \bar{\eta}, \zeta, \bar{\omega})$], *such that for all* ξ

$$S(\xi, \bar{\xi}, \xi, \bar{\xi}) \leq K \|\xi\|^4,$$

then

$$\sum_{\alpha,\beta} S(\xi_\alpha, \bar{\xi}_\alpha, \xi_\beta, \bar{\xi}_\beta) \leq \tfrac{1}{2}K[(\sum \|\xi_\alpha\|^2)^2 + \sum \|\xi_\alpha\|^4].$$

If $K \leq 0$, *then*

$$\sum_{\alpha,\beta} (\xi_\alpha, \bar{\xi}_\alpha, \xi_\beta, \bar{\xi}_\beta) \leq \frac{\nu+1}{2\nu} K(\sum \|\xi_\alpha\|^2)^2.$$

Proof. Let $P = Z_4^\nu$ and represent each $A \in P$ as $\{\epsilon_1, \ldots, \epsilon_\nu\}$ with $\epsilon_\alpha^4 = 1$. Let

$$\xi_A = \sum \epsilon_\alpha \xi_\alpha.$$

Then $\|\xi_A\|^2 = \sum \|\xi_\alpha\|^2$, and so

$$S(\xi_A, \bar{\xi}_A, \xi_A, \bar{\xi}_A) \leq K(\sum \|\xi_\alpha\|^2)^2.$$

Hence

$$K(\sum \|\xi_\alpha\|^2)^2 \geq \frac{1}{4^\nu} \sum_A S(\xi_A, \bar{\xi}_A, \xi_A, \bar{\xi}_A) = \frac{1}{4^\nu} \sum_A \epsilon_\alpha \bar{\epsilon}_\beta \epsilon_\gamma \bar{\epsilon}_\delta S(\xi_\alpha, \bar{\xi}_\beta, \xi_\gamma, \bar{\xi}_\delta)$$

$$= \sum_\alpha S(\xi_\alpha, \bar{\xi}_\alpha, \xi_\alpha, \bar{\xi}_\alpha) + \sum_{\alpha \neq \gamma} S(\xi_\alpha, \bar{\xi}_\alpha, \xi_\gamma, \bar{\xi}_\gamma) + S(\xi_\alpha, \bar{\xi}_\gamma, \xi_\gamma, \bar{\xi}_\alpha).$$

By the symmetry of S, we get

$$\sum_\alpha S(\xi_\alpha, \bar{\xi}_\alpha, \xi_\alpha, \bar{\xi}_\alpha) + 2 \sum_{\alpha \neq \gamma} S(\xi_\alpha, \bar{\xi}_\alpha, \xi_\gamma, \bar{\xi}_\gamma) \leq K(\sum \|\xi_\alpha\|^2)^2$$

and

$$2 \sum_{\alpha,\gamma} S(\xi_\alpha, \bar{\xi}_\alpha, \xi_\gamma, \bar{\xi}_\gamma) \leq K[(\sum \|\xi_\alpha\|^2)^2 + \sum \|\xi_\alpha\|^4].$$

Since $\nu \sum \|\xi_\alpha\|^4 \geq (\sum \|\xi_\alpha\|^2)^2$, we get

$$\sum_{\alpha,\gamma} S(\xi_\alpha, \bar{\xi}_\alpha, \xi_\gamma, \bar{\xi}_\gamma) \leq \frac{\nu+1}{2\nu} K(\sum \|\xi_\alpha\|^2)^2,$$

proving the lemma.

We now assume that M is complete with Ricci curvature bounded from below by k and let u be the function given by proposition 1. Since u is proper and non-negative, the regions $D_\epsilon = \{p: (1-\epsilon u) > 0\}$ have compact closure, and each point p of M belongs to all D_ϵ with $\epsilon < u(p)^{-1}$. Let f be a non-constant holomorphic map of M into a Kähler manifold with holomorphic sectional curvature bounded from below by $K<0$. We also suppose $k \leq 0$.

The function $\Lambda(1-\epsilon u)^2$ is a continuous non-negative function in $\bar{D}_\epsilon$, vanishes on the boundary and is not identically zero. Since $\bar{D}_\epsilon$ is compact, it must have a positive maximum at some point p. Let v be a smooth upper supporting function for u at p with $\|\nabla v\| \leq 1$, $\Delta v \leq 1$. Since $1-\epsilon v(p) = 1-\epsilon u(p)$ and $1-\epsilon v(q) \geq 1-u(q)$ in a neighborhood of p, we see that $\Lambda(1-\epsilon v)^2$ has a positive local maximum at p. Thus $\log \Lambda(1-\epsilon v)^2$ also has a local maximum there, and so at p

$$0 \geq \Delta \log \Lambda(1-\epsilon v)^2 = \Delta \log \Lambda - \frac{\epsilon \Delta v}{1-\epsilon v} - \frac{\epsilon^2 \|\nabla v\|^2}{(1-\epsilon v)^2}.$$

If $\epsilon < 1$, we then have

$$0 \geq -2k + \frac{\nu+1}{\nu} K\Lambda - \frac{2\epsilon}{(1-\epsilon v)^2}.$$

Since $K<0$,

$$\Lambda \leq \frac{2\nu}{\nu+1}\frac{k}{K} + \frac{\epsilon}{(1-\epsilon v)^2},$$

and so

$$(1-\epsilon v(p))^2 \Lambda(p) \leq \frac{2\nu}{\nu+1}\frac{k}{K} + \epsilon.$$

Now $v(p)=u(p)$, and $\Lambda(1-\epsilon u)^2$ has its maximum at p. Thus for any $q\in D_\epsilon$

$$(1-\epsilon u(q))^2\Lambda(q)\leq(1-\epsilon u(p))^2\Lambda(p)\leq\frac{2\nu k}{(\nu+1)K}+\epsilon,$$

or

$$\Lambda\leq\frac{2\nu}{\nu+1}\frac{k}{K}\frac{1}{(1-\epsilon u)^2}+\frac{\epsilon}{(1-\epsilon u)^2}$$

for all points in D_ϵ. If we fix q and let ϵ tend to zero, we have

$$\Lambda\leq\frac{2\nu}{\nu+1}\frac{k}{K},$$

thus proving the following Theorem:

THEOREM 1. *Let M be a complete Kähler manifold with Ricci curvature bounded from below by $k\leq 0$, and N a Kähler manifold with holomorphic sectional curvature bounded from above by $K<0$. Then for any holomorphic map $f: M\to N$ we have*

$$\|df\|^2\leq\Lambda(f)\leq\frac{2\nu}{\nu+1}\frac{k}{K},$$

where ν is the maximal rank of df.

If M and N are both the unit ball in $\mathbf{C}^n$ with the Kähler metric of constant holomorphic section curvature $-c^2$, and f is the identity map, then $\Lambda=n$, $k=[(n+1)/2]c^2$ and $K=-c^2$, and so $\Lambda=[2\nu/(\nu+1)](k/K)$. Of course $\|df\|^2=1<\Lambda$ unless $n=1$.

Observe that if $k=0$ in the Theorem then, $\|df\|^2\equiv 0$, and f is constant map. This gives the following corollary. The case where $\dim M=\dim N=1$ was originally obtained by Alfred Huber [5].

COROLLARY 1. *Let M be a complete Kähler manifold with non-negative Ricci curvature and N a Kähler manifold whose holomorphic sectional curvature is bounded above by $K<0$. Then there is no non-constant holomorphic map of M into N.*

In the proof of Theorem 1 we needed the assumption $k \leq 0$ to conclude $-k(1-\epsilon v)^2 \leq -k$. If, however, the Ricci curvature of M is bounded from below by a positive constant k, then M is compact. If f is a non-constant holomorphic map of M into N, then Λ must have a positive maximum at some point p. There we must have

$$0 \geq \Delta \log \Lambda \geq 2k - \frac{\nu+1}{\nu} K\Lambda,$$

and so $K>0$, proving the following corollary:

COROLLARY 2. *Let M be a complete Kähler manifold with Ricci curvature bounded from below by a positive constant k. Then M is compact, and there is no non-constant holomorphic map of M into a Kähler manifold with non-positive holomorphic sectional curvature.*

3. In this section we consider holomorphic maps between Hermitian manifolds. The study of curvature for a Hermitian manifold (M, g) is quite complicated. There are several different versions of the curvature tensor. Fortunately, they all give the same definition of holomorphic sectional curvature. If ξ is a tangent vector at a point p of M, then the holomorphic sectional curvature at p in the direction ξ is the Rieman sectional curvature of the section determined by ξ and $i\xi$. If φ is a holomorphic map of a disk Δ_a in $\mathbf{C}$ into M with $\varphi(0)=p$ and $\varphi'(0)=\xi$, then the pullback $\varphi * g = g_{\alpha\bar{\beta}}\varphi^{\alpha}_{\zeta}\bar{\varphi}^{\beta}_{\zeta}$ is a conformal metric in Δ_a whose Gaussian curvature at 0 is less than or equal to the holomorphic sectional curvature of M at p in the direction ξ, and there is a map φ for which it is equal. This is the only property of holomorphic sectional curvature for Hermitian manifolds that we shall use, and could in fact be taken as the definition of holomorphic sectional curvature. Note that the Gaussian curvature of the conformal metric $ds^2 = \rho^2 |d\zeta|^2$ is given by $-2\rho^{-2}\rho_{\zeta\bar{\zeta}}$.

Let (M, g) and (N, h) be two Hermitian manifolds, and $f: M \to N$ a non-constant holomorphic map. We assume that the holomorphic sectional curvature of M is bounded from below by a constant $k \leq 0$, and that the holomorphic sectional curvature of N is bounded from above by a constant $K<0$. We shall also assume that M satisfies the following condition.

(C). There is a continuous proper non-negative function u on M with the property that at each point p there is a smooth upper supporting function v with $\|\nabla v\| \leq 1$ and $v_{\alpha\bar{\beta}} \leq g_{\alpha\bar{\beta}}$ at p.

It follows from propositions 2 and 3 that M satisfies (C) if it is complete and the Riemann sectional curvature of M is bounded from below or if it is a complete Kähler manifold with holomorphic bisectional curvature bounded from below.

Let $\lambda = \lambda(p)$ be the function $\|df\|$. Thus for each tangent vector ξ at p we have $\|f^*\xi\| \leq \lambda(p)\,\|\xi\|$, and there is one tangent vector ξ for which equality holds. Let u be the function given by condition (C), and set $D_\epsilon = \{p \in M\colon \epsilon u(p) < 1\}$. Then $\bar{D}_\epsilon$ is compact. The function $\lambda(1-\epsilon u)$ is continuous on $\bar{D}_\epsilon$ and vanishes on the boundary of D_ϵ. Hence there is a point p in D_ϵ where it assumes its maximum. Let ξ be a tangent vector at p such that $\lambda(p) = \|f^*\xi\|$, $\|\xi\| = 1$, and let φ be a holomorphic map of a disk $\Delta_a \subset \mathbf{C}$ into M with $\varphi(0) = p$, $\varphi'(0) = \xi$ such that the conformal metric

$$\rho^2\,|d\zeta|^2 = g_{\alpha\bar{\beta}}\varphi^\alpha_\zeta \bar{\varphi}^\beta_{\bar{\zeta}}\, d\zeta\, d\bar{\zeta}$$

has Gaussian curvature at 0 equal to the holomorphic sectional curvature of M at p in the direction ξ. Since $\|\xi\| = 1$, $\rho(0) = 1$, and the Gaussian curvature of ρ at 0 is at least k.

Let $\sigma^2\,|d\zeta|^2$ be the conformal metric on Δ_a obtained by pulling back the metric h on N by the holomorphic map $f \circ \varphi$. Then $\sigma(0) = \|f^*\xi\| = \lambda(p)$ and the Gaussian curvature of σ at 0 is at most K. For an arbitrary $\zeta \in \Delta_a$ we have $\rho(\zeta) = \|\varphi'(\zeta)\|$ and $\sigma(\zeta) = \|f^*\varphi'(\zeta)\| \leq \lambda(\varphi(\zeta))\,\|\varphi'(\zeta)\| = \lambda(\varphi(\zeta))\rho(\zeta)$.

Let v be a smooth upper supporting function of u at p with $\|\nabla v\| \leq 1$ and $v_{\alpha\bar{\beta}} \leq g_{\alpha\bar{\beta}}$. Let us take Δ_a so small that $\varphi[\Delta_a]$ is contained in the domain of v. Then we have

$$\lambda(q)(1-\epsilon v)(q)) \leq \lambda(q)(1-\epsilon u(q)) \leq \lambda(p)(1-\epsilon u(p)) = \lambda(p)(1-\epsilon v(p)),$$

and so $\lambda(1-\epsilon v)$ has a local maximum at p. Hence

$$\frac{\sigma(\zeta)}{\rho(\zeta)}[1-\epsilon v(\varphi(\zeta))]$$

has a local maximum at $\zeta = 0$. Now

$$\left|\frac{\partial}{\partial \zeta} v(\varphi(\zeta))\right| = |v_\alpha \varphi^\alpha_\zeta| \leq \|\nabla v\|\,\|\varphi'(\zeta)\| = \rho(\zeta)$$

and

$$\frac{\partial^2 v(\varphi(\zeta))}{\partial\zeta\,\partial\bar{\zeta}} = v_{\alpha\bar{\beta}}\varphi^\alpha_\zeta \bar{\varphi}^\beta_{\bar{\zeta}} \leq g_{\alpha\bar{\beta}}\varphi^\alpha_\zeta \bar{\varphi}^\beta_{\bar{\zeta}} = \rho^2(\zeta).$$

If $\lambda(p)>0$, then $\log \sigma/\rho\,(1-\epsilon v)$ must also have a local maximum at $\zeta=0$, and we must have there

$$0 \geqslant \frac{\partial^2}{\partial\zeta\,\partial\bar{\zeta}}\left[\log\frac{\sigma}{\rho}(1-\epsilon v)\right]$$

$$\geqslant \frac{\partial^2}{\partial\zeta\,\partial\bar{\zeta}}(\log\sigma-\log\rho)-\frac{\epsilon v_{\zeta\bar{\zeta}}}{1-\epsilon v}-\frac{\epsilon^2 v_\zeta \bar{v}_\zeta}{(1-\epsilon v)^2}$$

$$\geqslant -K\sigma^2(0)+k\rho^2(0)-\frac{2\epsilon\rho^2(0)}{(1-\epsilon v)^2}=\left(-K\lambda^2(p)+k-\frac{2\epsilon}{(1-\epsilon v(p))^2}\right)\rho^2(0)$$

Thus at p we have

$$-K\lambda^2(p)(1-\epsilon u(p))^2 \leqslant -k(1-\epsilon u(p))^2+2\epsilon \leqslant -k+2\epsilon.$$

Thus for any $q\in D_\epsilon$ we have

$$-K\lambda^2(q)(1-\epsilon u(q))^2 \leqslant -k+2\epsilon$$

or

$$\lambda^2(q) \leqslant \frac{k}{K}(1-\epsilon u(q))^{-2}+\frac{2\epsilon}{K}(1-\epsilon u(q))^{-2}.$$

Since $q\in D_\epsilon$ whenever $\epsilon<[u(q)]^{-1}$, we may let ϵ tend to 0, getting $\lambda^2\leqslant k/K$. We have thus established the following Theorem:

THEOREM 2. *Let M and N be Hermitian manifolds with M complete and holomorphic sectional curvature bounded from below by a constant $k\leqslant 0$ and the holomorphic sectional curvature of N bounded from above by a constant $K<0$. Assume either that M has Riemann sectional curvature bounded from below or that M is Kähler with holomorphic bisectional curvature bounded from below. Then any holomorphic map $f: M\to N$ satisfies*

$$\|df\|^2 \leqslant k/K.$$

COROLLARY 1. *If M is a Hermitian manifold with non-negative holomorphic sectional curvature and either has Riemannian sectional curvature bounded from below or is Kähler with holomorphic bisectional curvature bounded* *from below, then there are no non-constant holomorphic maps of M into a Hermitian manifold with curvature bounded from above by $K<0$. In particular, there are no non-constant bounded holomorphic functions on M.*

In the proof of Theorem 2 we made use of the fact that k was non-positive to conclude $-k \leq -k(1-\epsilon u)^2$. I do not know whether or not a Hermitian manifold with holomorphic sectional curvature bounded below by $k>0$ and satisfying condition C can have non-constant maps into a manifold with non-negative sectional curvature. If, however, M is compact and has holomorphic sectional curvature bounded from below by k, then the above proof can be carried out with $u=v\equiv 0$, and the sign of k is irrelevant, and one obtains the inequality $-K\,\|df\|^2 \leq -k$. Thus if $K\leq 0$, we must have $k\leq 0$, and so there are no non-constant maps of a compact Hermitian manifold with holomorphic sectional curvature bounded from below by $k>0$ into a Hermitian manifold with non-positive holomorphic sectional curvature. In particular, if M is a complete Kähler manifold with holomorphic sectional curvature bounded from below by $k>0$, then M is compact.

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Received January 11, 1980