Minimal models for non-nilpotent spaces.

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Introduction

Minimal models in the sense of Quillen [12] and Sullivan [15] have become a standard method to study problems in rational homotopy theory, provided the spaces involved are simply connected or nilpotent (see, e.g. [1], [3], [11], amongst others). Recall that a space X of the homotopy type of a connected CW-complex is nilpotent if $\pi_1 X$ acts nilpotently on $\pi_i X$ for all $i \geq 1$.

The purpose of this note is to adapt this minimal model description to more general spaces of the homotopy type of a CW-complex, and to apply it to basic problems which are specific to the non-nilpotent situation.

In more detail, for a given CW-complex X we consider regular nilpotent covers $\tilde{X}_N$ together with the covering transformations, which determine an action of the group $G = \pi_1 X/N$ on the minimal model of $\tilde{X}_N$. Then we show that the model of $\tilde{X}_N$, together with this G-action, describes the "rational homotopy type of X" in a well defined sense, provided that the rational cohomology dimension of G is ≤ 1 (Theorem 2.3). This is the case, e.g. if G is either a finite or a free group. The proof of our result is based on a paper by G. Cooke [4]. As an immediate application we get a classification of homotopy types of CW-complexes having a given rational universal cover and a given finite or free fundamental group (Corollary 2.7). Moreover we construct minimal models of cell complexes whose (rational) attaching maps and two-skeleton are given (Proposition 3.1). This can be applied in particular to homology circles (e.g. higher knot complements) with infinite cyclic fundamental group. Our main application is concerned however with the explicit construction of a minimal model for the $\mathbf{Q}$-acyclic functor $A^{\mathbf{Q}}X$ ([5], [2]), where X is a CW-complex with $H_*(X, \mathbf{Q})$ of finite $\mathbf{Q}$-type and with finite fundamental group (Theorem 4.5). Recall that $A^{\mathbf{Q}}X$ is a $\mathbf{Q}$-acyclic space (i.e. $\tilde{H}_*(A^{\mathbf{Q}}X, \mathbf{Q}) = 0$) which is the homotopy fibre of the "rational plus construction" $\Phi : X \to X^+_{\mathbf{Q}}$ (cf. [2], [9]). Moreover Φ agrees with the rationalization of X in the sense of [2] as $\pi_1 X$ is finite. Thus given the rational homotopy type of X we can read off the rational homotopy groups of $A^{\mathbf{Q}}X$ from our model, and hence we get explicit information on $\pi_i X^+_{\mathbf{Q}}$. This in turn leads to information on the rational homotopy groups of the plus construction X^+ of Quillen [13]. The groups $\pi_i X^+$ are of geometric interest for any CW-complex X, as they may be interpreted as certain bordism groups [7]. Moreover a knowledge of $\pi_i X^+$ is useful for the classification of acyclic maps [9].

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§1. Homotopy actions on spaces and algebras

In this introductory section we recall some definitions and basic facts, and we define homotopy actions on graded differential algebras as well as minimal models of such actions.

(1.1) A homotopy action of a group G on a space $X \in \Pi\mathrm{CW}$, the homotopy category of connected CW-complexes, is a homomorphism $\alpha : G \to \mathscr{E}(X)$ from the group G to the group $\mathscr{E}(X)$ of homotopy classes of (free) homotopy equivalences of X. A homotopy action α of G on X is equivalent to a homotopy action β of G on Y if there exists a homotopy equivalence $f : X \to Y$ such that the diagram

$$\begin{array}{ccc} & \overset{\alpha}{\nearrow} & \mathscr{E}(X) \\ G & & \big\downarrow \scriptstyle{\mathscr{E}(f)} \\ & \underset{\beta}{\searrow} & \mathscr{E}(Y) \end{array}$$

is commutative.

Here $\mathscr{E}(f)$ is defined by $[g] \mapsto [fgf^{-1}]$, where $[g] \in \mathscr{E}(X)$ and f^{-1} is any homotopy inverse of f (cf. [4]).

(1.2) To define similar homotopy actions on algebras we first introduce some notation. For a treatment of graded differential algebras and minimal models in the sense of Quillen [12] and Sullivan [15] we refer, e.g. to [3], [1], [11], [8].

DGA is the category of differential graded, augmented, commutative, associative algebras defined over the rationals $\mathbf{Q}$ which are concentrated in non-negative degrees and have a cohomology differential (of degree $+1$). We restrict DGA to algebras A which are homologically connected, $H^0(A) = \mathbf{Q}$, and we just call the objects in this category "algebras."

DGLA is the category of differential graded Lie algebras defined over $\mathbf{Q}$ which are concentrated in non-negative degrees and have a homology differential. We just call objects in this category "Lie algebras."

For the appropriate homotopy categories HDGA, HDGLA of algebras or Lie algebras respectively, as well as for the pointed versions of each, we refer to [3], [11], [1].

(1.3) DEFINITION. A homotopy action of a (discrete) group G on an algebra $A \in \mathrm{DGA}$ is a homomorphism $\alpha: G \to \mathscr{E}(A)$ from G to the group $\mathscr{E}(A)$ of homotopy classes of automorphisms of A.

In the same way we define homotopy actions on Lie algebras. As in the following completely similar definitions and results hold for Lie algebras, we omit the corresponding statements.

(1.4) DEFINITION. A homotopy action α of G on $A \in \mathrm{DGA}$ is *equivalent* to a homotopy action β of G on $B \in \mathrm{DGA}$ if there exists a weak equivalence (=cohomology isomorphism) $f: A \to B \in \mathrm{DGA}$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha(g)} & A \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{\beta(g)} & B \end{array}$$

for every element $g \in G$ commutes up to homotopy.

(1.5) *Remark.* If one restricts to *cofibrant* algebras [3], every weak equivalence $f: A \to B \in \mathrm{DGA}$ has a homotopy inverse ([3], 6.5). Hence for cofibrant algebras the definition of equivalence of homotopy actions on algebras takes a form analogous to that on spaces. This in particular applies to *minimal* algebras ([3], 7.5).

(1.6) DEFINITION. A homotopy action α of G on $A \in \mathrm{DGA}$ is *minimal* if A is a minimal algebra.

(1.7) PROPOSITION. *Every homotopy action α of G on $A \in \mathrm{DGA}$ has a minimal model.*

Proof. Choose a minimal model $e: M \to A \in \mathrm{DGA}$, then we define a homotopy action β of G on M which is equivalent to α: For an automorphism $\alpha(g)$: $A \to A$, $g \in G$, there exists a map $M(\alpha(g)): M \to M \in \mathrm{DGA}$ (which is unique up to homotopy) such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha(g)} & A \\ \uparrow{\scriptstyle e} & & \uparrow{\scriptstyle e} \\ M & \xrightarrow{M(\alpha(g))} & M \end{array}$$

commutes up to homotopy ([3], 8.7). As the map $M(\alpha(g))$ is a weak equivalence, and M is minimal, it is an automorphism of M.

(1.8) For our main result we shall need a relation between homotopy actions on spaces in ΠCW and such actions on algebras in DGA. This is established by certain functors in ([3], §8). First recall that there are (adjoint) functors $F: \mathrm{HDGA} \to \mathrm{H}\mathcal{S}$ and $M: \mathrm{H}\mathcal{S} \to \mathrm{HDGA}$, where $\mathrm{H}\mathcal{S}$ denotes the homotopy category of connected simplicial sets. By taking the geometric realization we can replace this category by ΠCW. Note that these functors have also pointed versions. Let now $\alpha: G \to \mathscr{E}(X)$ denote a homotopy action where $X \in \Pi\mathrm{CW}$. Then applying the functor M we get a homotopy action $\beta: G \to \mathscr{E}(M_X)$, where M_X denotes a minimal model of X. Vice versa, a homotopy action β of a group G on an algebra $A \in \mathrm{DGA}$ gives rise to a homotopy action α of G on the space $F(A) \in \mathcal{S}$ (or in ΠCW).

§2. Differential algebras and non-nilpotent spaces

In this section to any space $X \in \Pi\mathrm{CW}$ we associate minimal models in terms of homotopy actions on suitable algebras. In the main result (Theorem 2.3) we give a criterium on the fundamental group of X, which allows a description of the "rational homotopy type" of X via minimal models. This generalizes the minimal model approach of Quillen and Sullivan given for the homotopy theory of rational nilpotent spaces. As an immediate application we get a classification of spaces with a prescribed rational universal cover and suitably given fundamental group.

Let $X \in \Pi\mathrm{CW}$ be given. Then to X we associate minimal models as follows: Consider fibrations (up to homotopy)

$$\tilde{X}_N \to X \to K(\pi_1 X/N, 1) \tag{2.1}$$

where $\tilde{X}_N$ is a nilpotent (e.g. a simply connected) covering space with respect to the nilpotent normal subgroup N of $\pi_1 X$. The covering determines a free topological action (and hence a homotopy action) α of $G = \pi_1 X/N$ on $\tilde{X}_N$. By taking a minimal model β of this action we thus have associated a minimal model to X (depending on the group N). By fibre-wise rationalization of (2.1) we get a fibration (up to homotopy)

$$\tilde{X}_{N(0)} \to \bar{X} \to K(G, 1). \tag{2.2}$$

Now suppose that $\tilde{X}_{N(0)}$ is of finite **Q**-type, i.e. the **Q**-vector spaces $H_n(X; \mathbf{Q})$, $n \geq 1$, are finite dimensional. Then the Sullivan or Quillen model of this space allows to reconstruct the homotopy type of $\tilde{X}_{N(0)}$ in a precise sense: In this case the functors F and M restrict to adjoint equivalences (cf. [3] for minimal algebras and [1], [11] for minimal Lie algebras). We now ask for conditions which allow to reconstruct, in a well defined sense, the fibre homotopy type of (2.2) (and hence the homotopy type of $\bar{X}$) from the minimal model $\beta: G \to \mathscr{E}(M_{\tilde{X}_N})$ of X.

Let $\bar{\beta}: G \to \mathscr{E}(\|F(M_Y)\|)$ be a realization of this action, where $Y = \tilde{X}_N$ and $\|F(M_Y)\|$ has the same homotopy type as $Y_{(0)}$. Then to obtain $Y_{(0)}$ as a covering space of $\bar{X}$ out of the action $\bar{\beta}$ we have first to turn $\bar{\beta}$ into an equivalent *free topological* action $\gamma: G \to \mathscr{E}(\bar{Y}_{(0)})$, where $\bar{Y}_{(0)} \simeq Y_{(0)}$. If this can be done in a unique way (up to equivalence of homotopy actions) we say "$\beta: G \to \mathscr{E}(M_{\tilde{X}_N})$ determines the rational homotopy type of X." Our main result now gives a sufficient condition when this is the case.

(2.3) THEOREM. *Let X be a space in* ΠCW and let $\tilde{X}_N$ be a regular nilpotent *cover with respect to the nilpotent group N. Suppose that $\tilde{X}_N$ is of finite* **Q**-*type and that $G = \pi_1 X/N$ has rational cohomology dimension ≤ 1. Then the associated minimal model $\beta: G \to \mathscr{E}(M_{\tilde{X}_N})$ determines the rational homotopy type of X.*

(2.4) *Remark.* There is a similar result valid for $M_{\tilde{X}_N}$ replaced by a Quillen model of $\tilde{X}_N$. If $\tilde{X}_N$ is simply connected no finiteness condition on $\tilde{X}_N$ is necessary [12].

Proof of Theorem 2.3. Let again denote the realization of β by $\bar{\beta}: G \to \mathscr{E}(\|F(M_Y)\|)$ where $Y \simeq \tilde{X}_N$. Then to $\bar{\beta}$ there can be associated a lifting problem [4]

$$\begin{array}{ccc} & & B_{G(Y_{(0)})} \\ & \nearrow & \downarrow B_\pi \\ K(G,1) & \xrightarrow{B_{\bar{\beta}}} & K(\mathscr{E}(Y_{(0)}),1) \end{array} \qquad (2.5)$$

Here $G(Y)$ denotes the space of free self-homotopy equivalences of the space Y. It is an associative H-space with $\pi_0 G(Y) = \mathscr{E}(Y)$. The identity component is denoted by $G_1(Y)$ and there is an exact sequence of H-spaces

$$G_1(Y) \longrightarrow G(Y) \xrightarrow{\pi} \pi_0 G(Y).$$

Since $\pi_1 B_{G(Y)} \to \pi_1 K(\mathscr{E}(Y), 1)$ is an isomorphism, there is a lifting over the two-skeleton of $K(G, 1)$ which is unique on the one-skeleton. Then it follows from the methods of [4] that the action $\bar{\beta}$ can be turned into a unique topological action equivalent to $\bar{\beta}$ if and only if the lifting problem (2.5) has exactly one solution (up to homotopy). This is the case in particular if $H^n(G, \{\pi_{n-2}(G_1(Y_{(0)}))\}) = 0 = H^{n-1}(G, \{\pi_{n-2}(G_1(Y_{(0)}))\})$ for all $n \geq 3$. This in turn is satisfied if G has rational cohomology dimension ≤ 1. Note that one can replace any topological action by an equivalent *free* topological action by taking a product with a contractible free G-space W: In the notation as before set $\bar{Y}_{(0)} = Y'_{(0)} \times W$, where $\beta' : G \to \mathscr{E}(Y'_{(0)})$ is a topological action equivalent to $\bar{\beta}$, and let G act on $\bar{Y}_{(0)}$ via the diagonal action. Hence under the conditions of Theorem 2.3 there is a well defined quotient space $\bar{Y}_{(0)}/G \simeq \bar{X}$ associated to the homotopy action β.

(2.6) *Remark.* In [15] Sullivan has given an algorithm which computes the groups $\pi_i G_1(Y_{(0)})$ for a rational nilpotent space $Y_{(0)}$ (with suitable finiteness assumptions).

(2.7) COROLLARY. *Let $X \in \Pi\mathrm{CW}$ be a simply connected rational space, and let G be either a finite or a free group. Then there is a one-to-one correspondence between spaces $Y \in \Pi\mathrm{CW}$ having a universal cover equivalent to X and with fundamental group G, and the set of equivalence classes of homotopy actions $\alpha : G \to \mathscr{E}(L_X)$, where L denotes the Quillen minimal model of X.*

§3. A minimal model for cell complexes

In this section we indicate a construction for a Lie algebra model of any cell complex whose two-skeleton and all higher (rational) attaching maps are explicitly known. As an immediate application we get an algorithm for the computation of rational homotopy groups of such spaces. This even works if the Lie algebra model thus constructed doesn't completely determine the rational homotopy type of X (in the sense of Theorem 2.3).

Let X be a space in $\Pi\mathrm{CW}$. Then we first consider the universal covering $\tilde{X} \to X$ of X, where $X = \tilde{X}/\pi$ and $\pi = \pi_1(X)$. If $f : A \to X$ is a map of a simply connected space A into X, a universal covering space for $X \cup_f CA$ can be constructed as follows: One maps $\pi \times A$ into X by choosing a lifting $\tilde{f} : A \to \tilde{X}$ and by sending (g, a) into $\tilde{f}_g(a) = g \cdot \tilde{f}(a)$, $g \in \pi$. Then the mapping cone of the map $\pi \times A \to \tilde{X}$ thus defined is a universal covering space of $X \cup_f CA$. Note that the map $\tilde{f}_g$ is not base-point preserving, but is (freely) equivalent to such a map if A and X are based. For an inductive construction of a Lie algebra model of X we start with $C_0^{\mathbf{Q}} X^2$, where X^n denotes the n-skeleton of X and $C_0^{\mathbf{Q}} X$ denotes the fibre-wise rationalization of the fibration (up to homotopy) $\tilde{X} \to X \to K(G, 1)$. We suppose that $\pi_2(X) \otimes \mathbf{Q}$ as a π-module is known. (This is the case, e.g. if either π

is a cyclic or a finitely generated free group, cf. [6].) Then the inductive step is given in the following

(3.1) PROPOSITION. *Let $\tilde{f}: S^{n+1} \to \tilde{X}$, $n \geq 1$, be a lifting of the attaching map f of an $(n+2)$-cell to X. If $\alpha: \pi \to \mathscr{E}(L_{\tilde{X}})$ is a minimal model of X, the minimal model for $X \cup_f e^{n+2}$ is quasi-isomorphic to $\beta: \pi \to \mathscr{E}(L)$, where β is defined as follows: $L = L_{\tilde{X}} \amalg \amalg_{g \in \pi} \mathbf{L}(y_{n+1})_g$ ($\mathbf{L}(y_{n+1})$ denotes the free Lie algebra on the generator y_{n+1}), and the differential d of L is given by $d|_{L_{\tilde{X}}} = \tilde{d}$, the differential in $\tilde{L}_{\tilde{X}}$, and $d(y_{n+1})_g = g \cdot x_n$, where $g \cdot x_n \in H_n(L_{\tilde{X}}) \cong \pi_{n+1}(\tilde{X}) \otimes \mathbf{Q}$ is represented by $\tilde{f}_g: S^{n+1} \to \tilde{X}$. Moreover the action of π on L is given by the action on generators.*

Proof. If π is finite this is a consequence of ([11, Proposition 8.11, or [1]) and the construction of the universal cover of a mapping cone as given above. If π is infinite one restricts the map $\pi \times S^{n+1} \to \tilde{X}$ to finite subcomplexes of $\pi \times S^{n+1}$ and applies ([11], Proposition 8.11) again. Since homology and homotopy commute with direct limits we get the result.

(3.2) EXAMPLES. (1) If X is a pseudo projective plane, i.e. the mapping cone of a map $f: S^1 \to S^1$ of degree q, $q \geq 2$, then the action of $\pi_1 X = \mathbf{Z}/q$ on $\tilde{X} \cong \bigvee_{q-1} S^2$ is explicitly known. The pseudo projective planes are basic examples of rationally acyclic spaces (i.e. $\tilde{H}_*(X; \mathbf{Q}) = 0$). If $p = 2$, i.e. if X is the real projective plane, this space is a two-stage in the sense of [5], [9]. For $p = 3$ the universal cover of X can be described as the space obtained from 3 closed discs by identifying boundaries. A covering transformation will be given by a cyclic permutation of the discs followed by a rotation by $2\pi/3$. Therefore the Lie algebra model of X is given by $\alpha: \mathbf{Z}/3 \to \mathscr{E}(L_{\tilde{X}})$, where $L_{\tilde{X}}$ is the free Lie algebra on generators a, b of degree 1, and the action is given by $\alpha(a) = b$, $\alpha(b) = -a - b$. As an interesting application of this description of the rational homotopy type of X one can show that the action of $\pi_1 X$ on $\pi_i(X) \otimes \mathbf{Q}$ is non-trivial for infinitely many dimensions i. This means that the $\mathbf{Q}$-acyclic decomposition ([5], [9]) of X is infinite.

(2) If X is a compact homology circle with $\pi_1 X \cong \mathbf{Z}$ we can use (3.1) and (1.7) to get a minimal model, which determines the rational homotopy type of X, according to Theorem 2.3. Although $\tilde{X}$ is not compact (as $\pi_1 X$ is infinite) the groups $\pi_1 X \otimes \mathbf{Q}$, $i \geq 2$, are always finite dimensional $\mathbf{Q}$-vector spaces. This follows from the classical fact that $H_i(\tilde{X}; \mathbf{Q})$ is a finite dimensional $\mathbf{Q}$-vector space (see Theorem 2.2 of [16]).

As a well known explicit example consider the mapping cone X of $\theta: S^2 \to S^2 \vee S^1$, given by $2a^g - a$, where $g \in \pi_1 S^1$ and $a \in \pi_2 S^2$ are generators. Then one can show that the minimal model of X is $\alpha: \mathbf{Z} \to \mathscr{E}(L_{S^2_{(0)}})$, where the action is given by $\alpha(g)(1) = 2^{-g}$, $g \in \mathbf{Z}$.

§4. The Q-acyclic functor

In [5] Dror has defined an acyclic functor $A:\Pi\mathrm{CW}\to\Pi\mathrm{CW}$ and for any $X\in\Pi\mathrm{CW}$ a natural map $a:AX\to X$ such that

(i) AX is an acyclic space, i.e. $\tilde{H}_*(AX;\mathbf{Z})=0$, and

(ii) The map a is, up to homotopy, universal for maps of acyclic spaces into X.

This notion as well as the step by step construction given in [5] may be relativized [2] to an R-acyclic functor A^R for a subring R of the rationals. If one concentrates on the case where $R=\mathbf{Q}$ and X is a space in $\Pi\mathrm{CW}$ with finite fundamental group and with $H^i(X;\mathbf{Q})$ a finite dimensional $\mathbf{Q}$-vector space for all i, one has a well known relation between $H^*(\tilde{X};\mathbf{Q})$ and $H^*(X;\mathbf{Q})$:

$$H^*(X;\mathbf{Q})\cong H^*(\tilde{X};\mathbf{Q})^G, \qquad G=\pi_1X,$$

where $H^*(\tilde{X};\mathbf{Q})^G$ is the invariant subalgebra of $H^*(\tilde{X};\mathbf{Q})$. This enables us to imitate the construction of $A^{\mathbf{Q}}X$ by a step by step construction of a minimal model for $A^{\mathbf{Q}}X$, which determines its rational homotopy type according to Theorem 2.3. In this way we get an effective tool to systematically compute the higher homotopy groups of $A^{\mathbf{Q}}X$ if π_1X is finite.

(4.1) For our construction of a model for $A^{\mathbf{Q}}X$ we first recall the construction of $A^{\mathbf{Q}}X$ itself (cf. [5] in the case $R=\mathbf{Z}$): $A^{\mathbf{Q}}X$ may be defined as $A^{\mathbf{Q}}X=\lim A^{\mathbf{Q}}_nX$, where

$$\cdots\to A^{\mathbf{Q}}_nX\to A^{\mathbf{Q}}_{n-1}X\to\cdots\to A^{\mathbf{Q}}_0X=X$$

is a tower of fibrations as follows:

(i) As π_1X is finite it is $\mathbf{Q}$-perfect and we put $A^{\mathbf{Q}}_1X=X$

(ii) $A^{\mathbf{Q}}_nX$ $(n>1)$ is defined as the fibre square

$$\begin{array}{ccc} A^{\mathbf{Q}}_nX & \dashrightarrow & \Lambda \\ \Big\downarrow & & \Big\downarrow \\ A^{\mathbf{Q}}_{n-1}X & \xrightarrow{\ p\ } & K(H_n(A^{\mathbf{Q}}_{n-1}X;\mathbf{Q}),n) \end{array} \tag{4.2}$$

Here the vertical map on the right is the universal path fibration and $p\in H^n(A^{\mathbf{Q}}_{n-1}X;H_n(A^{\mathbf{Q}}_{n-1}X;\mathbf{Q}))\cong\mathrm{Hom}\,(H_n(A^{\mathbf{Q}}_{n-1}X;\mathbf{Q}),H_n(A^{\mathbf{Q}}_{n-1}X;\mathbf{Q}))$ corresponds to the identity in the latter group. It is immediate from the construction that the fibre of $A^{\mathbf{Q}}_nX\to A^{\mathbf{Q}}_{n-1}X$ is $(n-2)$-connected. Hence

$$\pi_iA^{\mathbf{Q}}X\otimes\mathbf{Q}\cong\pi_iA^{\mathbf{Q}}_nX\otimes\mathbf{Q} \quad \text{if} \quad i\leq n-2.$$

Let now $\alpha: G \to \mathscr{E}(M_{\tilde{X}})$ be a minimal model of X, where $G = \pi_1 X$ is a finite group and $\tilde{X}$, the universal cover of X, is of finite $\mathbf{Q}$-type. Then according to (4.2) we construct a minimal model $\beta: G \to \mathscr{E}(M)$ for $A^{\mathbf{Q}}X$ as follows: Put $\beta_1 = \alpha: G \to \mathscr{E}(M_{\tilde{X}})$. For $n > 1$ $\beta_n: G \to \mathscr{E}(M_{Y_n})$, where Y_n denotes $\widetilde{A_n^{\mathbf{Q}}X}$, is obtained as a pushout

$$\begin{array}{ccc} L_{n-1}(W) & = & L_{n-1}(W) \\ \uparrow & & \uparrow \\ M_{Y_n} & \longleftarrow & L_n(W) \otimes_t L_{n-1}(W) \\ \uparrow & & \uparrow \\ M_{Y_{n-1}} & \xleftarrow{\ \tau\ } & L_n(W) \end{array} \tag{4.3}$$

Here $W = H_n(A_{n-1}^{\mathbf{Q}}X; \mathbf{Q}) \cong H^n(A_{n-1}^{\mathbf{Q}}X; \mathbf{Q}) \cong H^n(\widetilde{A_{n-1}^{\mathbf{Q}}X}; \mathbf{Q})^G$. Moreover τ corresponds to p in diagram (4.2) under the isomorphism

$$[L_n(W), M_Y] \cong H^n(Y; \mathbf{Q}) \otimes W^* \qquad \text{(cf. [8], p. 20)} \tag{4.4}$$

for a simply connected space Y of finite $\mathbf{Q}$-type. ($L_n(W)$ denotes the symmetric or exterior algebra on W as to whether n is even or odd, and $L_n(W) \otimes_i L_{n-1}(W)$ is an acyclic algebra.) Hence we obtain

THEOREM 4.5. *Let X in* ΠCW *be a space with finite fundamental group and with universal cover of finite* $\mathbf{Q}$*-type. Then*

(a) *the* $\mathbf{Q}$*-acyclic space* $A^{\mathbf{Q}}X$ *has a minimal model* $\alpha: G \to \mathscr{E}(M_Y)$, $Y = \widetilde{A^{\mathbf{Q}}X}$, *which determines the rational homotopy type of* $A^{\mathbf{Q}}X$, *and*

(b) α *may inductively be constructed as follows:*

(i) $\alpha_1 = \beta: G \to \mathscr{E}(M_{\tilde{X}})$, *where* β *is a minimal model of* X

(ii) *for* $n > 1$ $\alpha_n: G \to \mathscr{E}(M_{Y_n})$, $Y_n = \widetilde{A_n^{\mathbf{Q}}X}$, *is isomorphic to* $G \to \mathscr{E}(\bar{M})$, *where* $\bar{M} = M_{Y_{n-1}} \otimes_\tau L_{n-1}(W)$, $W \cong H^n(Y_{n-1}; \mathbf{Q})^G$, *and the differential* $\bar{d}$ *in* $\bar{M}$ *is given by* $\bar{d}(a \otimes 1) = d_M a \otimes 1$ *and* $\bar{d}(1 \otimes w) = \tau w \otimes 1$, $a \in M_{Y_{n-1}}$. *Here* τ *corresponds to* p *in* (4.2) *via* (4.4). *Moreover the* G*-action on* $\bar{M}$ *is induced from the action on* $M_{\tilde{X}}$ *and is trivial on new generators.*

(4.6) *Remark.* If $\alpha_n: G \to \mathscr{E}(\bar{M})$ is not already minimal we have to replace it by a minimal model (cf. [8], §8), so as to be able to read off the homotopy groups of $A_n^{\mathbf{Q}}X$ as the generators in the model.

§5. Explicit computations and examples

As we may see from Theorem 4.5, an effective computation of the homotopy groups of $A^{\mathbf{Q}}X$ depends on the possibility of an effective computation of

$W = H^n(\widetilde{A^{\mathbf{Q}}_{n-1}X}; \mathbf{Q})^G$, the invariant subalgebra of the cohomology algebra of $\widetilde{A^{\mathbf{Q}}_{n-1}X}$. If X is such that $\pi_1 X = G$ acts trivially on $\pi_i X \otimes \mathbf{Q}$ for all $i \geq 2$, the space $A^{\mathbf{Q}}X$ is rationally trivial. So the first interesting case arises if we have a non-trivial (=non-nilpotent) action of the finite group G on $\pi_n X \otimes \mathbf{Q}$ for exactly one dimension $n \geq 2$. Then the computation of $\pi_i A^{\mathbf{Q}} X \otimes \mathbf{Q}$ reduces to that of $\pi_i A^{\mathbf{Q}} Y \otimes \mathbf{Q}$, where Y is a two-stage Postnikov system

$$K(Q, n) \rightarrow Y \rightarrow K(G, 1) \tag{5.1}$$

with Q a finite dimensional $\mathbf{Q}$-vector space, and $n \geq 2$ (cf. [9], [10]). If n is even we have $H^*(Y; \mathbf{Q}) \cong H^*(K(Q, n); \mathbf{Q})^G \cong S(Q)^G$. According to Theorem 4.5 this leads to the classical problem of computing invariant elements generating the subalgebra $S(Q)^G$ of the symmetric algebra $S(Q)$ (see [14] for a recent exposition of this beautiful topic). Assuming Y is of the form (5.1) with n even one can express $\pi_i A^{\mathbf{Q}} X \otimes \mathbf{Q}$ in a range in terms of the cohomology ring $H^*(Y; \mathbf{Q}) \cong S(Q)^G$. The fibration (5.1) is determined (up to fibre homotopy type) by the G-action on Q, i.e. by a rational representation $\omega: G \rightarrow \mathrm{Aut}(Q)$, and without restriction of generality we can assume that ω has no invariant elements except 0 (cf. [10]). Moreover we restrict to the case $n = 2$ as the general case of even numbers n is similar. Then we have

(5.2) PROPOSITION. *Let $K(Q, 2) \rightarrow X \rightarrow K(G, 1)$ be a fibration determined by a finite dimensional rational representation $\omega: G \rightarrow \mathrm{Aut}(Q)$ of the finite group G. Suppose furthermore that ω has no invariant elements except* 0. *Then*

(a) $\pi_2 A^{\mathbf{Q}} X = Q$ *and* $\pi_i A^{\mathbf{Q}} X = \pi_{i+1} X^+_{\mathbf{Q}}$ *for* $i > 2$

(b) $\pi_4 X^+_{\mathbf{Q}} = H^4(K(Q, 2), \mathbf{Q})^G$, $\pi_5 X^+_{\mathbf{Q}} = 0$, $\pi_6 X^+_{\mathbf{Q}} = H^6(K(Q, 2), \mathbf{Q})^G$, $\pi_7 X^+_{\mathbf{Q}} \cong \mathrm{coker}\, \delta_8$, $\pi_8 X^+_{\mathbf{Q}} \cong \ker \delta_8$, *and* $\pi_9 X^+_{\mathbf{Q}} \cong \mathrm{coker}\, \delta_{10}$.

Here $\delta_i: \tilde{H}_i(X; \mathbf{Q}) \rightarrow (\tilde{H}_*(X; \mathbf{Q}) \otimes \tilde{H}_*(X; \mathbf{Q}))_i$, $i \leq 10$, *is the geometric diagonal.*

(5.3) *Remarks.* (i) This result could directly be obtained by (tedious) computations based on Theorem 4.5.

(ii) In principle there is a similar expression of $\pi_{10} X^+_{\mathbf{Q}}$ depending only on the cohomology ring $H^*(X; \mathbf{Q})$. For higher dimensions however one can not expect a general description of $\pi_i X^+_{\mathbf{Q}}$ entirely in terms of $H^*(X; \mathbf{Q})$.

Proof. Statement (a) is immediate whereas (b) follows from the EHP-sequence (3.8) in [1]

$$\cdots \longrightarrow \tilde{H}_{i+1}(X^+_{\mathbf{Q}}; \mathbf{Q}) \xrightarrow{\delta_{i+1}} \Lambda(i-1) \longrightarrow \pi_i X^+_{\mathbf{Q}} \otimes \mathbf{Q} \longrightarrow \tilde{H}_i(X^+_{\mathbf{Q}}; \mathbf{Q}) \longrightarrow \cdots$$

Here $\Lambda(i) \cong (\tilde{H}_*(X^+_{\mathbf{Q}};\mathbf{Q})\otimes\tilde{H}_*(X^+_{\mathbf{Q}};\mathbf{Q}))_{i+2}$ and the sequence is valid for $i \leq 9$ as $X^+_{\mathbf{Q}}$ is 3-connected.

As an illustration of our description of $A^{\mathbf{Q}}X$ by a minimal model we explicitly determine it in the following simple example:

(5.4) EXAMPLE. Let $\omega:\mathbf{Z}/2 \to \mathrm{Aut}(Q)$, $Q \cong \mathbf{Q}^2$, be the representation given by $\omega(1,1)=(-1,-1)$. Then $H^*(X;\mathbf{Q}) = H^*(K(Q,2);\mathbf{Q})^{\mathbf{Z}/2} \cong P_{\mathbf{Q}}[u,v,w]/(uv-w^2)$ with $\deg u = \deg v = \deg w = 4$. From (5.2) we see that $\pi_2 A^{\mathbf{Q}}X = \mathbf{Q}^2$, $\pi_3 A^{\mathbf{Q}}X = \mathbf{Q}^3$ and $\pi_6 A^{\mathbf{Q}}X = \mathbf{Q}$, and by a method of [10] one can even show that these are the only non-vanishing homotopy groups. Hence

$$M_A\widetilde{\mathbf{Q}_X} \cong S(a,b)\otimes E(\alpha,\beta,\gamma)\otimes S(\delta)$$

where $\deg a = \deg b = 2$, $\deg\alpha = \deg\beta = \deg\gamma = 3$ and $\deg\delta = 6$. The action $\omega:\mathbf{Z}/2 \to \mathscr{E}(M_A\widetilde{\mathbf{Q}_X})$ is given by $\omega(a) = -a$, $\omega(b) = -b$ and is trivial on the other generators. For the differential we have to follow up the construction as given in Theorem 4.5: One finds that $d\alpha = a^2$, $d\beta = b^2$, $d\gamma = ab$, $d\delta = a^2\beta - ab\gamma$.

(5.5) *Remarks.* (a) Following this pattern it is straightforward to determine models for the $\mathbf{Q}$-acyclic spaces studied in ([10], §4). Moreover, using the algorithm as given in Theorem 4.5 one can do explicit calculations of $\pi_i A^{\mathbf{Q}}X$ beyond the range given in (5.2) in many cases where the method of [10] doesn't apply. It is not clear however what is the actual range where Theorem 4.5 explicitly works, since it is not known to what extent one can explicitly determine generators of invariant subalgebras of cohomology algebras.

(b) In principle one could also determine $\pi_i A^{\mathbf{Q}}X$ by a Serre spectral sequence argument, again using Dror's construction of $A^{\mathbf{Q}}X$. However, beyond computing cohomology algebras explicitly, this depends on the possibility of passing from cohomology to homotopy and vice versa. Therefore our approach seems to be much more adequate as our model keeps the entire cohomology and homotopy information on $A^{\mathbf{Q}}_n X$ simultaneously.

References

[1] BAUES, H. J. and LEMAIRE, J. M., *Minimal Models in Homotopy Theory*, Math. Ann. *225* (1977), 219–242.

[2] BOUSFIELD, A. K. and KAN, D. M., *Homotopy Limits, Completions and Localizations, Lecture Notes in Mathematics 304,* Berlin–Heidelberg–New York: Springer 1972

[3] BOUSFIELD, A. K. and GUGENHEIM, V. K. A. M., *On PL De Rham theory and rational homotopy type*, Mem. AMS vol. 8, *179* (2) (1976).

[4] COOKE, G., *Replacing homotopy actions by topological actions*, Trans AMS *237* (1978), 391–406.
[5] DROR, E., *Acyclic spaces*, Topology *11* (1972), 339–348.
[6] DYER, M. N. and SIERADSKI, A. J., *Trees of Homotopy Types of Two-Dimensional CW-Complexes*, Comm. Math. Helv. *48* (1973), 31–44.
[7] HAUSMANN, J-C. and VOGEL, P., *The plus construction and lifting maps from manifolds*, Proc. Symp. Pure Math. *32* (1978), 67–76.
[8] LEHMANN, D., *Théorie homotopique des formes différentielles, astérisque* 45, société mathématique de france (1977).
[9] MEIER, W., *Acyclic Maps and Knot Complements*, Math. Ann. *243* (1979), 247–259.
[10] MEIER, W. and STREBEL, R., *Homotopy groups of acyclic spaces*, Quart. J. Math. Oxford (to appear).
[11] NEISENDORFER, J., *Lie algebras, coalgebras and rational homotopy theory for nilpotent spaces*, Pacific Jour. Math. *74* (1978), 429–460.
[12] QUILLEN, D., *Rational homotopy theory*, Ann. of Math. *90* (1969), 205–295.
[13] ——, *Cohomology of groups*, Actes Congrès Int. Math. Nice T. *2* (1970), 47–51.
[14] STANLEY, R. P., *Invariants of finite groups and their applications to combinatorics*, Bull. (New Series) AMS *1* (1979), 475–511.
[15] SULLIVAN, D., *Infinitesimal computations in Topology*, Publ. Math. IHES *47* (1978), 269–331.
[16] SUMNERS, D. W., *Polynomial invariants and the integral homology of knots and links*, Invent. Math. *15* (1972), 78–90.

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