An asymptotic series for the mean value of Dirichlet L-functions.

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An asymptotic series for the mean value of Dirichlet *L*-functions

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1. Introduction

Let χ denote a typical Dirichlet character (mod q), where $q>1$, and let $L(s,\chi)$ be the corresponding L-function. These functions have many applications to the distribution of primes and other arithmetic objects in the various arithmetic progressions to modulus q. For such applications one frequently needs information on the mean values

$$\sum_{\chi(\bmod q)} |L(\tfrac{1}{2}+it,\chi)|^{2k} \tag{1}$$

and

$$\sum_{\chi(\bmod q)} \int_0^T |L(\tfrac{1}{2}+it,\chi)|^{2k}\,dt \tag{2}$$

for a positive integer k. Although one requires a certain degree of uniformity in t or T, an arbitrary power of $2+|t|$ or $2+T$ in any error terms will usually suffice. Moreover it is normally sufficient to have a good upper bound rather than an asymptotic formula. In this respect the estimate of Montgomery [4; Theorem 10.1], namely

$$\sideset{}{^*}\sum_{\chi(\bmod q)} \int_0^T |L(\tfrac{1}{2}+it,\chi)|^4\,dt \ll \phi(q)T(\log qT)^4, \qquad (T\geq 2),$$

where $\sum^*$ denotes summation over primitive characters only, is the best currently available. None the less, it is of theoretical interest to investigate the mean values (1) and (2) more closely, and it is this line of enquiry we shall pursue here, by examining the behaviour of the sum (1), with $k=1$, for large values of q. There are many respects in which the behaviour of $\zeta(\frac{1}{2}+it)$, for varying t, resembles that of $L(\frac{1}{2}+it,\chi)$ with respect to q. In particular one might expect the behaviour of

(1), when $k=1$, to be similar to that of

$$I=\int_0^T |\zeta(\tfrac{1}{2}+it)|^2\,dt.$$

This is in fact not so, for it is known that

$$I=T\left(\log\frac{T}{2\pi}+2\gamma-1\right)+E(T),$$

in which, by Balasubramanian [1] we have $E(T)\ll T^{\frac{1}{3}+\varepsilon}$, and by Good [2] we have $E(T)=\Omega(T^{1/4})$. In contrast we shall prove:—

THEOREM. *We have*

$$\sum_{\chi(\bmod q)} |L(\tfrac{1}{2},\chi)|^2=\frac{\phi(q)}{q}\sum_{k\mid q}\mu(q/k)T(k), \tag{3}$$

where $T(k)$ has the asymptotic expansion

$$T(k)=k\left(\log\frac{k}{8\pi}+\gamma\right)+2\zeta(\tfrac{1}{2})^2k^{1/2}+\sum_{n=0}^{2N-1}c_nk^{-n/2}+O(k^{-N}), \tag{4}$$

for any $N\geqslant 1$. Here the c_n are numerical constants and γ is Euler's constant.

COROLLARY. *If $q=p$ is prime then we have an asymptotic expansion*

$$\sum_{\chi(\bmod p)} |L(\tfrac{1}{2},\chi)|^2=(p-1)\left(\log\frac{p}{8\pi}+\gamma\right)+2\zeta(\tfrac{1}{2})^2p^{1/2}+\sum_{n=0}^{2N-1}d_np^{-n/2}+O(p^{-N}), \tag{5}$$

for any $N\geqslant 1$.

The behaviour is therefore much closer to that of the mean value

$$\int_0^\infty |\zeta(\tfrac{1}{2}+it)|^2\,e^{-2t/T}\,dt,$$

which has the asymptotic series

$$\frac{\log\dfrac{T}{4\pi}+\gamma}{2\sin\dfrac{1}{T}}+\sum_{n=0}^{N-1}c_n'T^{-n}+O(T^{-N}),$$

for which see Kober [3]. However one should note the occurrence of the half-integer powers in the expansion (4). Note also that for general q the error terms in (4) will decrease at different rates for the different factors k of q. Thus we cannot give an asymptotic series for the sum (3) itself, except when, as in the corollary, q has no small prime factors.

Instead of examining sums of the form (3) one could treat the expressions

$$\sum_{\chi \,(\mathrm{mod}\, q)}^{*} |L(\tfrac{1}{2}+it, \chi)|^2,$$

or

$$\sum_{\chi \,(\mathrm{mod}\, q)} |L(\tfrac{1}{2}+it, \chi^*)|^2,$$

where χ^* is the primitive character which induces χ. However these present additional difficulties without simplifying the form of the results. Thus if $q=p$ is prime and $t=0$, for example, the above sums differ from (5) by $O(1)$, so that the $p^{1/2}$ term will still be present.

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2. Functional equations

Let

$$T_1(q, s) = \sum_{\chi \,(\mathrm{mod}\, q)} L(s, \chi) L(1-s, \bar{\chi}).$$

Here $L(s, \chi)$ is holomorphic for non-principal χ. If χ is principal and $q \neq 1$ then $L(s, \chi)$ will have a simple pole at $s=1$ and $L(1-s, \bar{\chi})$ will have a zero; moreover $L(s, \chi)$ will be regular elsewhere. Thus $T_1(q, s)$ is holomorphic for $q>1$. We shall also need to know that $T_1(q, s)$ satisfies the growth condition

$$T_1(q, \sigma+it) \ll (1+|t|)^2, \qquad (-1 \leq \sigma \leq 2) \tag{6}$$

for fixed q; but this follows from a trivial bound for $L(s, \chi)$. Finally, since $\bar{\chi}$ runs over all characters (mod q) as χ does, we find

$$T_1(q, s) = T_1(q, 1-s). \tag{7}$$

We now express $L(s,\chi)$ in terms of the Hurwitz Zeta-function

$$\zeta(s,\alpha)=\sum_{n=0}^{\infty}(n+\alpha)^{-s},\qquad (0<\alpha\leqslant 1,\ \mathrm{Re}\,(s)>1)$$

and apply the functional equation (for which see Titchmarsh [5; §2.17]) for $\zeta(1-s,\alpha)$. This yields for $\mathrm{Re}\,(s)>1$,

$$L(s,\chi)L(1-s,\bar\chi)=\Big(q^{-s}\sum_{u=1}^{q}\chi(u)\zeta(s,u/q)\Big)\Big(q^{s-1}\sum_{v=1}^{q}\bar\chi(v)\zeta(1-s,v/q)\Big)$$
$$=q^{-1}\sum_{u,v}\chi(u)\bar\chi(v)\zeta(s,u/q)\left\{\frac{2\Gamma(s)}{(2\pi)^s}\sum_{l=1}^{\infty}\frac{\sin(2\pi lv/q+\pi(1-s)/2)}{l^s}\right\}.$$

We now sum over χ, using the orthogonality relation

$$\sum_{\chi(\mathrm{mod}\,q)}\chi(u)\bar\chi(v)=\begin{cases}\phi(q), & u\equiv v(\mathrm{mod}\,q),\quad (uv,q)=1,\\ 0, & \text{otherwise.}\end{cases}$$

We then obtain

$$T_1(q,s)=\frac{\phi(q)}{q}\sum_{\substack{v=1\\(v,q)=1}}^{q}\zeta(s,v/q)\left\{\frac{2\Gamma(s)}{(2\pi)^s}\sum_{l=1}^{\infty}\frac{\sin(2\pi lv/q+\pi(1-s)/2)}{l^s}\right\}$$
$$=\frac{\phi(q)}{q}2\Gamma(s)\Big(\frac{q}{2\pi}\Big)^s\sum_{\substack{v=1\\(v,q)=1}}^{q}\ \sum_{\substack{h=1\\h\equiv v(\mathrm{mod}\,q)}}^{\infty}h^{-s}\sum_{l=1}^{\infty}\frac{\sin(2\pi lv/q+\pi(1-s)/2)}{l^s}$$
$$=\frac{\phi(q)}{q}2\Gamma(s)\Big(\frac{q}{2\pi}\Big)^s\sum_{\substack{h=1\\(h,q)=1}}^{\infty}h^{-s}\sum_{l=1}^{\infty}\frac{\sin(2\pi lh/q+\pi(1-s)/2)}{l^s}$$
$$=\frac{\phi(q)}{q}2\Gamma(s)\Big(\frac{q}{2\pi}\Big)^s\sum_{h=1}^{\infty}\Big(\sum_{\substack{d|q\\d|h}}\mu(d)\Big)h^{-s}\sum_{l=1}^{\infty}\frac{\sin(2\pi lh/q+\pi(1-s)/2)}{l^s}$$
$$=\frac{\phi(q)}{q}2\Gamma(s)\Big(\frac{q}{2\pi}\Big)^s\sum_{d|q}\mu(d)d^{-s}\sum_{j,l=1}^{\infty}(jl)^{-s}\sin(2\pi djl/q+\pi(1-s)/2).$$

Now writing $d=q/k$ we see that

$$T_1(q,s)=\frac{\phi(q)}{q}\sum_{k|q}\mu(q/k)T(k,s),\tag{8}$$

where

$$T(k, s) = 2\Gamma(s)\left(\frac{k}{2\pi}\right)^s \sum_{n=1}^{\infty} d(n)n^{-s} \sin(2\pi n/k + \pi(1-s)/2). \tag{9}$$

Clearly the sum on the left of (3) is $T_1(q, \frac{1}{2})$.

3. A weighted sum

The sum (9) for $T(k, s)$ does not converge absolutely if $s = \frac{1}{2}$, so we use the functional equation (7) to produce a weighted sum. Let

$$F(s) = s^{-1} \cos(\pi s) \exp(s^2).$$

Then $F(s)$ is an odd function of s and is holomorphic, except for a simple pole at the origin, of residue 1. Moreover it satisfies the growth condition

$$F(\sigma + it) \ll e^{-2|t|}, \qquad |t| \geq 1 \tag{10}$$

in any fixed vertical strip. Now consider the integral

$$I = \frac{1}{2\pi i} \int_{(1)} T_1(q, s + \tfrac{1}{2}) F(s)\, ds,$$

where the symbol (σ) denotes integration along a line from $\sigma - i\infty$ to $\sigma + i\infty$. After moving the line of integration to (-1) and allowing for the pole at $s = 0$, we obtain

$$I = T_1(q, \tfrac{1}{2}) + \frac{1}{2\pi i} \int_{(-1)} T_1(q, s + \tfrac{1}{2}) F(s)\, ds. \tag{11}$$

Then, using (7) together with the relation $F(-s) = -F(s)$, we deduce that $T_1(q, \frac{1}{2}) = 2I$.

We evaluate I by termwise integration, using (8) and (9). This yields

$$T_1(q, \tfrac{1}{2}) = 2I = \frac{\phi(q)}{q} \sum_{k \mid q} \mu(q/k) T(k),$$

where

$$T(k)=4\sum_{1}^{\infty} d(n)K_1\left(\frac{2\pi n}{k}\right) \tag{12}$$

and

$$\begin{aligned} K_1(x) &= \frac{1}{2\pi i}\int_{(1)} \Gamma(s+\tfrac{1}{2})x^{-s-\frac{1}{2}}\sin\left(x+\frac{\pi}{4}-\frac{\pi s}{2}\right)F(s)\,ds \\ &= \mathrm{Re}\left\{\frac{1}{2\pi i}\int_{(1)} \Gamma(s+\tfrac{1}{2})x^{-s-\frac{1}{2}}e^{ix-i\pi/4-i\pi s/2}F(s)\,ds\right\} \\ &= \mathrm{Re}\,(K(x)), \end{aligned}$$

say. Here

$$K(x)=e^{-i\pi/4}e^{ix}x^{-1/2}J(x) \tag{13}$$

and

$$J(x)=\frac{1}{2\pi i}\int_{(1)} \Gamma(s+\tfrac{1}{2})x^{-s}e^{-i\pi s/2}F(s)\,ds. \tag{14}$$

Our choice of $F(s)$ ensures that the integrand of $J(x)$ is holomorphic except for a simple pole at $s=0$, with residue $\Gamma(\frac{1}{2})=\pi^{1/2}$. Thus, on moving the line of integration to $(\pm M)$, where $M>0$, we find

$$J(x)\ll x^{-M}, \qquad (x\geqslant 1), \tag{15}$$

$$J(x)-\pi^{1/2}\ll x^{M}, \qquad (0<x\leqslant 1), \tag{16}$$

for any $M>0$. Similarly, after differentiating under the sign l (>0) times, we obtain

$$J^{(l)}(x)\ll\begin{cases} x^{-M}, & (x\geqslant 1), \\ x^{M}, & (0<x\leqslant 1), \end{cases} \tag{17}$$

there now being no pole at $s=0$. From these bounds it follows in particular that

$$K(x)\ll\begin{cases} x^{-M}, & (x\geqslant 1), \\ x^{-1/2}, & (0<x\leqslant 1), \end{cases} \tag{18}$$

$$K'(x)\ll\begin{cases} x^{-M}, & (x\geqslant 1), \\ x^{-3/2}, & (0<x\leqslant 1). \end{cases} \tag{19}$$

We now sum (12) by parts, writing

$$\sum_{n\le x} d(n) = D(x) = x(\log x + 2\gamma - 1) + \Delta(x)$$

whether $x \geq 1$ or not. This yields

$$T(k) = 4\,\mathrm{Re}\,(S(k)),$$

with

$$S(k) = -\frac{2\pi}{k}\int_0^\infty D(x)K'\left(\frac{2\pi x}{k}\right)dx$$

$$= -\frac{2\pi}{k}\int_0^\infty x(\log x + 2\gamma - 1)K'\left(\frac{2\pi x}{k}\right)dx - \frac{2\pi}{k}\int_0^\infty \Delta(x)K'\left(\frac{2\pi x}{k}\right)dx, \qquad (20)$$

the integrals converging absolutely, by (19).

4. The leading terms

An application of (19), together with Voronoï's bound $\Delta(x) \ll x^{\frac{1}{3}+\varepsilon}$, shows that the second term in (20) is $O(k^{1/2})$. On the other hand, the first term in (20) becomes, on substituting $x = ky/(2\pi)$,

$$-\frac{k}{2\pi}\int_0^\infty y\left(\log\frac{k}{2\pi} + \log y + 2\gamma - 1\right)K'(y)\,dy = A\frac{k}{2\pi}\log\frac{k}{2\pi} + B\frac{k}{2\pi},$$

where the constants A and B are given by

$$A = -\int_0^\infty yK'(y)\,dy, \qquad B = -\int_0^\infty y(\log y + 2\gamma - 1)K'(y)\,dy.$$

Integration by parts yields

$$A = \int_0^\infty K(y)\,dy, \qquad B = 2\gamma A + \int_0^\infty K(y)\log y\,dy.$$

Put $L(y) = 1$ or $\log y$. Then by (13) and (14), with the line of integration moved to

$(\frac{1}{4})$, we have

$$\int_0^1 K(y)L(y)\,dy = \frac{1}{2\pi i}\int_0^1 \int_{(1/4)} L(y)\Gamma(s+\tfrac{1}{2})y^{-s-\frac{1}{2}}e^{iy-i\pi/4-i\pi s/2}F(s)\,ds\,dy$$

$$= \frac{1}{2\pi i}\int_{(1/4)} \Gamma(s+\tfrac{1}{2})e^{-i\pi/4-i\pi s/2}F(s)\int_0^1 L(y)y^{-s-\frac{1}{2}}e^{iy}\,dy\,ds, \qquad (21)$$

the interchange of integrations being justified by absolute convergence (it is for precisely this reason that we move the line of integration to $(\frac{1}{4})$). Similarly we find

$$\int_1^\infty K(y)L(y)\,dy = \frac{1}{2\pi i}\int_{(1)} \Gamma(s+\tfrac{1}{2})e^{-i\pi/4-i\pi s/2}F(s)\,\alpha(s)\,ds, \qquad (22)$$

where

$$\alpha(s) = \int_1^\infty L(y)y^{-s-\frac{1}{2}}e^{iy}\,dy.$$

We next modify the path of integration in the last integral so as to run from 1 to $i\infty$. We then see that $\alpha(s)$ is regular for all s and satisfies

$$\alpha(s) \ll \exp\left(\frac{\pi}{2}|t|\right), \qquad (\tfrac{1}{4} \leqslant \sigma \leqslant 1),$$

for $s = \sigma + it$. We may therefore move the line of integration in (22) back to $(\frac{1}{4})$ and add the result to (21) to obtain

$$\int_0^\infty K(y)L(y)\,dy = \frac{1}{2\pi i}\int_{(1/4)} \Gamma(s+\tfrac{1}{2})e^{-i\pi/4-i\pi s/2}F(s)\beta(s)\,ds,$$

where

$$\beta(s) = \int_0^1 L(y)y^{-s-\frac{1}{2}}e^{iy}\,dy + \int_1^{i\infty} L(y)y^{-s-\frac{1}{2}}e^{iy}\,dy$$

$$= \int_0^{i\infty} L(y)y^{-s-\frac{1}{2}}e^{iy}\,dy.$$

We therefore see that $\beta(s) = i^{\frac{1}{2}-s}\Gamma(\frac{1}{2}-s)$ for $L(y) = 1$ and

$$\beta(s) = i^{\frac{1}{2}-s}\left(\frac{i\pi}{2}\Gamma(\tfrac{1}{2}-s) + \Gamma'(\tfrac{1}{2}-s)\right)$$

for $L(y)=\log y$. Now

$$\Gamma(\tfrac{1}{2}+s)\Gamma(\tfrac{1}{2}-s)=\pi\sec(\pi s) \tag{23}$$

so that

$$A=\frac{1}{2\pi i}\int_{(1/4)} e^{-i\pi s}\pi\sec(\pi s)F(s)\,ds$$

and

$$\int_0^\infty K(y)\log y\,dy=\frac{i\pi}{2}A+\frac{1}{2\pi i}\int_{(1/4)} e^{-i\pi s}\pi\sec(\pi s)\frac{\Gamma'(\frac{1}{2}-s)}{\Gamma(\frac{1}{2}-s)}F(s)\,ds.$$

Thus

$$\mathrm{Re}\,(A)=\frac{1}{2\pi i}\int_{(1/4)}\pi F(s)\,ds$$

and

$$\mathrm{Re}\left(\int_0^\infty K(y)\log y\,dy\right)=\frac{1}{2\pi i}\int_{(1/4)}\frac{\pi^2}{2}\tan(\pi s)F(s)\,ds$$
$$+\frac{1}{2\pi i}\int_{(1/4)}\pi\frac{\Gamma'(\frac{1}{2}-s)}{\Gamma(\frac{1}{2}-s)}F(s)\,ds.$$

By logarithmic differentiation of the relation (23) we find

$$\frac{\Gamma'(\frac{1}{2}+s)}{\Gamma(\frac{1}{2}+s)}-\frac{\Gamma'(\frac{1}{2}-s)}{\Gamma(\frac{1}{2}-s)}=\pi\tan(\pi s),$$

so that it follows that

$$G(s)=\frac{\pi^2}{2}\tan(\pi s)+\pi\frac{\Gamma(\frac{1}{2}-s)}{\Gamma(\frac{1}{2}-s)}$$

is an even function of s. Then

$$I=\frac{1}{2\pi i}\int_{(1/4)}G(s)F(s)\,ds=G(0)+\frac{1}{2\pi i}\int_{(-1/4)}G(s)F(s)\,ds$$
$$=G(0)+\frac{1}{2\pi i}\int_{(1/4)}G(-w)F(-w)\,dw=G(0)-\frac{1}{2\pi i}\int_{(1/4)}G(w)F(w)\,dw,$$

whence

$$I=\tfrac{1}{2}G(0)=-\frac{\pi}{2}(\gamma+\log 4).$$

Similarly

$$\frac{1}{2\pi i}\int_{(1/4)}\pi F(s)\,ds=\frac{\pi}{2}.$$

Thus, finally, we have

$$4\operatorname{Re}\left(A\frac{k}{2\pi}\log\frac{k}{2\pi}+B\frac{k}{2\pi}\right)=k\left(\log\frac{k}{8\pi}+\gamma\right)$$

as required.

5. Lower order terms

On writing

$$H(x)=e^{-i\pi/4}e^{ix}x^{-1/2}$$

and

$$\delta(x)=\begin{cases}1, & (0<x\leqslant 1),\\ 0, & (x\geqslant 1),\end{cases}$$

the estimates (15), (16) and (17) yield

$$K^{(l)}\left(\frac{2\pi x}{k}\right)=H^{(l)}\left(\frac{2\pi x}{k}\right)\Gamma(\tfrac{1}{2})\delta\left(\frac{2\pi x}{k}\right)+M_l\left(\frac{2\pi x}{k}\right),$$

where for any $M>0$ we have

$$M_l(x)\ll\begin{cases}x^{-M}, & (x\geqslant 1),\\ x^{M}, & (0<x\leqslant 1).\end{cases}\qquad (24)$$

In particular it follows that the second term of (20) is

$$-\frac{2\pi}{k}\int_0^\infty \Delta(x)K'\left(\frac{2\pi x}{k}\right)dx = -\left(\frac{k}{2\pi}\right)^{1/2}\Gamma(\tfrac{1}{2})e^{-i\pi/4}\int_0^{k/(2\pi)}\Delta(x)\frac{d}{dx}(e^{2\pi i x/k}x^{-1/2})\,dx$$

$$+O\left(\int_0^{k/2\pi} k^{-1}\,|\Delta(x)|\,dx\right)+O\left(\int_{k/2\pi}^\infty k\,|\Delta(x)|\,x^{-2}\,dx\right)$$

$$=\left(\frac{k}{8}\right)^{1/2} e^{-i\pi/4}\int_0^{k/2\pi}\Delta(x)(x^{-3/2}+O(x^{-1/2}k^{-1}))\,dx$$

$$+O(k^{\frac{1}{3}+\varepsilon})$$

$$=\frac{k^{1/2}(1-i)}{4}\int_0^\infty \Delta(x)x^{-3/2}\,dx+O(k^{\frac{1}{3}+\varepsilon}).$$

On noting that (see Titchmarsh [5; (12.5.5)])

$$\zeta^2(s)=s\int_0^\infty \Delta(x)x^{-s-1}\,dx, \qquad (\tfrac{1}{3}<\mathrm{Re}\,(s)<1),$$

we obtain $2\zeta^2(\frac{1}{2})$ for the coefficient of $k^{1/2}$ in (4), as required.

We now define, inductively, $\Delta_1(x)=\Delta(x)$ and

$$\Delta_{l+1}(x)=\int_0^x \Delta_l(t)\,dt.$$

Then, for $l\geq 3$, we have

$$\Delta_l(x)=\frac{1}{2\pi i}\int_{(3/4)}\zeta^2(s)\frac{x^{s+l-1}}{s(s+1)\cdots(s+l-1)}\,ds. \tag{25}$$

Here we have

$$\zeta^2(\sigma+it)\ll(1+|t|)^{l-1-\varepsilon}, \qquad (1-\tfrac{1}{2}l+\tfrac{1}{2}\varepsilon\leq\sigma\leq\tfrac{3}{4}),$$

so that, on moving the line of integration to $(1-\frac{1}{2}l+\frac{1}{2}\varepsilon)$, we obtain

$$\Delta_l(x)=\sum_{1\leq n<l/2}K_{l,n}x^{l-n}+O(x^{\frac{1}{2}(l+\varepsilon)}) \tag{26}$$

for certain constants $K_{l,n}$. Next, on integrating by parts $l-1$ times, we find

$$-\frac{2\pi}{k}\int_0^\infty \Delta(x)K'\left(\frac{2\pi x}{k}\right)dx=(-1)^l\left(\frac{2\pi}{k}\right)^l\int_0^\infty \Delta_l(x)K^{(l)}\left(\frac{2\pi x}{k}\right)dx$$

$$=(-1)^l\left(\frac{2\pi}{k}\right)^l\Gamma(\tfrac{1}{2})\int_0^{k/(2\pi)}\Delta_l(x)H^{(l)}\left(\frac{2\pi x}{k}\right)dx$$

$$+(-1)^l\left(\frac{2\pi}{k}\right)^l\int_0^\infty \Delta_l(x)M_l\left(\frac{2\pi x}{k}\right)dx.$$

We now write

$$H^{(l)}(y)=y^{-\frac{1}{2}-l}e^{iy}p_1(y),$$

where $p_1(y)$ is a polynomial of degree l. Moreover

$$e^{iy}=p_2(y)+y^{l+1}p_3(y),$$

where $p_2(y)$ is also a polynomial of degree l, and where $p_3(y)\ll 1$ for $0\leqslant y\leqslant 1$. Combining these, we may write

$$H^{(l)}(y)=y^{-\frac{1}{2}-l}p(y)+q(y),$$

where $p(y)$ is a polynomial of degree at most l, and $q(y)\ll 1$ for $0\leqslant y\leqslant 1$. Finally let

$$N_l(y)=M_l(y)+\Gamma(\tfrac{1}{2})q(y)\delta(y),$$

so that, by (24)

$$N_l(y)\ll\begin{cases} y^{-M}, & (y\geqslant 1),\\ 1, & (0<y\leqslant 1).\end{cases} \tag{27}$$

We have thus to consider

$$I_{l,m}=k^{-l}\int_0^{k/(2\pi)}\Delta_l(x)\left(\frac{x}{k}\right)^{-\frac{1}{2}-l+m}dx$$

for $0 \leqslant m \leqslant l$, and

$$I_l = k^{-l}\int_0^\infty \Delta_l(x) N_l\left(\frac{2\pi x}{k}\right) dx.$$

We aim to show that each of these expressions can be written in the form

$$\sum_{-1 \leqslant n \leqslant l-3} C_{l,n} k^{-n/2} + O(k^{1-(l-\varepsilon)/2}),$$

with constants $C_{l,n}$ depending on l and n alone. On choosing $l = 2N+3$, this will show that

$$T(k) = k\left(\log\frac{k}{8\pi} + \gamma\right) + 2\zeta(\tfrac{1}{2})^2 k^{1/2} + \sum_{n=0}^{2N-1} c_{n,N} k^{-n/2} + O(k^{-N}).$$

However it is clear that the constants $c_{n,N}$ must be independent of N, so that the theorem will follow.

In the case of $I_{l,m}$ we first examine

$$k^{-m+\frac{1}{2}}\int_0^1 \Delta_l(x) x^{-\frac{1}{2}-l+m}\, dx.$$

Since $\Delta_l(x) \ll x^{l-\frac{1}{4}}$, by (25), the integral converges. Thus this contribution to $I_{l,m}$ is of the required form. For the remaining part, namely

$$k^{-m+\frac{1}{2}}\int_1^{k/(2\pi)} \Delta_l(x) x^{-\frac{1}{2}-l+m}\, dx, \tag{28}$$

the main terms of (26) yield

$$\sum_n k^{-m+\frac{1}{2}}\int_1^{k/(2\pi)} K_{l,n} x^{m-n-\frac{1}{2}}\, dx = \sum_{1 \leqslant n < l/2} K_{l,n}(m-n+\tfrac{1}{2})^{-1}\left(\left(\frac{k}{2\pi}\right)^{m-n+\frac{1}{2}} - 1\right)k^{-m+\frac{1}{2}},$$

which is of the required form. As to the error term in (26), $E_l(x)$ say, this contributes to (28) a quantity $\ll k^{1+(\varepsilon-l)/2}$ if $m \geqslant (l-1)/2$, and otherwise

$$k^{-m+\frac{1}{2}}\int_1^\infty E_l(x) x^{-\frac{1}{2}-l+m}\, dx + O\left(k^{-m+\frac{1}{2}}\int_{k/(2\pi)}^\infty x^{(\varepsilon-1-l)/2+m}\, dx\right)$$
$$= C'_{l,m} k^{-m+\frac{1}{2}} + O(k^{1+(\varepsilon-l)/2}),$$

which again is of the required form.

For I_l the leading terms of (26) yield

$$\sum_n K_{l,n}k^{-l}\int_0^\infty x^{l-n}N_l\left(\frac{2\pi x}{k}\right)dx = \sum_{1\le n<l/2} K_{l,n}(2\pi)^{n-l-1}k^{1-n}\int_0^\infty y^{l-n}N_l(y)\,dy.$$

The bound (27) shows that the integrals here converge, so that the above expression has the form we seek. Finally the error term of (26) contributes

$$\ll k^{-l}\int_0^\infty x^{(l+\varepsilon)/2}\left|N_l\left(\frac{2\pi x}{k}\right)\right|dx$$

$$\ll k^{-l}\cdot k^{1+(l+\varepsilon)/2}\int_0^\infty y^{(l+\varepsilon)/2}\,|N_l(y)|\,dy \ll k^{1+(\varepsilon-l)/2},$$

and this too has the required shape. The proof of the theorem is now complete.

6. The corollary

The theorem shows that the sum on the left of (5) is

$$\frac{p-1}{p}(T(p)-T(1)).$$

Here, since $T(1)$ is a numerical constant, given by (12), the term $-(p-1)p^{-1}T(1)$ may be incorporated in the asymptotic expansion. Moreover the factor $1-p^{-1}$ in front of $T(p)$ merely changes the constants in the asymptotic expansion.

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