Closed similarity manifolds.

Comment. Math. Helvetici 55 (1980) 576–582 Birkhäuser Verlag, Basel

Closed similarity manifolds

DAVID FRIED*

We classify similarity structures on closed manifolds. A *similarity structure* on a manifold M is defined by an atlas of charts with gluing functions that are the restrictions of similarity tranformations (those affine isomorphisms that change distance by a constant factor) to open subsets of Euclidean space. Thus M has a preferred conformal structure. The universal cover $\tilde{M}$ is locally Euclidean since one may choose a preferred inner product at a basepoint and parallel translate it over $\tilde{M}$ without ambiguity. Similarity manifolds are the affine manifolds (i.e. manifolds with a given connection of zero curvature and zero tension) with a "parallel protractor" in the sense that Euclidean manifolds are affine manifolds with a "parallel ruler."

The classification we obtain in Theorem 2 below was given (with one omission) by Kuiper but under the implicit hypothesis that the development map D cover its image [K]. Here D is an affine immersion from $\tilde{M}$ to Euclidean space E^d (these properties determine $D: \tilde{M} \to E^d$ up to composition by a similarity of E^d). For general affine structures, the development map doesn't cover its image [S–Th] but it will follow from our classification that D does cover its image for similarity structures. To show this covering property directly would require essentially all of our demonstration, so it is natural to proceed directly to the classification without using [K].

In addition to extending the classification of [K] to all similarity structures, our methods show that similarity manifolds which aren't Euclidean admit a natural metric μ in the preferred conformal class, unlike Euclidean manifolds where the preferred metric can be renormalized by any scale factor. This situation is somewhat reminiscent of the case of Riemannian manifolds of constant curvature K, where the metric can be normalized so that $|K| = 1$ unless the metric is flat ($K = 0$). The non-Euclidean similarity manifolds possess a natural volume which we compute in Theorem 2. Our classification of non-Euclidean closed similarity manifolds of M of dimension $d \geq 3$ shows they correspond 1-1 to the following

* Partially supported by National Science Foundation Grant #MCS-8003622.

arbitrary data

(1) an isometry of a d-1 dimensional Clifford–Klein space

(2) a positive real value of the natural volume.

Since similarity manifolds arise in the behavior of certain hyperbolic manifolds at infinity [Th] we see that they are closely connected with Riemannian manifolds of constant curvature K for K positive, negative or zero.

We thank Dennis Sullivan and Nico Kuiper for their helpful comments. We gratefully acknowledge the support of I.H.E.S. during the preparation of this paper.

1. Incomplete implies radiant

The uniqueness of D determines similarities $\phi(g)\in \mathrm{Sim}\,(E^d)$, $g\in\pi_1 M$, with the property that $\phi(g)\circ D = D\circ g: \tilde{M}\to E^d$. This *holonomy* homomorphism $\phi:\pi_1 M\to \mathrm{Sim}\,(E^d)$ is crucial to the study of similarity structures. We call the similarity structure *reducible* if $\phi(\pi_1 M)$ fixes some affine subspace of E^d and *radiant* when this subspace is a single point [FGH]. Choosing the fixed point as origin, radiant similarity structures are based on the group of linear similarities $\mathrm{Sim}_0\,(E^d) = \mathrm{Sim}\,(E^d)\cap \mathrm{Gl}\,(d, \mathbf{R})$. Since we later show that (geodesically) complete similarity manifolds are Euclidean, the following theorem shows the structure group of a similarity manifold may always be reduced to either $\mathrm{Sim}_0\,(E^d)$ or the Euclidean group $\mathrm{Euc}\,(E^d)$.

THEOREM 1. *A connected closed similarity manifold which is incomplete must be radiant.*

Proof. Fix a development map $D:\tilde{M}\to E^d$. This defines a Euclidean metric on $\tilde{M}$ and determines scale factors $\alpha(g)\in\mathbf{R}^+$ for $g\in\pi_1 M$ by the rule $\|gv\| = \alpha(g)\,\|v\|$, $v\in T\tilde{M}$. Here $\alpha:\pi_1 M\to\mathbf{R}^+$ is a homomorphism related to the holonomy ϕ by the equation $\alpha(g)^d = |\det\phi(g)|$.

For each $\tilde{m}\in\tilde{M}$ let $D_{\tilde{m}}$ be the largest open disc in $T_{\tilde{m}}\tilde{M}$ on which exp is defined. Let $r(\tilde{m})\in(0,\infty]$ be the radius of $D_{\tilde{m}}$. We see that $r(\tilde{m})\geq r(\tilde{n}) - \mathrm{dist}\,(\tilde{m},\tilde{n})$ (Fig. 1). Thus r satisfies $|r(x)-r(y)|\leq \mathrm{dist}\,(x, y)$.

Since M is incomplete, r is finite at some point and hence finite on all of $\tilde{M}$. Clearly r is continuous and $r(g\tilde{m}) = \alpha(g)r(\tilde{m})$ for all $g\in\pi_1 M$. Thus

$$\frac{\|Tg(y)\|}{r(g\tilde{m})}: T\tilde{M}\to[0,\infty)$$

is independent of g and defines a continuous metric μ on M such that

(1) μ is in the preferred conformal class

(2) for each $m \in M$, the unit disc in $T_m M$ is the largest disc (relative to any preferred inner product at m) on which exp is defined.

For any fixed $m \in M$ there is a $v_0 \in \partial D_m$ such that $\gamma(t) = \exp(tv_0)$ is defined for $0 \le t < 1$ but not for $t = 1$. The μ-speed of the affine ray $\gamma(t)$ is $(\mu(\gamma'(t), \gamma'(t)))^{1/2} = 1/(1-t)$ that is inversely proportional to the distance from $\gamma(t)$ to $\partial(\exp D_{\tilde{m}})$ (Fig. 2). Since $\gamma(t)$ has infinite μ-length it passes by some point $p \in M$ infinitely often and arbitrarily closely. Fixing some orthonormal frame F_m at m there is an orthonormal frame F_p at p and times $t_i \nearrow 1$ such that F_p is the limit of $(1-t_i) \cdot (\gamma|_{[0,t_i]} F_m)$ which denotes the orthonormal frame obtained by parallel translating F_m along $\gamma(t)$ for time t_i and scaling down by $(1-t_i)$. Let g_{ij} denote the element of $\pi_1(M, p)$ approximated by $\gamma \mid [t_i, t_j]$.

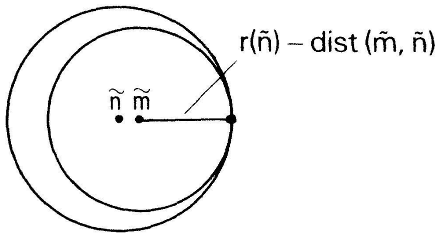

Fig. 1.

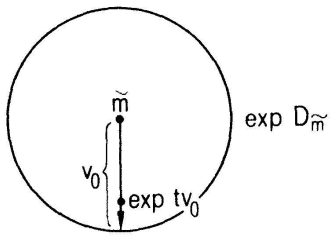

Fig. 2.

Now lift everything to $\tilde{M}$ and fix $\tilde{w} \in T_{\tilde{m}}\tilde{M}$ with inner product $\tilde{\mu}(\tilde{w}, \tilde{v}_0) < 1$. For $i \gg 0$ and $j \gg i$, g_{ij} carries $\tilde{\gamma}(t_i)$ close to $\tilde{\gamma}(t_j)$ in the $\tilde{\mu}$ metric. The holonomy $\phi(g_{ij})$ is a very sharp contraction $(\alpha(g_{ij}) \approx (1-t_j)/(1-t_i))$ with almost no rotation (Fig. 3). Using the local coordinate given by D, we see that (for $j \gg i \gg 0$) $\exp(t \cdot g_{ij}\tilde{w})$ lies within $\exp D_{\tilde{m}}$ for all $t \in [0, 1]$. Thus, applying g_{ij}^{-1}, we see that $\exp \tilde{w}$ is defined.

This shows that the vector v_0 considered above is unique, since all $w \in \bar{D}_m - v_0$ satisfy $\mu(w, v_0) < 1$. So we have a vector field X on M whose value at $m \in M$ is

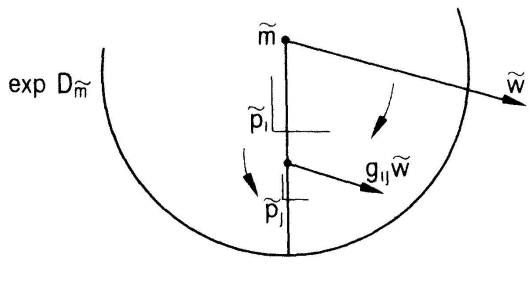

Fig. 3. (Note that $\partial(\exp D_{\tilde{m}})$ is perpendicular to the radius γ)

the shortest vector $v_0 \in T_m M$ for which exp is undefined. We have seen that each $\tilde{m} \in \tilde{M}$ has a natural affine halfspace neighborhood $H_{\tilde{m}} = \{\exp \tilde{w} \mid \tilde{\mu}(\tilde{w}, \tilde{v}_0) < 1\}$.

Let $\tilde{X}$ denote the vector field on $\tilde{M}$ that covers X. We now study the nonempty set $I_{\tilde{m}} = \{\tilde{w} \in T_{\tilde{m}}\tilde{M} \mid \tilde{\mu}(\tilde{w}, \tilde{X}(\tilde{m})) = 1$ and $\exp \tilde{w}$ undefined$\}$ and show it is an affine subspace of $T_{\tilde{m}}\tilde{M}$.

Suppose $\tilde{w} \in J_{\tilde{m}} = \{\tilde{w} \mid \tilde{\mu}(\tilde{w}, \tilde{X}(\tilde{m})) = 1$ and $\exp \tilde{w}$ is defined$\}$. Let $D^* : T\tilde{M} \to E^d$ denote the mapping which is an affine isomorphism on each tangent space and which satisfies $D^*\tilde{v} = D(\exp \tilde{v})$ whenever $\exp \tilde{v}$ is defined.

LEMMA 1. *$D^*(\tilde{X}(\tilde{m})) \in \partial(DH_{\tilde{n}})$, where $\tilde{n} = \exp \tilde{w}$.*

Proof of lemma 1. Choose coordinates so $D^*(\tilde{X}(\tilde{m})) = 0$. Since $H_{\tilde{n}}$ and $H_{\tilde{m}}$ meet (indeed $\tilde{n} \in H_{\tilde{m}}$) and exp is undefined at $\tilde{X}(\tilde{m})$, we cannot have $0 \in DH_{\tilde{n}}$. Assume $0 \notin \partial(DH_{\tilde{n}})$. Choose $j \gg i \gg 0$ so that $\phi(g_{ij})$ carries $DH_{\tilde{n}}$ very close to 0 and rotates $DH_{\tilde{n}}$ very little. Then one sees that $g_{ij}H_{\tilde{n}}$ is a convex region that contains $\exp D_{\tilde{n}}$ and that $\overline{\exp D_{\tilde{n}}}$ is compact. This contradiction proves the lemma (Fig. 4).

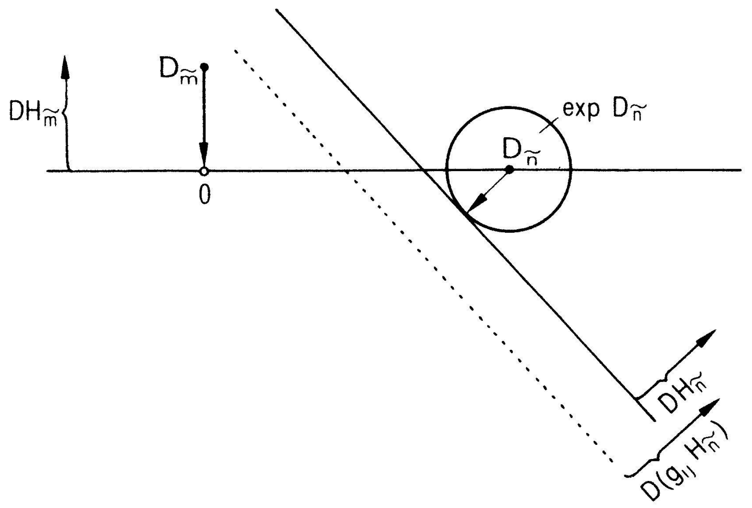

Fig. 4.

Setting $\tilde{m}' = \exp(\tilde{w} - \tilde{X}(\tilde{m}))$ for $\tilde{w} \in I_m$, we see that $H_{\tilde{m}'} = H_{\tilde{m}}$ (because $\tilde{X}(\tilde{m}')$ and $\tilde{X}(\tilde{m})$ are parallel vectors and $\tilde{w} - \tilde{X}(\tilde{m})$ is parallel to $\partial H_{\tilde{m}}$). So Lemma 1 shows that $D(I_{\tilde{m}}) \subset \partial DH_{\tilde{n}}$ for all $\tilde{n} \in \exp J_{\tilde{m}}$. The convexity of $H_{\tilde{n}}$ shows that whenever $\tilde{w} \in I_{\tilde{m}}$ and $\tilde{w}' \in J_{\tilde{m}}$ all the vectors $t\tilde{w} + (1-t)\tilde{w}'$, $0 < t < 1$, belong to $J_{\tilde{m}}$. This clearly implies that $I_{\tilde{m}}$ is an affine subspace of $\{\tilde{\mu}(\tilde{w}, \tilde{X}(\tilde{m})) = 1\}$.

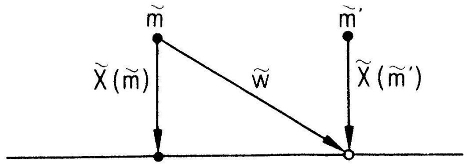

Fig. 5.

Considering the compact set $\{\tilde{w} \in J_{\tilde{m}} \mid \mu(\tilde{w}-\tilde{X}(\tilde{m}), \tilde{w}-\tilde{X}(\tilde{m}))=1$ and $\mu(\tilde{w}-\tilde{X}, I_{\tilde{m}})=0\} = C_{\tilde{m}}$ and the open union $\bigcup_{\tilde{w} \in C_{\tilde{m}}} H_{\exp \tilde{\omega}} \subset \tilde{M}$ we see that $E(I_{\tilde{m}})$ is locally constant. Thus $I = E(I_{\tilde{m}})$ is constant and always outside im D. Since I is intrinsically described, it is clearly invariant under $\phi(\pi_1 M)$.

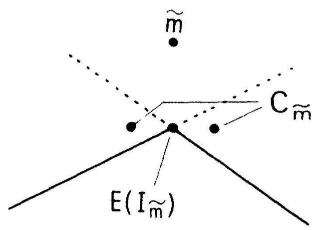

Fig. 6.

We now show I is a point. The vectorfield X on M is seen to correspond to the vector field Y on E^m which assigns to each $x \in E^m$ the vector $Y(x)$ from x to I which is perpendicular to I. It is now easily checked that $\operatorname{div}_\mu X = \dim(I)$. Since $\operatorname{Vol}_\mu(M)$ is finite we must have $\dim I = 0$. Q.E.D.

The preceding argument raises the hope that incompleteness may be a useful property in the study of affine manifolds.

2. The classification

Using Theorem 1, we may classify similarity manifolds in terms of the well-understood Riemannian manifolds of constant nonnegative curvature [W].

THEOREM 2. *A closed connected similarity manifold M is either Euclidean (if it is complete) or radiant (if it isn't complete). All radiant similarity manifolds M are constructed in the following ways.*

(1) $d \leq 2$: *$\tilde{M}$ is $\mathbf{R}^+$ or $\tilde{\mathbf{C}}^*$. Via e^z, the action of $\pi_1 M$ on $\tilde{M}$ by similarities corresponds to a uniform discrete group of rigid motions of $\mathbf{R}$ or $\mathbf{C}$ (translations or glide reflections), in the coordinate z in $\mathbf{R}$ or $\mathbf{C}$. This change of coordinates associates a Euclidean manifold to M in a natural way.*

(2) $d \geq 3$: *M is the quotient of $E^d - 0$ by a group $K \rtimes \mathbf{Z}$, where $K \subset O(d)$ is finite and $\mathbf{Z}$ is generated by an expansion $g \in \operatorname{Sim}_0(E^d)$.*

Topologically, radiant similarity manifolds are precisely the mapping tori of isometries of Clifford–Klein spaces. For $d \geq 3$, this isometry and the volume of M (relative to the natural metric μ introduced in Theorem 1) are natural invariants that classify M, and the volume can be freely chosen in $\mathbf{R}^+$. The value of $\operatorname{Vol}_\mu(M)$ *for $d \geqslant 3$ is*

$$\frac{(\log|\det g|)(\operatorname{vol} S^{d-1})}{d(\operatorname{card} K)}.$$

All similarity manifolds are finitely covered by T^d or $S^{d-1} \times S^1$.

Proof. When M is Euclidean the Hopf–Rinow Theorem shows M is complete. If M is complete then $M = E^d/\pi_1 M$ so $\pi_1 M$ acts without fixed points, the holonomy contains only isometries and M is Euclidean [K]. Theorem 1 shows that incomplete M are radiant. Conversely when M is radiant, the fixed point $e \in E^d$ lies outside the developing image $\operatorname{im}(D)$ (see [FGH, Thm. 3.3] for a correct proof) and so M isn't complete.

When M is radiant, we choose $e \in E^d$ as origin and find that the holonomy preserves the complete metric $ds/\|x\|$ on $E^d - 0$. This induces a metric on M which is clearly just the metric μ constructed in Theorem 1.

It's known [e.g. Th, Thm. 3.6] that when G is a group of analytic isometries of a complete simply connected Riemannian manifold X and M is a closed (G, X) manifold the development map $D: \tilde{M} \to X$ is a homeomorphism. Let G be $\widetilde{\mathrm{Sim}_0(E^d)}$. Choosing X appropriately, we find that $\tilde{M}$ is $\mathbf{R}^+$, $\tilde{\mathbf{C}}^*$ or $E^d - 0$ $(d \geq 3)$.

It remains to analyze the action of $\pi_1 M$ on $\tilde{M}$. The case $d \leq 2$ is easy, since the only fixed point free rigid motions of $\mathbf{R}$ and $\mathbf{R}^2$ are translations and glide reflections. For $d \geq 3$, the subgroup $\alpha(\pi_1 M) \subset \mathbf{R}^+$ must be discrete for $\pi_1 M$ to act discontinuously (since $O(d) = \widetilde{O(d)}$ is compact, $\alpha(g)$ near 1 implies $\phi(g^n)$ near 1 for some $n > 0$). As $\pi_1 M$ acts uniformly, $\alpha(\pi_1 M)$ is infinite cyclic. The kernel $K = \phi(\pi_1 M) \cap O(d)$ acts properly discontinuously on S^{d-1} and hence is finite. The map $\log\|x\|: E^d - 0 \to \mathbf{R}$ induces a fibration $f: M \to S^1$ with fiber the Clifford–Klein space S^{d-1}/K. Transverse to f is the radiant vectorfield $\tilde{X}$ considered in Theorem 1, given in coordinates by $\tilde{X}(\tilde{m}) = -\tilde{m}$. The flow ϕ generated by X permutes the fibers of f and has return map corresponding to the isometry $g/(\alpha(g))$ of S^{d-1}/K.

In dimensions $d \leq 2$, the only mapping tori of Clifford–Klein spaces are S^1, T^2 and the Klein bottle. These admit Euclidean metrics and e^z gives rise to radiant similarity structures on these manifolds. For $d \geq 3$, an isometry $h: S^{d-1}/K \hookleftarrow$ of a Clifford–Klein space gives rise to a one-parameter family of radiant similarity structures on the mapping torus M_h as follows. Lift h to $h' \in O(d)$ and let G be the subgroup of $\mathrm{Sim}_0(E^d)$ generated by K and $g = \alpha h'$, $\alpha > 1$. Then G acts properly discontinuously on $E^d - 0$ and quotient is the desired similarity manifold.

The volumes are easily computed by passing to a finite cover to reduce to $K = \{1\}$ and considering the fundamental domain $\{1 \leq \|x\| \leq \alpha(g)\}$. It follows that the parameter $\alpha > 1$ may be replaced by $\mathrm{vol}_\mu(M) > 0$.

Since Bieberbach proved that flat manifolds M are covered by T^d [W], we have shown all portions of Theorem 2. Q.E.D.

We note that the existence of a finite cover with abelian fundamental group allows one to apply [S] and [FGH] to affine structures on M. For instance, when $d \geq 3$ and M is an incomplete closed connected similarity manifold, every *affine* structure on M is radiant. Also, perturbations of similarity structures within the class of all affine structures are well-understood.

REFERENCES

[FGH] FRIED, DAVID, GOLDMAN, WILLIAM and HIRSCH, MORRIS. *Affine Manifolds with Nilpotent Holonomy*, to appear.

[K] KUIPER, N. *Compact Spaces with a Local Structure Determined by the Group of Similarity Transformations in E^m*, Indag. Math. Vol. XII, Fasc. 4, 1950.

[S] SMILLIE, JOHN. *Affinely Flat Manifolds*. Ph.D. Thesis, Univ. of Chicago, 1977.

[S-Th] SULLIVAN, DENNIS and THURSTON, W. *Manifolds with Canonical Coordinate Charts: Some Examples*. I.H.E.S. preprint 1979.

[Th] THURSTON, W. Princeton University notes, 1977.

[W] WOLF, J. *Spaces of Constant Curvature*, McGraw-Hill, 1967.

University of California at Santa Cruz

Received January 10/July 1 1980