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The coefficients of quasiconformality of tori in n-space

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Introduction

Let D and D' be domains in $\bar{\mathbf{R}}^n$, $n \geq 3$, the one point compactification of euclidean n-space $\mathbf{R}^n$. Next let f be a homeomorphism of D onto D'. With each such f we can associate two *dilatations*

$$K_I(f) = \sup_{\Gamma} \frac{M(f(\Gamma))}{M(\Gamma)}, \qquad K_O(f) = \sup_{\Gamma} \frac{M(\Gamma)}{M(f(\Gamma))}. \tag{1}$$

Here $M(\Gamma)$ denotes the n-module of the curve family Γ, see [13], and the suprema are taken over all families Γ of curves which lie in D with $M(\Gamma) \neq 0, \infty$. These dilatations satisfy the inequalities

$$K_I(f) \leq K_O(f)^{n-1}, \qquad K_O(f) \leq K_I(f)^{n-1} \tag{2}$$

and reduce simultaneously to 1 if and only if f is a conformal mapping, i.e. a Möbius transformation since $n \geq 3$. The mapping f is *quasiconformal* if one, and hence both, of the dilatations is finite. Moreover, when $f: D \to D'$ is a diffeomorphism of domains in R^n it is easy to show that

$$K_I(f) = \sup_{x \in D} \frac{|J(x,f)|}{l(f'(x))^n}, \qquad K_O(f) = \sup_{x \in D} \frac{|f'(x)|^n}{|J(x,f)|} \tag{3}$$

where $J(x, f)$ denotes the Jacobian of f at x, while $|f'(x)|$ is the norm of the linear mapping $f'(x)$ and $l(f'(x)) = \min\{|f'(x)h| : |h| = 1\}$; [13].

The *inner* and *outer coefficient* (*of quasiconformality*) of the ordered pair (D, D') of domains in $\bar{\mathbf{R}}^n$ are defined as

$$K_I(D, D') = \inf K_I(f), \qquad K_O(D, D') = \inf K_O(f) \tag{4}$$

where the infima are taken over all homeomorphisms f of D onto D'. It follows from (1), (2), and (3) that

$$\begin{cases} 1 \le K_I(D, D'), K_O(D, D') \le \infty \\ K_I(D, D') = K_O(D', D) \\ K_I(D, D') \le K_O^{n-1}(D, D'). \end{cases} \tag{5}$$

The problem of characterizing domains with finite coefficients and that of determining these coefficients are rather complicated in n-space. For Jordan domains some results in this direction have been obtained in [9], [11] and [4]. Gehring [7] has also determined the outer and inner coefficient when D and D' are circular tori in $\mathbf{R}^3$, i.e. cartesian products of an open disc and a circle, while Väisälä [13] solves the problem for spherical ring domains in $\mathbf{R}^n$, i.e. cartesian products of an open interval and an $(n-1)$-dimensional sphere. In the present paper we first extend Gehring's result in 3-space to n-space, the circular tori being cartesian products of an open $(n-1)$-dimensional ball and a circle. Next we consider the more general case when D and D' are the cartesian products of an $(n-k)$-ball and a k-sphere. Both the inner and outer coefficients are determined, see Theorem 4.

The standard procedure for determining the coefficients is the following: (i) Find a lower bound for $K_I(D, D')$ ($K_O(D, D')$) using appropriate curve families. (ii) Show that this bound is sharp by constructing a diffeomorphism $f: D \to D'$ such that $K_I(f)$ ($K_O(f)$), calculated from (3), equals the bound in (i). For the case of circular tori in n-space we are able to follow this procedure exactly and thus generalize Gehring's method in [7]. It may be of interest to observe that some relations become more transparent in n-space where explicit computations have to be replaced by more conceptual arguments. For the general case the standard procedure does not seem to work and we have treated this by introducing surface families instead of curve families. This method does not appear to have been used on coefficient problems before.

The results on the outer coefficient appeared in the author's thesis [10] while the results on the inner coefficients in the general case are new. The author wishes to express her thanks to Professor F. W. Gehring for suggesting this problem and for many helpful discussions.

A word on notation

We refer to [13] for all definitions and notations not explicitly stated.

For each positive integer p let Ω_p denote the p-dimensional Lebesgue measure of B^p, and let ω_p denote the $(p-1)$-dimensional Lebesque measure of S^{p-1}. Next, $V_p(a)$ will denote the volume of $B^p(ae_1, 1)$, $a>1$, with respect to hyperbolic density in the p-dimensional half-space containing $B^p(ae_1, 1)$. Similarly, $v_p(a)$ will denote the $(p-1)$-dimensional hyperbolic volume of $S^p(ae_1, 1)$.

We let $(r, \theta, x_{k+2}, \ldots, x_n)$ with $k=1, 2, \ldots, n-1$ denote polar coordinates of $x=\sum_{i=1}^n x_i e_i$ in $\mathbf{R}^n$. Here

$$\theta=(\theta_1, \theta_2, \ldots, \theta_k), \quad r\geq 0, \quad 0\leq\theta_k<2\pi, \quad 0\leq\theta_i<\pi; \quad 1\leq i\leq k-1.$$

These coordinates are related by the formulas: $x_1=r\cos\theta_1$, $x_2=r\sin\theta_1\cos\theta_2$, $x_3=r\sin\theta_1\sin\theta_2\cos\theta_3, \ldots, x_k=r\sin\theta_1\sin\theta_2\cdots\sin\theta_{k-1}\cos\theta_k$, $x_{k+1}=r\sin\theta_1\sin\theta_2\cdots\sin\theta_{k-1}\sin\theta_k$. We identify the half-space $\theta=0$ in $\mathbf{R}^n$ with $\mathbf{R}_+^{n-k}$, the subspace $x_{k+2}=\cdots=x_n=0$ in $\mathbf{R}^n$ with $\mathbf{R}^{k+1}$, and speak of the ball $B^{n-k}(ae_1, 1)$ in $\theta=0$, the sphere S^k in $x_{k+2}=\cdots=x_n=0$ etc. Finally a domain D in $\bar{\mathbf{R}}^n$ is called a *k-torus* if it can be mapped conformally onto

$$T(k, a)=\{(r, \theta, x_{k+2}, \ldots, x_n):(r-a)^2+x_{k+2}^2+\cdots+x_n^2<1\} \tag{6}$$

for some $a>1$, and we call a its *modulus*. Note that m_p, $0\leq p\leq n$, denotes p-dimensional Hausdorff measure in $\mathbf{R}^n$.

Lower bounds for the coefficients of 1-tori

We consider the family of Jordan curves in $T=T(1, a)$ and ∂T, respectively, which are not homotopic to 0 in $\bar{T}$. Let Γ_T and $\Gamma_{\partial T}$ denote these families.

LEMMA 1.
(i) $M(\Gamma_T)=\pi^{1-n}V_{n-1}(a)$
(ii) $M^{\partial T}(\Gamma_{\partial T})=\pi^{2-n}v_{n-2}(a)$,
where $M^{\partial T}$ denotes the $(n-1)$-modulus with respect to ∂T.

Proof. For (i) suppose that ρ is an admissible density for Γ_T. For each fixed point $x=(r, 0, x_3, \ldots, x_n)\in B^{n-1}(ae_1, 1)$ the circular path γ_x given by $\gamma_x(\theta)=(r, \theta, x_3, \ldots, x_n)$, $\theta\in[0, 2\pi]$, is in Γ_T (to see this consider for example the projection in the x_1, x_2-plane). Thus we get, using Hölder's inequality,

$$1\leq\left(\int_0^{2\pi}\rho(r, \theta, x_3, \ldots, x_n)r\,d\theta\right)^n\leq(2\pi r)^{n-1}\int_0^{2\pi}\rho^n r\,d\theta$$

and integrating over $B^{n-1}(ae_1, 1)$ we obtain

$$\int_{\mathbf{R}^n} \rho^n \, dm_n \geq \int_{B^{n-1}(ae_1, 1)} \left(\int_0^{2\pi} \rho^n r \, d\theta \right) dm_{n-1} \geq \pi^{1-n} V_{n-1}(a) \tag{7}$$

and hence $M(\Gamma_T) \geq \pi^{1-n} V_{n-1}(a)$. On the other hand consider the function ρ_0 which is equal to $1/2\pi r$ in T and 0 in $\underset{\sim}{C}(T)$. Then $\int_\gamma \rho_0 \, ds \geq 1$ for all $\gamma \in \Gamma_T$ since $ds \geq r \, d\theta$ and γ intersects the half space $\theta = t$ for all $t \in [0, 2\pi)$. Clearly ρ_0 gives equality in (7) and we conclude that (i) is valid.

To prove (ii) we argue in exactly the same manner: Suppose that ρ is an admissible density for $\Gamma_{\partial T}$. The circular path, generated by revolving $x = (r, 0, x_3, \ldots, x_n) \in \partial B^{n-1}(ae_1, 1)$, is in $\Gamma_{\partial T}$. Hence, by Hölder's inequality

$$1 \leq \left(\int_0^{2\pi} \rho(r, \theta, x_3, \ldots, x_n) r \, d\theta \right)^{n-1} \leq (2\pi r)^{n-2} \int_0^{2\pi} \rho^{n-1} r \, d\theta$$

and thus

$$\int_{\partial T} \rho^{n-1} \, dm_{n-1} \geq \pi^{2-n} v_{n-2}(a). \tag{8}$$

Next $\rho_0 : \partial T \to [0, \infty)$ given by $\rho_0(x) = 1/2\pi r$ is admissible and gives equality in (8). Thus (ii) follows.

PROPOSITION 1. *Given $1 < a < b$, let D, D' be 1-tori of modulus a and b, respectively. Then*

(i) $K_O(D, D') \geq \dfrac{V_{n-1}(a)}{V_{n-1}(b)}$

(ii) $K_I(D, D') \geq \left[\dfrac{v_{n-2}(a)}{v_{n-2}(b)} \right]^{1/(n-2)}$.

Proof. We may assume that $D = T(1, a)$, $D' = T(1, b)$ as in (6).

(i) This follows directly from (1), (4) and Lemma 1(i).

(ii) By the Boundary Correspondence Theorem [9], [10] (see also Notices Amer. Math. Soc. *19*, A-317, 1972) each quasiconformal mapping $f: T(1, a) \to T(1, b)$ can be extended to a homeomorphism of $\overline{T(1, a)}$ onto $\overline{T(1, b)}$ and the induced boundary mapping f_* is an $(n-1)$-dimensional quasiconformal mapping with $K_I(f_*) \leq K_I(f)$. Moreover

$$K_I(f_*) \geq K_O(f_*)^{1/(n-2)} \geq \left[\frac{M^{\partial T(1,a)}(\Gamma_{\partial T(1,a)})}{M^{\partial T(1,b)}(\Gamma_{\partial T(1,b)})} \right]^{1/(n-2)}$$

by the surface versions of (1) and (2). Thus (ii) follows from (4) and Lemma 1(ii).

Remark. The above procedure is Gehring's method in [7] carried over to n-space. It is not hard to see that we instead of the curve family $\Gamma_{\partial T}$ could have used the curve family "perpendicular" to this one, i.e. consisting of Jordan curves in ∂T which are not homotopic to 0 in $\complement T$. It turns out that this family is the right one for further generalisations.

Modulus inequality for surfaces

In this section we give some results on surfaces which will be used to obtain lower bounds for the coefficients in the general case.

We shall follow Agard [1] and restrict ourselves to the following class of "parametic p-surfaces" in $\mathbf{R}^n$: We say that a continuous mapping from some open set G in $\mathbf{R}^p$, $1 \leq p \leq n-1$, into $\mathbf{R}^n$ is (a locally p-dimensional) *quasiconformal surface* if there exists for each $u_0 \in G$ a neighborhood $U = U(u_0)$ such that σ has the properties (i), (ii), (iii) and (iv) below in U.

(i) The partial derivatives of σ are absolutely continuous on lines ([13]) and L^p-integrable.

(ii) σ is totally differentiable a.e.

(iii) $J_\sigma(u) = \left(\sum_{i_1 < \cdots < i_p} \left| \frac{\partial(\sigma_{i_1}, \ldots, \sigma_{i_p})}{\partial(u_1, \ldots, u_p)} \right|^2 \right)^{1/2} \neq 0$ a.e.

(iv) There exists a constant $Q = Q(U)$ such that

$$\|\sigma'(u)\|^p \leq Q J_\sigma(u) \quad \text{a.e.}$$

In defining the *modulus* $M(\Sigma)$ we declare a non-negative Borel measurable function ρ in $\mathbf{R}^n$ to be *admissible* for a family of quasiconformal surfaces Σ if $\int_{\sigma_G} (\rho \circ \sigma)^p J_\sigma \, dm_p \geq 1$ for all $\sigma \in \Sigma$. We denote the class of admissible functions by $A(\Sigma)$, and we then set

$$M(\Sigma) = \inf \left\{ \int_{\mathbf{R}^n} \rho^n \, dm_n : \rho \in A(\Sigma) \right\}.$$

This is an example of the more general n/p-module of a system of measures defined by Fuglede [3]. In particular M is monotone, countably subadditive and has the "minorizing property."

By Theorem 6 in [1] we have

THEOREM 1. *Suppose that* $f: D \to D'$ *is a quasiconformal mapping of domains in* $\mathbf{R}^n$, *and that* Σ *is a family of quasiconformal surfaces in D. Then there is a family* $\Sigma_0 \subset \Sigma$, *with* $M(\Sigma - \Sigma_0) = 0$, *of mappings* σ *such that* $\sigma^* = f \circ \sigma$ *is quasiconformal surface and*

$$M(\Sigma_0) \le K_O(f) M(\Sigma_0^*).$$

Remark. This theorem holds for surfaces satisfying only conditions (i) and (iv) (slightly rephrased) as proved by Reimann [12] and pointed out by Agard [1]. For quasiconformal surfaces Agard has also established the modulus inequality for surface area based on Lebesgue area while it is not yet established with respect to Hausdorff measure.

We want to establish the analogous result for quasiconformal mappings of smooth hypersurfaces in $\mathbf{R}^n$. For this we first generalize the concept of the modulus to families of surfaces in an $(n-1)$-dimensional C^1-manifold S in $\mathbf{R}^n$: If Σ is a family of surfaces in S, then the *modulus of* Σ *wrt to* S is given by

$$M^S(\Sigma) = \inf \left\{ \int_S \rho^{n-1} \, dm_{n-1} : \rho \in A(\Sigma) \right\}.$$

Next, suppose $f: S \to S'$ is a homeomorphism of $(n-1)$-dimensional C^1-manifolds in $\mathbf{R}^n$. For each $\varepsilon > 0$ and each point p on the manifold we have an $\mathbf{R}^n$-neighborhood U and a bi-Lipschitzian diffeomorphism $i_p: U \to U'$ such that i_p maps the manifold into $\mathbf{R}^{n-1}$ and $K_O(i_p) < 1 + \varepsilon$, see 17.12 [13].

Now for $x \in S$, let $x' = f(x)$. We say that f is *quasiconformal* if there exists a K, $1 \le K < \infty$, such that for each $\varepsilon > 0$ there is a corresponding map $g_x = i_{x'} \circ f \circ i_x^{-1}$ satisfying

$$\sup_{x \in S} K_O(g_x) < \infty, \qquad \operatorname*{ess\,sup}_{x \in S} K_O(g_x) \le K + \varepsilon.$$

The smallest $K \ge 1$ for which the above is true is called the *outer dilatation* of f and is denoted by $K_O(f)$. The inner dilatation $K_I(f)$ is similarly defined and as before f is said to be *quasiconformal* if one (and hence both) of the dilatations is finite. There are several characterizations of these concepts, see [9], [10]. We shall only need the following result.

THEOREM 2. *If $f: S \to S'$ is a quasiconformal map then*

$$K_O(f) = \operatorname{ess\,sup} \frac{L(x, f)^{n-1}}{\mu_f'(x)}$$

where

$$L(x, f) = \limsup_{y \to x} \frac{|f(y) - f(x)|}{|y - x|}, \qquad \mu_f'(x) = \limsup_{r \to 0} \frac{m_{n-1} f(S \cap \bar{B}^n(x, r))}{m_{n-1}(S \cap \bar{B}^n(x, r))}.$$

Furthermore, f is m_{n-1}-absolutely continuous so that if $g: S' \to [0, \infty]$ is a Borel function then

$$\int_{S'} g \, dm_{n-1} = \int_S (g \circ f)(\mu_f') \, dm_{n-1}.$$

Before we can prove the surface generalization of Theorem 1 we need some more preliminary results.

LEMMA 1. *If $\sigma: G \to \mathbf{R}^n$ is a quasiconformal surface and f is diffeomorphism defined in some domain containing $\sigma(G)$ then $f \circ \sigma$ is a quasiconformal surface.*

Proof. This is an immediate consequence of the definition. In addition to chain rules we use the fact that f is locally Lipschitzian for (i), and for (iii) the following lemma (stated for C^1-maps for later reference).

LEMMA 2. *Given $\sigma: G \subset \mathbf{R}^p \to \mathbf{R}^n$ and a C^1-map f in $\mathbf{R}^n$ of a domain containing $\sigma(G)$. Then $f' = O_1 D O_2$ where O_1, O_2 are orthogonal matrices and $D = \operatorname{diag}(d_1, \ldots, d_n)$ with $0 \le d_1 \le \cdots \le d_n$, and at points where σ is differentiable we have*

$$J_{f\circ\sigma}^2 = J_\sigma^2 \left\{ \sum_{\gamma \in C(n,p)} \left(\prod_{i \in \gamma} d_i^2 \right) O_\gamma^2 \right\} \quad \textit{with} \quad \sum_{\gamma \in C(n,p)} O_\gamma^2 = 1$$

where $C(n, p)$ is the set of naturally ordered subsets of p elements from the integers $\{1, 2, \ldots, n\}$.

Proof. See proof of Lemma 0 in [1].

LEMMA 3. *Given $\sigma: G \subset \mathbf{R}^p \to \mathbf{R}^n$ and a nonnegative Borel function f. Then*

$$\int_{\mathbf{R}^n} f(x) m_0(\sigma, x, G)\, dm_p(x) \geq \int_G f(\sigma(u)) J_\sigma(u)\, dm_p(u)$$

whenever σ is m_p – a.e. *differentiable with equality if σ is m_p-absolutely continuous.*

Proof. See p. 38 [1]. The proof is based on Theorem 5.3 [2] and the validity of the area formula for Lipschitzian maps.

LEMMA 4. *Suppose that S is an $(n-1)$-dimensional C^1-manifold in $\mathbf{R}^n$ and that $i: U \to U'$ is a bi-Lipschitzian map, i.e. there exists a $C>0$ such that $C^{-1}|y-x| < |f(y)-f(x)| < C|y-x|$ for all $x, y \in S$. Assume further that $i(S \cap U)$ is a domain $D \subset \mathbf{R}^{n-1}$, that Σ is a family of quasiconformal p-surfaces in $S \cap U$ and let $\Sigma' = \{i \circ \sigma : \sigma \in \Sigma\}$. Then*

$$C^{-2(n-1)} M^S(\Sigma) \leq M^P(\Sigma') \leq C^{2(n-1)} M^S(\Sigma)$$

and $M^S(\Sigma) = 0$ if and only if $M^P(\Sigma') = 0$.

Proof. If $\rho \in A(\Sigma')$ then $C\rho \circ i \in A(\Sigma)$ since $J_{i\circ\sigma} \leq C^p J_\sigma$ by Lemma 2. Thus

$$M^S(\Sigma) \leq C^{n-1} \int_S (\rho \circ i)^{n-1}\, dm_{n-1} = C^{n-1} \int_P \rho^{n-1} J_{i^{-1}}\, dm_{n-1}$$

by Lemma 4. The first half of the inequality follows. The second half is proved in exactly the same manner.

THEOREM 3. *Suppose $f: S \to S'$ is a quasiconformal mapping of $(n-1)$-dimensional C^1-manifolds and let Σ be a family of quasiconformal surfaces in S. Then there is a family $\Sigma_0 \subset \Sigma$, with $M(\Sigma - \Sigma_0) = 0$, of mappings σ such that $\sigma^* = f \circ \sigma$ is a quasiconformal surface, and*

$$M^S(\Sigma_0) \leq K_O(f) M^{S'}(\Sigma_0^*).$$

Proof. Given $x \in S$ let $i_x: U_x \to U'_x$ and $i_{x'}: U_{x'} \to U'_{x'}$ be the diffeomorphisms introduced earlier. In particular $g_x = i_{x'} \circ f \circ i_x^{-1}$ is a quasiconformal map. Let σ_x denote the restriction of σ to $\sigma^{-1}(U_x)$, set $\Sigma_x = \{\sigma_x\}$, and consider $i_x(\Sigma_x)$ under g_x. By Theorem 1 followed by Lemmas 1 and 4 there is a family $\Sigma_{x,0} \subset \Sigma_x$ with $M^{U_x}(\Sigma_x - \Sigma_{x,0}) = 0$ of mappings σ_x such that $\sigma_x^* = f \circ \sigma_x$ is a quasiconformal surface. Next let $\{U_{x_i}\}$ be a countable covering of S and let $\Sigma_1 =$

$\{\sigma\in\Sigma:(\exists i\in N)(\sigma_{x_i}\in\Sigma_{x_i}-\Sigma_{x_i,0})\}$. It follows from the countable subadditivity and the "minorizing property" of M that $M^S(\Sigma_1)=0$. Hence we can choose $\Sigma_0=\Sigma-\Sigma_1$. To prove the inequality let $\rho'\in A(\Sigma'_0)$ and define $\rho: S\to[0,\infty]$ by $\rho(x)=\rho'(f(x))L(x,f)$. For each $\sigma\in\Sigma_0$ we obtain

$$\int_G(\rho\circ\sigma)^p J_\sigma\,dm_p=\int_G\rho'(f(\sigma(u))^p L(\sigma(u),\cdot f)^p J_\sigma(u)\,dm_p(u)$$

$$\geq\int_G\rho'((f\circ\sigma)(u))^p J_{f\circ\sigma}(u)\,dm_p(u)\geq 1.$$

Thus $\rho\in A(\Sigma_0)$ and

$$M^S(\Sigma_0)\leq\int_S\rho^{n-1}\,dm_{n-1}=\int_S\rho'(x)^{n-1}L(x,f)\,dm_{n-1}(x)$$

$$\leq K_O(f)\int_S\rho^{n-1}\mu'_f\,dm_{n-1}=K_O(f)\int_{S'}(\rho')^{n-1}\,dm_{n-1}$$

by Theorem 2. Since $\rho'\in A(\Sigma'_0)$ was arbitrary we have the desired modulus inequality.

Lower bounds for the coefficients of k-tori

In addition to the Lemmas 2 and 3 we shall need the following lemma for our considerations.

LEMMA 5. *Let $f: T(k,a)\to T(k,b)$ be a quasiconformal mapping of k-tori, and let $f_*:\partial T(k,a)\to\partial T(k,b)$ be the induced boundary mapping. Then*

(i) *for m_{n-k}-a.e. $x=(r,\theta,x_{k+1},\dots,x_n)\in B^{n-k}(a,1)$ the map f restricted to the spheres $S_x=S^k(r)+x_{k+2}e_{k+2}+\cdots+x_ne_n$ is m_k-absolutely continuous.*

(ii) *for m_k-a.e. $x\in S^k$ the map f_* restricted to the $(n-k-1)$-spheres $S_x=\partial B^{n-k}(a,1)+ax$ is m_{n-k}-absolutely continuous.*

Proof. The proof of Theorem 8 in [1], which is based on a method used by Gehring [5], uses the fact that every quasiconformal mapping in n-space has finite linear dilatations at each point. From Lemma 1 p. 12 [8] it follows that the same is true for quasiconformal boundary mappings. The proof of Theorem 8 [1] can be carried over with the obvious modifications except for the following

COVERING LEMMA. *Let E be a subset of a sphere S_x as in* (i) *or* (ii). *Then* *there exists for each $t>0$ a sequence of open n-balls $B_1, B_2, \ldots, B_q$ $(q=q(t))$ such that*

(i) $E\subset \bigcup_{j=1}^q B_j\cap S_x$

(ii) $\bigcup_{j=1}^q B_j\cap S_x\subset E(t)=\{y\in S_x : \text{dist}\,(y, E)\le t\}$

(iii) *No point in $\mathbf{R}^n$ lies in more than N of the B_j where N is independent of t. (In particular no point of S_x lies in more than N of the $B_j\cap S_x$.)*

Proof of covering lemma. Fix $t>0$ and pick a finite sequence $y_1, y_2, \ldots$ as follows: Let y_1 be an arbitrary point in E. If $B^n(y_1, t)\not\supset E$ pick $y_2\in E-B^n(y_1, t)$. If $[B^n(y_1, t)\cup B^n(y_2, t)]\not\supset E$ pick $y_3\in E-[B^n(y_1, t)\cup B^n(y_2, t)]$ etc. The process must stop after a finite number of steps, i.e. there exists a q such that $\bigcup_{j=1}^q B^n(y_j, t)\supset E$, since for each q' the union $\bigcup_{j=1}^{q'} B^n(y_j, t/2)$ is disjoint and so $q'm_p(B^n(y_1, t/2)\cap S_x)\le m_p(S_x)$ where $p=k$ in the case (a), $p=n-k-1$ in the case (b). This proves (i) and (ii). To prove (iii) let y be an arbitrary point in $\mathbf{R}^n$. If $y\in B^n(y_j, t)$ then $B^n(y_j, t)\subset B^n(y, 2t)$. Again, by considering the $B^n(y_j, t/2)$ we see that if y belongs to N' of the $B^n(y_j, t)$ we must have $N'\Omega_n(t/2)^n\le\Omega_n(2t)^n$ and so $N'\le 4^n$.

PROPOSITION 2. *Given $1<a<b$, let D, D' be k-tori of modulus a and b, respectively. Then*

$$\text{(i)}\quad K_O(D, D')\ge\frac{V_{n-k}(a)}{V_{n-k}(b)}$$

$$\text{(ii)}\quad K_I(D, D')\ge\left[\frac{v_{n-k-1}(a)}{v_{n-k-1}(b)}\right]^{k/(n-k-1)},\qquad k\ne n-1$$

$$K_I(D, D')\ge\left[\frac{V_1(a)}{V_1(b)}\right]^{n-1},\qquad k=n-1.$$

Proof. We may assume that $D=T(k, a)$ and $D'=T(k, b)$ as in (6). Let f be an arbitrary quasiconformal mapping of $T(k, a)$ onto $T(k, b)$.

(i) We consider for each $x\in B^{n-k}(ae_1, 1)$ such that f restricted to $S_x=S^k(r)+x_{k+2}+\cdots+x_ne_n$ is m_k-absolutely continuous the spherical projection $\sigma_x:\mathbf{R}^k\to S^k(r)+x_{k+2}e_{k+2}+\cdots+x_ne_n$. The hypothesis of Theorem 1 is satisfied for $f: T(k, a)\to T(k, b)$ and the family Σ of the σ_x above. Thus (i) is established if we can show that for some positive constant c

(a) $M(\Sigma)\ge cV_{n-k}(a)$

(b) $M(\Sigma_0^*)\le cV_{n-k}(b)$.

Proof of (a): Suppose that $\rho\in A(\Sigma)$. In this case (punctured spheres) we can perform the surface integration using Hausdorff measure, cf. Lemma 4. Fixing a parametric σ_x we get by Hölder's inequality

$$1 \leq \left(\int_{S_x} \rho^k r^k \, dm_k \right)^{n/k} \leq \omega_k^{(n-k)/k} r^{n-k} \int_{S_x} \rho^n r^k \, dm_k.$$

Integrating over $B^{n-k}(ae_1, 1)$ we obtain

$$\begin{aligned} \int_{\mathbf{R}^n} \rho^n \, dm_n &\geq \int_{B^{n-k}(ae_1, 1)} \left(\int_{S_x} \rho^n r^k \, dm_k \right) dm_{n-k} \\ &\geq 2^{n-k} \omega_k^{(k-n)/n} V_{n-k}(a) \end{aligned}$$

and hence (a) with $c = 2^{n-k} \omega_k^{(k-n)/n}$.

Proof of (b): The function ρ which is equal to $1/\omega_k^{1/k} r$ in $T(k, b)$ and 0 in $\complement T(k, b)$ gives $\int_{\mathbf{R}^n} \rho^n \, dm_n = c V_{n-k}(b)$.

It remains, however, to prove that $\rho \in A(\Sigma_0^*)$. Denoting the orthogonal projection of $\mathbf{R}^n$ onto $\mathbf{R}^{k+1}$ by P it follows directly from the definition of J_σ that

$$\int_{\mathbf{R}^k} (\rho^k \circ \sigma_x^*) J_{\sigma_x^*} \, dm_k \geq \int_{\mathbf{R}^k} (\rho^k \circ P \circ \sigma_x^*) J_{P \circ \sigma_x^*} \, dm_k.$$

Next, let S be the central projection of $\mathbf{R}^{k+1} - \{0\}$ onto S^k, i.e. $S(u) = u/|u|$. The derivative of S with respect to the natural orthogonal system based on spherical coordinates has matrix $D = \operatorname{diag}(0, 1/r, \ldots, 1/r)$. Thus from Lemma 2

$$\int_{\mathbf{R}^k} (\rho^k \circ P \circ \sigma_x^*) J_{P \circ \sigma_x^*} \, dm_k \geq \frac{1}{\omega_k} \int_{\mathbf{R}^k} J_{S \circ P \circ \sigma_x^*} \, dm_k.$$

Now, the function $S \circ P \circ \sigma_x^*$ is m_k-absolutely continuous as a composition of m_k-absolutely continuous functions. Lemma 3 can be applied and we conclude

$$\int_{\mathbf{R}^k} (\rho^k \circ \sigma_x^*) J_{\sigma_x^*} \, dm_k \geq \frac{1}{\omega_k} \int_{(S \circ P \circ \sigma_x^*)(\mathbf{R}^k)} dm_k \geq 1$$

where the last inequality follows from simple topological considerations: By assumption $\overline{\sigma_x^*(\mathbf{R}^k)}$ (one point compactification of $\sigma_x^*(\mathbf{R}^k)$) is not contractible in $T(k, b)$ and hence $\overline{\sigma_x^*(\mathbf{R}^k)}$ meets all the $(n-k)$-dimensional half-spaces $\theta =$ constant. Thus $(S \circ P)(\overline{\sigma_x^*(\mathbf{R}^k)}) = S^k$ and $(S \circ P \circ \sigma_x^*)(\mathbf{R}^k)$ is S^k possibly punctured at one point.

(ii) For $k \neq n-1$. Consider the induced boundary map $f_* : \partial T(k, a) \to \partial T(k, b)$.

Since $K_I(f_*) \le K_I(f)$ by the Boundary Correspondence Theorem [9], [10] and $K_I(f_*) = K_O(f_*^{-1})$ it is sufficient to show

$$K_O(f_*^{-1}) \ge \left[\frac{v_{n-k-1}(a)}{v_{n-k-1}(b)}\right]^{k/(n-k-1)}.$$

We associate with each $x \in S^k$ such that f_*^{-1} restricted to $S_x = \partial B^{n-k}(b, 1) + bx$ is m_{n-k}-absolutely continuous the spherical projection $\sigma_x : \mathbf{R}^{n-k-1} \to S_x$. The hypothesis of Theorem 3 is satisfied for $f_*^{-1} : \partial T(k, b) \to \partial T(k, a)$ and the family Σ of the σ_x above. We now follow the procedure from (i) and note that (ii) is established if we can show that for some positive constant c

(a) $M^{\partial T(k,b)}(\Sigma) \ge c v_{n-k-1}(b)^{-k/(n-k-1)}$

(b) $M^{\partial T(k,a)}(\Sigma_0^*) \le c v_{n-k-1}(a)^{-k/(n-k-1)}$.

Proof of (a): Suppose $\rho \in A(\Sigma)$. Fixing a parametric surface S_x we get by Hölder's inequality

$$1 \le \left(\int_{S_x} \rho^{n-k-1}\, dm_{n-k-1}\right)^{(n-1)/(n-k-1)} \le 2^k v_{n-k-1}(b)^{k/(n-k-1)} \int_{S_x} \rho^{n-1} r^k\, dm_{n-k-1}$$

and hence

$$\int_{T(k,b)} \rho^{n-1}\, dm_{n-1} \ge 2^{-k} \omega_k v_{n-k-1}(b)^{-k/(n-k-1)}.$$

Proof of (b): The function ρ which is equal to $v_{n-k-1}(a)^{-1/(n-k-1)}(2r)^{-1}$ in $\partial T(k, a)$ and 0 in $\complement\, \partial T(k, a)$ gives $\int_{\partial T(k,a)} \rho^{n-1}\, dm_{n-1} = 2^{-k}\omega_k v_{n-k-1}(a)^{-k/(n-k-1)}$. Again it remains to prove that $\rho \in A(\Sigma_0^*)$. Consider the polar projection $P(r, \theta, x_{k+2}, \ldots, x_n) = (r, 0, x_{k+2}, \ldots, x_n)$. It follows that

$$\int_{\mathbf{R}^{n-k-1}} (\rho^{n-k-1} \circ \sigma_x^*) J_{\sigma_x}\, dm_{n-k-1} \ge \int_{\mathbf{R}^{n-k-1}} (\rho^{n-k-1} \circ P \circ \sigma_x^*) J_{P\circ\sigma_x}\, dm_{n-k-1}.$$

Since the function $P \circ \sigma_x^*$ is m_{n-k-1}-absolutely continuous Lemma 3 can be applied and we conclude as before that the value of the integral is larger than 1.

(ii) for $k = n-1$. The tori are spherical rings and we get the bound by considering the family of curves which join the boundary spheres, see Theorem 39.1 [13].

Extremal mappings

We show that the lower bounds given in Proposition 2 are sharp by constructing a pair of extremal mappings. In this construction we make use of the symmetry properties of the domains and the fact that the bounds are given in terms of hyperbolic volumes.

To be more precise, first let g_1, g_2 be Möbius transformations of the $(n-k)$-dimensional half-space $\mathbf{R}_+^{n-k}$ onto the unit ball B^{n-k} so that $g_1(B^{n-k}(ae_1, 1)) = B^{n-k}(c)$, $g_2(B^{n-k}(be_1, 1)) = B^{n-k}(d)$ for some $c, d < 1$ $(c > d)$. Next suppose we are given a diffeomorphism $h:[0, c) \to [0, d)$. We use h to define a mapping $f: T(k, a) \to T(k, b)$ in two steps as follows:

(i) $k: B^{n-k}(c) \to B^{n-k}(d)$ is defined by

$$k(x) = \begin{cases} \dfrac{x}{|x|} h(|x|) & \text{if} \quad 0 < |x| < c \\ 0 \quad \text{if} \quad x = 0 \end{cases}$$

(ii) $g_2^{-1} \circ k \circ g_1 : B^{n-k}(ae_1, 1) \to B^{n-k}(be_1, 1)$ is extended in the obvious way to $f: T(k, a) \to T(k, b)$, i.e. $f(r, \theta, x_{k+2}, \ldots, x_n) = (r', \theta, x'_{k+2}, \ldots, x'_n)$ where $(r', x'_{k+2}, \ldots, x'_n) = g_2^{-1} \circ k \circ g_1(r, x_{k+2}, \ldots, x_n)$

LEMMA 6. *For $f: T(k, a) \to T(k, b)$ constructed from a diffeomorphism $h:[0, c) \to [0, d)$ as above the following holds*

(i) *If h has the property*

$$\max\left(\frac{h(t)}{t}, h'(t)\right) \leq \frac{1 - h(t)^2}{1 - t^2} \quad \textit{for} \quad t \in (0, c) \tag{9}$$

then

$$K_O(f) = \sup_{t \in (0, c)} \frac{1}{h'(t)} \left(\frac{t}{h(t)}\right)^{n-k-1} \left(\frac{1 - h(t)^2}{1 - t^2}\right)^{n-k}$$

(ii) *If h has the property*

$$\max\left(\frac{1 - h(t)^2}{1 - t^2}, h'(t)\right) \geq \frac{h(t)}{t} \quad \textit{for} \quad t \in (0, c) \tag{10}$$

then

$$K_I(f) = \sup_{t \in (0, c)} h'(t) \left(\frac{t}{h(t)}\right)^{k+1} \left(\frac{1 - h(t)^2}{1 - t^2}\right)^{k}.$$

Proof. The function f is a diffeomorphism and the dilatations are given by the formulas (3). By symmetry it is enough to consider points in $B^{n-k}(ae_1, 1)$ and hence in $B^{n-k}(ae_1, 1)-g_1^{-1}(0)$. Thus the problem is to determine the semi-axes of $f'(x)$ when $x \in B^{n-k}(ae_1, 1)-g_1^{-1}(0)$.

We have $f \mid B^{n-k}(ae_1, 1)=g_2^{-1}\circ k \circ g_1$ and we set $v=g_1(x)$, $w=k(v)$, and $y=g_2(w)$. Let us first determine the semi-axes of $f'(x)$ in $\mathbf{R}^{n-k}$. Since k is a radial map induced by h the $(n-k)$ semi-axes of $k'(v)$ are $h'(|v|)$ and $h(|v|)/|v|$ where $h(|v|)/|v|$ occurs $(n-k-1)$ times. Thus returning to the map f the corresponding semi-axes of $f'(x)$ are

$$\frac{|g_1'(x)|}{|g_2'(y)|}h'(|v|) \quad \text{and} \quad \frac{|g_1'(x)|}{|g_2'(x)|}\frac{h(|v|)}{|v|}.$$

Next, suppose that x and $y=f(x)$ have polar coordinates $(r, \theta, x_{k+2}, \ldots, x_n)$ and $(r', \theta, y_{k+2}, \ldots, y_n)$ respectively. Then the last k semi-axes are

$$\frac{r'\,d\theta_i}{r\,d\theta_i}=\frac{2r}{2r}=\frac{|g_1'(x)|}{|g_2'(x)|}\frac{1-|w|^2}{1-|v|^2}$$

where the last equality holds since

$$\frac{1}{2r} \quad \text{and} \quad \frac{|g_1'(x)|}{1-|g_1(x)|^2}$$

both represent the density function for the hyperbolic metric in $\mathbf{R}_+^{n-k}$.

The results follow by substitution in (3).

THEOREM 4. *Given $1<a<b$, let D and D' be k-tori of modulus a and b, respectively. Then*

(i) $$K_O(D, D')=\frac{V_{n-k}(a)}{V_{n-k}(b)}$$

(ii) $$K_I(D, D')=\left[\frac{v_1(a)}{v_1(b)}\right]^k \quad \textit{for} \quad k\neq n-1$$

$$K_I(D, D')=\left[\frac{V_1(a)}{V_1(b)}\right]^{n-1} \quad \textit{for} \quad k=n-1.$$

Proof. We may again assume that $D=T(k, a)$, $D'=T(k, b)$.

(i) It is sufficient by Proposition 2 and Lemma 6 to construct an h satisfying (9) and so that

$$\frac{1}{h'(t)}\left(\frac{1-h(t)^2}{1-t^2}\right)^{n-1}\left(\frac{t}{h(t)}\right)^{n-2}=\frac{V_{n-k}(a)}{V_{n-k}(b)}. \tag{11}$$

Setting $L=V_{n-k}(a)/V_{n-k}(b)$ we can also write

$$L=\int_0^c \frac{s^{n-k-1}}{(1-s^2)^{n-k}}\,ds\Big/\int_0^d \frac{s^{n-k-1}}{(1-s^2)^{n-k}}\,ds$$

where c and d are as in Lemma 6. Let therefore $h:[0,c)\to[0,d)$ be given by

$$L\int_0^{h(t)} \frac{s^{n-k-1}}{(1-s^2)^{n-k}}\,ds=\int_0^t \frac{s^{n-k-1}}{(1-s^2)^{n-k}}\,ds. \tag{12}$$

Then it is not hard to see that h is a diffeomorphism of $[0,c)$ onto $[0,d)$. Furthermore, differentiation of (12) gives (11). It remains to show that (9) is satisfied. That $h(t)/t\le 1-h(t)^2/1-t^2$ is obvious since $h(t)\le t$. That $h'(t)\le 1-h(t)^2/1-t^2$ is equivalent to

$$\left(\frac{1-h(t)^2}{h(t)}\right)^{n-k-1}\int_0^{h(t)} \frac{s^{n-k-1}}{(1-s^2)^{n-k}}\,ds\le\left(\frac{1-t^2}{t}\right)^{n-k-1}\int_0^t \frac{s^{n-k-1}}{(1-s^2)^{n-k}}\,ds$$

and this follows since

$$f(x)=\left(\frac{1-x^2}{x}\right)^{n-k-1}\int_0^x \frac{s^{n-k-1}}{(1-s^2)^{n-k}}\,ds$$

is increasing in $(0,c)$.

(ii) For $k\neq n-1$. We observe that

$$v_{n-k-1}(a)^{k/(n-k-1)}\omega_{n-k-1}\left(\frac{c}{1-c^2}\right)^k=\frac{\omega_{n-k-1}}{\omega_1}\,v_1(a)^k.$$

Hence it suffices, by Proposition 2 and Lemma 6, to construct an h satisfying (10) and so that

$$h'(t)\left(\frac{t}{h(t)}\right)^{k+1}\left(\frac{1-h(t)^2}{1-t^2}\right)^k=\left[\frac{v_1(a)}{v_1(b)}\right]^k=\left[\frac{c}{d}\left(\frac{1-d^2}{1-c^2}\right)\right]^k. \tag{13}$$

Let therefore $h:[0, c]\to[0, d)$ be given by

$$\int_{h(t)}^{d} \frac{(1-s^2)^k}{s^{k+1}}\,ds = \left[\frac{c}{d}\left(\frac{1-d^2}{1-c^2}\right)\right]^k \int_t^c \frac{(1-s^2)^k}{s^{k+1}}\,ds. \tag{14}$$

Again h is a diffeomorphism of $[0, c)$ onto $[0, d)$. That

$$\frac{h(t)}{t} \le \frac{1-h(t)^2}{1-t^2}$$

is trivial since $h(t)<t$. It remains to prove that $h(t)/t \le h'(t)$ for $t\in(0, c)$, or equivalently

$$\frac{t}{h(t)}\,\frac{1-h(t)^2}{1-t^2} \le \frac{c}{d}\,\frac{1-d^2}{1-c^2}.$$

For this we observe that (14) implies

$$\int_{h(t)}^{d} \left(\frac{1-s^2}{s}\right)^{k-1} ds \le \left[\frac{c}{d}\left(\frac{1-d^2}{1-c^2}\right)\right]^k \int_t^c \left(\frac{1-s^2}{s}\right)^{k-1} ds.$$

Now, from the above inequality and (14) we obtain

$$\int_{h(t)}^{d} \left[\frac{(1-s^2)^k}{s^{k+1}} + 2\left(\frac{1-s^2}{s}\right)^{k-1}\right] ds \le \left[\frac{c}{d}\left(\frac{1-d^2}{1-c^2}\right)\right]^k \int_t^c \left[\frac{(1-s^2)^k}{s^{k+1}} + 2\left(\frac{1-s^2}{s}\right)^{k-1}\right] ds.$$

The integrands equal

$$-\frac{1}{k}\frac{d}{ds}\left(\frac{1-s^2}{s}\right)^k$$

and the result follows.

(ii) for $k=n-1$. Computation shows that f in (i) for $k=n-1$ is given by $f(x)=(a-1)^{-V_1(a)/V_1(b)}(b-1)\,|x|^{V_1(a)/V_1(b)-1}x$. The map is radial and

$$K_I(f) = \left[\frac{V_1(b)}{V_1(a)}\right]^{n-1}.$$

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