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Curvature, diameter and Betti numbers

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We give an upper bound for the Betti numbers of a compact Riemannian manifold in terms of its diameter and the lower bound of the sectional curvatures. This estimate in particular shows that most manifolds admit no metrics of non-negative sectional curvature.

§0. Introduction

0.1. *Sectional curvature*

Let V denote a compact (without boundary) connected Riemannian manifold of dimension n. We denote by K the sectional curvature of V and we set $\inf K = \inf_\tau K(\tau)$ where τ runs over all tangent 2-planes in V. One calls V *a manifold of non-negative curvature* if $\inf K \geqslant 0$. This condition has the following geometric meaning.

An n-dimensional Riemannian manifold has non-negative curvature iff for each point $v \in V$ there is a positive number ε and a map f of the n-dimensional Euclidean ε-ball B into V with the following two properties:

(a) *f sends B diffeomorphicly onto the ε-ball in V with the center v.*

(b) *The map f is distance non-increasing, that is for any two points x and y in B one has.*

$$\operatorname{dist}(f(x), f(y)) \leqslant \operatorname{dist}(x, y),$$

where the first "dist" denotes the Riemannian distance in V and the second one is the Euclidean distance in $B \subset \mathbf{R}^n$.

Such a map f when it exists, is unique and it coincides with the so called exponential map (see [2], [4], [17]). In particular, f sends the center of B to v.

Observe, that the more general condition $\inf K \geqslant k$, $k \in (-\infty, +\infty)$, can be also interpreted geometrically. One should only use an ε-ball in the space of constant curvature k instead of the Euclidean ball B. For $k > 0$ one takes the sphere of radius $k^{-1/2}$ and for $k < 0$ one uses the hyperbolic space of curvature k.

Examples. Most known manifolds of non-negative curvature have the group theoretic origin. For instance, if V admits a smooth transitive action of a compact Lie group, then there is a Riemannian metric on V of non-negative curvature (see [4]). For each dimension ≥ 3 there are infinitely many homotopy types of such manifolds. Among other examples we mention only an exotic 7-sphere with a metric of non-negative curvature (see [8]) and the connected sum of two copies of the complex projective space (see [3]).

Counterexamples. The first topological obstruction for the existence of a metric of non-negative curvature on a compact manifold V was found by Bochner (see [1]).

Let V be a compact n-dimensional Riemannian manifold of non-negative curvature. Then $\mathrm{Dim}\, H_1(V, \mathbf{R}) \leq n$ *and the equality takes place only if V is flat.*

In fact, this theorem of Bochner remains true for a manifold V *with non-negative Ricci curvature. Furthermore, the universal covering of every manifold of non-negative curvature metrically splits into the product of* $\mathbf{R}^m$ *and a compact simply connected manifold* V^{n-m} (see [4], [6]).

This theorem reduces the problem to the case when the fundamental group $\pi_1(V)$ is finite.

There is another general obstruction for the existence of metrics of non-negative curvature (see [18]) and, in fact, this obstruction already appears for the manifolds with positive *scalar* curvature. Without going into details we mention only a few facts.

There are exotic 9-spheres that carry no metrics of positive scalar curvature (see [16]). *In particular they admit no metrics of non-negative sectional curvature.*

The product of an arbitrary manifold by the sphere S^m, $m \geq 2$, *admits a metric of positive scalar curvature. Furthermore, connected sums of manifolds of positive scalar curvature admit metrics with positive scalar curvature* (see [13], [19]).

We shall see below that most of these manifolds admit no metrics with non-negative *sectional* curvature.

Non compact manifolds. Every open manifold admits a *noncomplete* metric with positive sectional curvature (see [9]). On the other hand, when such a V is complete it must be homeomorphic to $\mathbf{R}^n$ (see [7]). When the curvature of a complete manifold V is non-negative, then V is homeomorphic to a vector bundle over a compact manifold of non-negative curvature (see [5]). This theorem brings us back to the compact case.

0.2. *Estimates for Betti numbers*

Fix a field F and denote by $b_i = b_i(V; F)$ the dimension over F of the homology group $H_i(V; F)$.

0.2.A. *There exists a constant $\mathscr{C}=\mathscr{C}(n)$, such that every compact connected n-dimensional Riemannian manifold V of non-negative sectional curvature satisfies*

$$\sum_0^n b_i \leqslant \mathscr{C}.$$

COROLLARY. *The connected sums of sufficiently many copies of the products of spheres $S^p \times S^{n-p}$, $0<p<n$, or of the complex projective spaces, admit no metrics of non-negative curvature.*

Remarks. The n-dimensional torus is, probably, topologically the largest manifold of non-negative curvature, but our estimate for $\mathscr{C}(n)$ is very far from $2^n=\sum_0^n b_i(T^n)$. Even for $b_1(V,\mathbf{Z}_p)$ we can not get the expected estimate $b_1(V,\mathbf{Z}_p)\leqslant n$.

Let us replace now the condition $\inf K \geqslant 0$ by $\inf K \geqslant -\kappa^2$, $\kappa \geqslant 0$, and denote by D the diameter of V.

0.2B. *There exists a constant $\mathscr{C}=\mathscr{C}(n)$ such that every compact connected manifold V satisfies*

$$\sum_0^n b_i \leqslant \mathscr{C}^{1+\kappa D}.$$

Remarks

(a) When $\kappa=0$ this theorem reduces to 0.2.A.

(b) The minimal number of generators of the fundamental group $\pi_1(V)$ is also bounded from above by $\mathscr{C}^{1+\kappa D}$ (see [10]).

(c) The connected sum of k copies of the product $S^p\times S^{n-p}$ can be equipped with a metric such that $2k+2=\sum_0^n b_i \geqslant (1.01)^{1+\kappa D}$.

(d) The theorem 0.2B can be, probably, generalized to the manifolds with the *Ricci* curvature bounded from below, that is with $\inf_t$ (Ric $(t,t)\geqslant -\delta^2$, where t runs over all unite tangent vectors in V. But all known results on estimating topology of V by δD are tied up with the non torsion part of the fundamental group. For example, one can show that $b_1(V;\mathbf{R})\leqslant n-1+\mathscr{C}^{\delta D}$ (this generalizes Bochner's theorem) but it is unknown whether this estimate holds for $b_1(V,\mathbf{Z}_2)$, even when V has positive Ricci curvature. We shall discuss the π_1-related estimates elsewhere (see also [11], [12]).

The proof of the theorem 0.2.A and 0.2.B is given in §1–§3. The curvature assumption essentially appears in this proof only once, in §1 for an analysis of the critical points of the Riemannian distance function as in [15]. This analysis is based on Toponogov's comparison theorem (see §1). Although the curvature assumption is also present for estimating the number of small balls needed for a covering of a larger ball (compare to [20]), we could equally use for this purpose the Ricci curvature instead of the sectional curvature.

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§1. Distance function

1.1. *Critical points*

Take a complete Riemannian manifold and a point x in V. Denote by $\mathrm{dist}_x : V \to \mathbf{R}_+$ the distance function $\mathrm{dist}_x(y) = \mathrm{dist}(x, y)$, $y \in V$. This function is not smooth but one can develop a complete Morse theory for this kind of a function. We shall need here only a few simple facts.

A point $y \in V$, $y \neq x$, is called *critical* for the function dist_x, or simply for x, if for every non-zero tangent vector $t \in T_y(V)$ there is a minimizing geodesic segment γ between x and y, such that the angle between t and γ at y is at most $\pi/2$. Recall, that a segment γ between x and y is called minimizing if

$$\mathrm{length}(\gamma) = \mathrm{dist}(x, y).$$

If a point $y_0 \in V$ is *not* critical for x, then there is a neighbourhood U of y_0 and a smooth vector field in U, that is $t(y)$, $y \in U$, such that for every point $y \in U$ the angle between the vector $t(y) \in T_y(V)$ and an arbitrary minimizing segment between x and y is an *acute* angle. It follows that the function dist_x is strictly decreasing along each integral curve of the field $t(y)$. This leads to the following fact that is a slight modification of a result of Grove–Shiohama [15].

ISOTOPY LEMMA. *Take two concentric balls B_1 and $B_2 \subset B_1$ in V centered at $x \in V$ and suppose that the closed annulus A between these balls, that is $A = Cl(B_1 \backslash B_2)$, contains no critical points of the function* dist_x. *Then there exists an isotopy of V which sends B_1 into B_2 and which is fixed outside any given neighbourhood of B_1.*

Proof. With the local fields $t(y)$ above one constructs a field $\bar{t}$ on V which has its support in a small neighborhood of A and such that the function dist_x strictly decreases along the integral curves of $\bar{t}$. This field performs the required isotopy.

1.2. *Comparison theorems*

Take three points x, y_1 and y_2 in V and take some minimizing segments γ_1 and γ_2 joining x with y_1 and with y_2 correspondingly. Denote by α the angle between γ_1 and γ_2 at x. Let l_1 denote length $(\gamma_1) = \text{dist}(x, y_1)$ and let l_2 denote length $(\gamma_2) = \text{dist}(x, y_2)$.

Toponogov's theorem (see [4], [17]). *If V is a complete manifold of non-negative curvature then*

$$\text{dist}(y_1, y_2) \leq \sqrt{l_1^2 + l_2^2 - 2l_1 l_2 \cos \alpha}.$$

Notice that for the Euclidean space $\mathbf{R}^n$ this inequality becomes an equality. We shall later use Topagonov's inequality only in the following two cases.

1.2.A. *Let $l_1 \geq l_2$ and let $\alpha \leq \pi/2$. Then*

$$\text{dist}(y_1, y_2) \leq l_1 + \tfrac{1}{2} l_2.$$

1.2.B. *Let again $l_1 \geq l_2$ and suppose that $\alpha \leq \frac{1}{6} \simeq \pi/18$. Then*

$$\text{dist}(y_1, y_2) \leq l_1 - \tfrac{3}{4} l_2.$$

Toponogov's inequality generalizes to all complete manifolds (see [4], [17]). In particular one has.

1.2.C. *If* $\inf K \geq -\kappa^2$, $\kappa \geq 0$, *and if the product $l_1 \kappa$ is sufficiently small, for example, if $l_1 \kappa \leq 10^{-10}$, then the inequalities* 1.2.A *and* 1.2.B *hold true.*

1.3. *An inequality for a critical point*

Take three points x, y and z in V and suppose that y is a critical point for x. Suppose further that $\text{dist}(z, x) \geq 2\,\text{dist}(x, y)$.

If V has non-negative curvature, then

$$\text{dist}(z, x) \leq \text{dist}(z, y) + \tfrac{1}{2}\,\text{dist}(x, y). \qquad (*)$$

Proof. Take a minimizing segment γ_1 between z and y. According to the definition of the critical point there is a minimizing segment γ_2 between x and y such that the angle between γ_1 and γ_2 at y is at most $\pi/2$. The inequality $\text{dist}(z, x) \geq 2\,\text{dist}(x, y)$ implies that length $(\gamma_1) \geq$ length (γ_2) and so we can apply 1.2.A.

Notice, that by the remark 1.2.C *the inequality* (*) *holds for a manifold* V *with* $\inf K \geq -\kappa^2$ if $\kappa(\text{dist}(z, x)) \leq 10^{-10}$.

1.4. *An inequality for two critical points*

Take a point $x \in V$ and two critical points y_1 and y_2 for the distance function dist_x. Take some minimizing segments γ_1 and γ_2 joining x with y_1 and y_2 correspondingly and denote by α the angle between γ_1 and γ_2 at x.

If $\inf \kappa \geq 0$ *and if* $l_1 = \text{dist}(x, y_1) \geq 2l_2 = 2\,\text{dist}(x, y_2)$, *then* $\alpha > \frac{1}{6}$

Proof. If $\alpha \leq \frac{1}{6}$, then, by 1.2.B, we have

$$\text{dist}(y_1, y_2) \leq l_1 - \tfrac{3}{4}l_2.$$

Now we use the inequality (*) above with y_1 in place of z and with y_2 in place of y. We get

$$l_1 = \text{dist}(y_1, x) \leq \text{dist}(y_1, y_2) + \tfrac{1}{2}l_2, \qquad l_2 = \text{dist}(x, y_2).$$

It follows that $l_2 = 0$, that is $x = y_2$, but this is not allowed by the definition of a critical point.

1.5. *Non compact manifolds*

Let us start with an obvious fact.

1.5.A. *Let* $t_1, \dots, t_k$ *be non zero vectors in* $\mathbf{R}^n$, *such that the angle between any two of these vectors is at least* $\frac{1}{6}$. *Then the number k of these vectors does not exceed a universal constant,* $\text{const}_n < (100)^n$.

Consider now a complete n-dimensional manifold V of non-negative sectional curvature and the distance function at a point x in V.

All critical points of the function dist_x *are contained in a compact ball around* x.

Indeed, we could find otherwise some critical points $y_1, \dots, y_k$ such that $k > (100)^n$ and $\text{dist}(x, y_i) \geq 2\,\text{dist}(x, y_j)$ for all $1 \leq i < j \leq k$. Take some minimizing segments $\gamma_1, \dots, \gamma_k$ between x and $y_1, \dots, y_k$ and denote by $t_1, \dots, t_k$ their tangent vectors at x. According to 1.5.A some of these angles must be less than $\frac{1}{6}$, but this contradicts to 1.4.

As a corollary we get a weak version of a theorem of Cheeger–Gromoll (see [4], [5]).

The manifold V has "finite topological type" that is V is homeomorphic to the interior of a compact manifold with boundary.

Proof. Use the isotopy lemma in 1.1.

Our argument generalizes to a class of manifolds whose curvatures are "not very negative at infinity." Since this is a digression we leave the proof of the following theorem to the reader.

Take a point x in a complete manifold V and denote by $K_-(R)$ the infinimum of the sectional curvature of V outside the R-ball centered at x.

If $R^2K_-(R)\to 0$ as $R\to\infty$, then the function $\mathrm{dist}_x : V\to \mathbf{R}_+$ *has its all critical points contained in a compact ball. In particular V is homeomorphic to the interior of a compact manifold V with boundary.*

It will become clear later that the boundary V_0 of V is rather special. It must satisfy the inequality.

$$\sum_0^{n-1} b_i(V_0)\leqslant \mathscr{C}=\mathscr{C}(n).$$

§2. Coverings by balls

2.1. *Volumes of balls*

Let V be a complete n-dimensional manifold, such that $\inf K\geqslant -\kappa^2$. Denote by $b(R)$ the volume of a radius R ball in the hyperbolic space with curvature $-\kappa^2$. Take two concentric balls B_1 and $B_2\subset B_1$ in V of radii R_1 and R_2. The volumes of these balls are related as follows.

$$\frac{\mathrm{Vol}\,(B_1)}{\mathrm{Vol}\,(B_2)}\leqslant\frac{b(R_1)}{b(R_2)}. \tag{*}$$

See [2] for the proof. Notice that (*) also holds for $\inf_t \mathrm{Ric}\,(t,t)\geqslant -((n-1)\kappa)^2$ (see [2]). When V has non-negative curvature the inequality (*) says that

$$\frac{\mathrm{Vol}\,(B_1)}{\mathrm{Vol}\,(B_2)}\leqslant\frac{R_1^n}{R_2^n}.$$

If the balls are not supposed to be concentric the inequality (*) takes the following form

$$\frac{\mathrm{Vol}\,(B_1)}{\mathrm{Vol}\,(B_2)}\leqslant\frac{b(R_1+2d)}{b(R_2)}, \tag{**}$$

where d denotes the distance between the centers of B_1 and B_2. We shall use only the following two crude corrollaries of (**).

2.1.A. *If the balls* B_1 *and* B_2 *of the radii* R_1 *and* $R_2 \leq R_1$ *have a non-empty intersection then the inequality*

$$\frac{\operatorname{Vol}(B_1)}{\operatorname{Vol}(B_2)} \leq \frac{(10R_1)^n}{R_2^n} \tag{***}$$

holds in the following two cases

(a) $\operatorname{Inf} K \geq 0$,

(b) $\operatorname{Inf}(K) \geq -\kappa^2$ *and the product* κR_1 *is sufficiently small, for example,* $\kappa R_1 \leq \exp(-n^n)$.

2.1.B. *Let* V *be a compact manifold of diameter* D *and let* $\operatorname{Inf} K = -\kappa^2$. *Then each* ε *ball* B *in* V *satisfies the following inequality.*

$$\frac{\operatorname{Vol}(V)}{\operatorname{Vol}(B)} \leq 10^n D^n \varepsilon^{-n} \exp(n\kappa D).$$

2.2. *Minimal coverings*

Take some sets $\{B_i\}_{i=1,\ldots,N}$ in V and denote first by I the set of all multiindices $(i_1 < i_2 < \cdots < i_l)$, $l = 1, \ldots, N$. Denote by I_+ the subset of I consisting of all multiindices $(i_1, \ldots, i_l)$, such that the intersection $\bigcap_1^l B_{i_j}$, $j = 1, \ldots, l$, is not empty. The number of the elements in I_+ is called the *index* of the system $\{B_i\}$. Clearly, the index takes the values between N and 2^N.

Take a ball B in V of radius R and cover it by some ε-balls, $0 < \varepsilon \leq R$, as follows. Take the *maximal* system of points x_i in B such that the distance between any two of them is greater than $\varepsilon/2$. In this case the ε-balls R_i around x_i cover B. We call such a system $\{B_i\}$ *a minimal* ε*-covering of* B. We want to estimate from above the index of such a covering in terms of the ratio $\varepsilon^{-1}R$ and $\operatorname{Inf} K$.

Here and in future for a ball B of radius r we denote by λB, $\lambda > 0$ the concentric ball of radius λr.

Let us return to our minimal ε-covering $\{B_i\}$, $i = 1, \ldots, N$. Observe that the balls $\frac{1}{4}B_i$ are disjoint and they are all contained in the ball $2B$. Now we invoke 2.1.A and conclude.

2.2.A. *If* Inf $K \geqslant 0$, *or more generally, if* inf $K \geqslant -\kappa^2$ *and* $2\kappa R \leqslant \exp(-n^n)$, *then the number* N *of the balls* B_i *does not exceed* const $(n, \varepsilon^{-1}R) \leqslant (80\varepsilon^{-1}R)^n$. *Therefore the index of the covering does not exceed* 2^M, $M = (80\varepsilon^{-1}R)^n$.

This Lemma gives, in particular, a reasonable upper bound for the indices of minimal covering of V for $K \geqslant 0$, but in the general case of inf $K \geqslant -k^2$, $K > 0$, the following sharper estimate is needed.

2.2.B. *Let* V *be as in* 2.1.B *and let* $\{B_i\}$, $i = 1, \ldots, N$, *be a minimal* ε*-covering of* V. *Take a number* $\lambda > 1$ *and let the product* $\lambda\varepsilon\kappa$ *be sufficiently small, for example,* $4\lambda\varepsilon\kappa < \exp(-n^n)$. *Then the index of the concentric covering* $\{\lambda B_i\}$ *does not exceed*

$$2^M 80^n D^n \varepsilon^{-n} \exp(n\kappa D), \qquad M = (160\,\lambda)^n.$$

Proof. First, we conclude as above that $N \leqslant 80^n D^n \varepsilon^{-n} \exp(n\kappa D)$. Now, if some balls λB_i intersect a fixed ball λB_{i_0}, then the centers of these balls must be contained in the ball $2\lambda B_{i_0}$ and so, by 2.2.A, each ball is involved in no more than 2^M intersections. Q.E.D.

2.3. *Topological Lemma*

Let V be an arbitrary complete Riemannian manifold of dimension n. Fix a coefficient field F and define *the content* of a ball B in V as the rank of the inclusion homomorphism

$$H_*(\tfrac{1}{5}B; F) \to H_*(B; F).$$

The number $\frac{1}{5}$ plays no essential role here, but it is convenient for our further constructions. Observe that the homology $H_*(\frac{1}{5}B; F)$ may be not finitely generated but the content of B is finite just the same. Notice also that balls of radii $>$ diam V are equal to V and so

$$\text{cont}(V) = \sum_0^n b_i(V; F).$$

Take a ball B and cover the concentric ball $\frac{1}{5}B$ by some open balls B_i,

$i=1,\ldots,N$, all of the same radius. Consider also the concentric coverings $\{\lambda_j B_i\}$, $j=0,1,\ldots,n+1$, $\lambda_j=10^j$. Suppose that all balls $5\lambda_j R_i$, $j=1,\ldots,n+1$, $i=1,\ldots,N$, are contained in B and let the contents of all these balls be bounded by a constant p, that is

$$\operatorname{Cont}(5\lambda_j B_i)\leqslant p, \qquad j=0,\ldots,n+1, \qquad i=1,\ldots,N.$$

Denote by J the index of the system $\{5\lambda_{n+1}B_i\}$, $i=1,\ldots,N$.

The content of B satisfies the following inequality

$$\operatorname{Cont}(B)\leqslant(n+1)pJ.$$

Proof. The ranks of the inclusion homomorphisms between all non-empty intersections of our balls,

$$H_*(\lambda_j B_{i_1}\cap\cdots\cap\lambda_j B_{i_k})\to H_*(\lambda_{j+1}B_{i_1}\cap\cdots\cap\lambda_{j+1}B_{i_k}),$$

are estimated in terms of contents by interpolating pairs of balls,

$$\lambda_j B_{i_1}\cap\cdots\cap\lambda_j B_{i_k}\subset\lambda_j B_{i_1}\subset 5\lambda_j B_{i_1}\subset\lambda_{j+1}B_{i_1}\cap\cdots\cap\lambda_{j+1}B_{i_k}.$$

Then Leray's spectral sequence applies. See Appendix for the details.

2.4. *Main covering lemmas*

We return to the ball B of radius R as in 2.2 and we assume that $\inf K\geqslant 0$, or more generally, that $2\kappa R\leqslant\exp(-n^n)$, for $-\kappa^2\leqslant\inf K$.

2.4.A. *Let for some number $p>0$, the content of each ball of radius $r\leqslant 0.01R$ which intersects the ball $\frac{1}{5}B$ is bounded from above by p. Then*

$$\operatorname{Cont}(B)\leqslant(n+1)pJ, \quad \text{for} \quad J=2^M \quad \text{and} \quad M=8^n\,10^{n^2+4n}.$$

Remark. The numbers 0.01 and $\frac{1}{4}$ play no role here, but we shall need them later on.

Proof. According to 2.2.A the ball $\frac{1}{5}B$ can be covered by M balls B_i of radius $2\cdot 10^{-n-4}R$ and the topological lemma applies.

2.4.B. *Let V be a compact manifold of diameter D and let $\inf K\geqslant-\kappa^2$. Let p and ε_0 be positive numbers such that each ε-ball, $\varepsilon\leqslant\varepsilon_0$, has content at most p, and* *such that* $\varepsilon_0\kappa \leqslant \exp(n^{-n})$. *Then*

$$\operatorname{Cont}(V) = \sum_0^n b_i(V; F) \leqslant (n+1)pJ, \quad \textit{for} \quad J = \text{const}\, D^n \varepsilon_0^{-n} \exp(n\kappa D),$$

where for const = const (n) *one can take* 2^M, $M = 10^{n^2+5n}$.

Proof. Use a minimal ε-covering of V for $\varepsilon = 5 \cdot 10^{-n^{-3}}\varepsilon_0$ and apply 2.2.B and 2.3.

§3. Proof of the theorems 0.2.A and 0.2B

3.1 *Rank and Corank*

A ball B of radius R in a complete manifold V is called $\frac{1}{10}$-*critical* if there is a point $y \in V$ such that it is critical for the center x of B and such that $\operatorname{dist}(x, y) = 10R$.

Now let V have non-negative curvature. Take an arbitrary set A and define its *corank*, corank (A), as the maximal integer k, such that there exist some $\frac{1}{10}$-critical balls $B_1, \dots, B_k$ with the following two properties.

(1) The radii R_i of B_i satisfy the inequalities $R_i \geqslant 3R_{i+1}$, $i = 1, \dots, k-1$.

(2) The intersection $\bigcap_1^k B_i$ contains A.

There exists a positive integer $k_0 \leqslant (100)^n$, $n = \dim(V)$, *such that for every set* A *we have*

$$\operatorname{corank}(A) \leqslant k_0.$$

Remark. This proposition bounds the number of "essential directions" in V and the condition $K \geqslant 0$ is crucial.

Proof. Let $x_i \in V$ denote the centers of the balls B_i, $i = 1, \dots, k$, and let y_i be the corresponding critical points for x_i with $\operatorname{dist}(x_i, y_i) = 10R_i$. Take a point z in $A \subset \bigcap_1^k B_i$ and join it by shortest segments γ_i with each of the points y_i.

If $k > (100)^n$, then there are two segments, γ_{i_1} and γ_{i_2}, $i_2 > i_1$, such that the angle between them at z is at most $\frac{1}{6}$ (see 1.5.A.). Now we argue as in Section 1.4. Set

$$l_1 = \operatorname{dist}(z, y_{i_1}) = \operatorname{length}(\gamma_{i_1}), \qquad l_2 = \operatorname{dist}(z, y_{i_2}) = \operatorname{length}(\gamma_{i_2}),$$

$$r_1 = \operatorname{dist}(z, x_{i_1}) \leqslant R_{i_1}, \qquad r_2 = \operatorname{dist}(z, x_{i_2}) \leqslant R_{i_2}, \qquad l = \operatorname{dist}(y_{i_1}, y_{i_2}).$$

The triangle inequality implies that

$$l_1 \geqslant 10R_{i_1} - r_1 \geqslant 9R_{i_1}, \qquad l_2 \leqslant 10R_{i_2} + r_2 \leqslant 11R_{i_2}.$$

Since $R_{i_1} \geqslant 3R_{i_2}$, we conclude that $l_1 > l_2$. Using 1.2.B, we get

$$l \leqslant l_1 - \tfrac{3}{4}l_2. \tag{*}$$

Let d denote the distance between x_{i_2} and y_{i_1}, $d = \operatorname{dist}(x_{i_2}, y_{i_1})$. By the triangle inequality we have

$$d = l_1 - r_2 \geqslant 10R_{i_1} - r_1 - r_2 \geqslant 8R_{i_1} \geqslant 24R_{i_2} \geqslant 20R_{i_2} = 2\operatorname{dist}(x_{i_2}, y_{i_2}),$$

and so we can apply the inequality (*) in 1.3 with y_{i_1} in place of z and with x_{i_2} and y_{i_2} in place of x and y. We get $d \leqslant l + 5R_{i_2}$, and by the triangle inequality we have

$$l_1 \leqslant d + r_2 \leqslant d + R_{i_2} \leqslant l + 6R_{i_2}. \tag{**}$$

The triangle inequality also implies that $l_2 \geqslant 10R_{i_2} - r_{i_2} \geqslant 9R_{i_2}$, and together with (*) this yields $l \leqslant l_1 - \frac{27}{4}R_{i_2}$, but this contradicts to (**). Q.E.D.

Now, if we have a manifold V with $\inf K \geqslant -\kappa^2$, we change the notion of the corank by adding the condition $2R_1\kappa \leqslant 10^{-10}$ and the inequality $\operatorname{corank}(A) \leqslant k_0 \leqslant (100)^n$ holds true. Now we set:

$$k_0 = \sup_{A \subset V} \operatorname{corank}(A), \quad \text{and} \quad \operatorname{rank}(A) = k_0 - \operatorname{corank}(A).$$

3.2. *Inductive lemmas*

Let B be a ball in V of rank zero. Then the content of this ball (see 3.2) is equal to one. In fact, if we look at the distance function dist_x, where x is the center of B, we shall see that it has no critical points in B; otherwise, for a sufficiently small concentric ball εB, we would get a contradiction, $\operatorname{corank}(\varepsilon B) \geqslant \operatorname{corank}(B) + 1$. The isotopy lemma (see 1.1) now shows that B is contractible and so $\operatorname{cont}(B) = 1$.

Denote by $\mathfrak{B}(k)$ the set of all balls in V of rank $\leqslant k$ and let p_k denote the upper bound

$$\sup_{B \in \mathfrak{B}(k)} \operatorname{cont}(B).$$

3.2.A. *Let V be a complete n-dimensional manifold of non-negative curvature. Then for each* $k=0,1,2,\ldots$, *the number* p_{k+1} *satisfies the inequality* $p_{k+1} \leqslant (n+1)Jp_k$, *where the constant* $J=J(n)$ *is the same as in* 2.4.A.

Since $k \leqslant k_0 \leqslant (100)^n$, this lemma shows that

$$\sum_0^n b_i = \operatorname{cont}(V) \leqslant ((n+1)J)^{100^n},$$

and this implies theorem 0.2.A. The proof of 3.2.A is given in the next section.

Notice that the lemma 3.2.A and its proof immediately extend to the general case of $\inf K \leqslant -\kappa^2 < 0$ if one modifies the definition of the numbers p_k by replacing the set $\mathcal{B}(k)$ by the subset consisting of the balls of radius $<\varepsilon_0$ for $\varepsilon_0 = \frac{1}{2}\kappa^{-1}\exp(-n^n)$. In view of 2.4.B, this general form of 3.2.A yields theorem 0.2.B.

3.3. *Incompressible balls*

Let V be a complete Riemannian manifold. A ball B in V of radius $R>0$ is called *compressible* if there exists a ball B' in V of radius $R' \leqslant \frac{1}{2}R$, such that B' is contained in B and such that there is an isotopy of V which is fixed outside B and which sends the ball $\frac{1}{5}B$ into $\frac{1}{5}B'$. It is clear that $\operatorname{cont}(B') \geqslant \operatorname{cont}(B)$, and so we conclude.

Each ball B contains an incompressible ball B_0 *such that* $\operatorname{cont}(B_0) \geqslant \operatorname{cont}(B)$. Now the inclusion $B_0 \subset B$ implies $\operatorname{rank}(B_0) \leqslant \operatorname{rank}(B)$, and lemma 3.2.A becomes equivalent to the following more special lemma.

3.3.A *Let V be as in* 3.2.A, *and let B be an incompressible radius R ball in V of rank* $k+1$, $k=0,1,\ldots$. *Then*

$$\operatorname{cont}(B) \leqslant (n+1)Jp_k.$$

Proof. According to 2.4.A, we only have to show that each ball $\tilde{B}$, of radius $r<0.01R$ with the center at a point $\tilde{x}$ in the ball $\frac{1}{4}B$, has rank at most k. Look at the distance function $\operatorname{dist}_{\tilde{x}}$ and let us find an appropriate critical point of this function. Take the concentric ball $\tilde{B}' = (R/2r)\tilde{B}$ of radius R/2. This ball is contained in B but it contains the ball $\frac{1}{5}B$. Since by our hypothesis the ball B can not be compressed to $\tilde{B}'$, we conclude, in view of the isotopy (see 1.1), that the function $\operatorname{dist}_{\tilde{x}}$ must have a critical point $\tilde{y}$ such that

$$\tfrac{1}{10}R \leqslant \operatorname{dist}(\tilde{x}, \tilde{y}) \leqslant \tfrac{1}{2}R.$$

Now, by the definition of rank, there are some $\frac{1}{10}$-critical balls $B_1, \ldots, B_l$, $l = k_0 - k$, containing B and we take for B_{l+1} the ball concentric to B of radius $\frac{1}{10}$ dist (x, y). This ball contains $\tilde{B}$ and its radius is at least ten times less than the radius of the minimal of the balls $B_1, \ldots, B_l$. So the conditions (1) and (2) in 3.1 are met and rank $(\tilde{B}) <$ rank (B). Q.E.D.

Appendix: Leray spectral sequence

(1) *Filtered and graded spaces.* Recall, that a filtered vector space $\{F^i X\}$, $i = 0, 1, \ldots, n+1$, is defined as a decreasing sequence of subspaces

$$X = F^0 X \supset F^1 X \supset \cdots \supset F^n X \supset F^{n+1} X = \{0\}$$

The associated graded space to a filtered space $\{F^i X\}$ is the space

$$\operatorname{Gr} X = \bigoplus_{i=0}^{n} \operatorname{Gr}^i X,$$

where

$$\operatorname{Gr}^i X = F^i X / F^{i+1} X$$

A homomorphism f between two filtered spaces $\{F^i X\}$ and $\{F^i Y\}$ is, by definition, a linear map $f: X \to Y$ such that it sends each subspace $F^i X$ to $F^i Y$, $i = 0, \ldots, n+1$. Every such f gives rise to a *graded homomorphism* $\operatorname{Gr} f$, that is a linear map $\operatorname{Gr} X \to \operatorname{Gr} Y$ which sends each $\operatorname{Gr}^i X$ to $\operatorname{Gr}^i Y$. It is clear that

$$\operatorname{rank}(\operatorname{Gr} f) \leq \operatorname{rank}(f),$$

but the equality does not in general hold.

Now, consider a sequence of filtered spaces $\{F^i X_j\}$, $i, j = 0, 1, \ldots, n+1$, and a sequence of homorphisms

$$f_j : \{F^i X_j\} \to \{F^i X_{j+1}\}.$$

Denote by $f: \{F^i X_0\} \to \{F^i X_{n+1}\}$ the composition, $f = f_n \circ f_{n-1} \circ \cdots \circ f_0$.

LEMMA. *The rank of the homomorphism f satisfies the following inequality*

$$\operatorname{rank}(f) \leq \sum_{j=0}^{n} \operatorname{rank}(\operatorname{Gr} f_j)$$

Proof. A standard induction reduces the lemma to the case of $n=1$. Now, we have the following commutative diagram where the horizontal lines are exact

$$\begin{array}{ccccc} F^1X_0 & \to & X_0 & \to & \mathrm{Gr}^0 X_0 \\ \downarrow & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle \mathrm{Gr}^0 f_0} \\ F^1X_1 & \to & X_1 & \to & \mathrm{Gr}^0 X_1 \\ {\scriptstyle F^1f_1}\downarrow & & \downarrow{\scriptstyle f_1} & & \downarrow \\ F^1X_2 & \to & X_2 & \to & \mathrm{Gr}^0 X_2. \end{array}$$

It is clear that

$$\operatorname{rank}(f_1 \circ f_0) \leq \operatorname{rank}(\mathrm{Gr}^0 f_0) + \operatorname{rank}(F^1 f_1).$$

(2) *Coverings and spectral sequences.* Let us recall some relevant fact on Leray's sequence (see [14]). For a set A in an n-dimensional manifold V we denote by $H_*(A)$ the total homology of A over a fixed coefficient field F,

$$H_*(A) = \bigoplus_0^n H_i(A; F).$$

Let $B_1, \ldots, B_N$ be some open sets in V. Then the homology of the union $A = \bigcup_1^N B_k$ carries a natural filtration $\{F^i H_*(A)\}$, $i = 0, 1, \ldots, n+1$. This is *not* the filtration associated to the grading $H_* = \bigoplus_0^n H_i$. If we take some larger open sets $B'_k \supset B_k$, then for their union A' the inclusion map $H_*(A) \to H_*(A')$ is a (filtered) homomorphism.

With the sets B_k above one associates Leray's spectral sequence that is a sequence $E_1, E_2, \ldots,$ of vector spaces with the following properties.

(i) For each multiindex $\mu = \{i_1, \ldots, i_l\} \in I_+$ (see 2.2) we denote by H_*^μ the homology of the intersection $B_{i_1} \cap B_{i_2} \cap \cdots \cap B_{i_l}$. Then the space E_1 is isomorphic to $\bigoplus_{\mu \in I_+} H_*^\mu$.

(ii) Each space $E_1, E_2, \ldots,$ has an additional structure of a complex, that is there are differentials $d_1: E_1 \to E_1$, $d_1^2 = 0$, $d_2: E_2 \to E_2$, $d_2^2 = 0, \ldots$. Furthermore each space E_{i+1} is obtained as the homology group of (E_i, d_i).

These structures are functional, that is the inclusions $B_i \to B'_i$ induce some homomorphisms $E_i \to E'_i$ which commute with d_i. The first homomorphism, $E_1 \to E'_1$, corresponds to the inclusion homomorphisms

$$f_\mu: H_*(B_{i_1} \cap \cdots \cap B_{i_l}) \to H_*(B'_{i_1} \cap \cdots \cap B'_{i_l}), \qquad \mu = (i_1, \ldots, i_l).$$

In particular, *the rank of the homomorphism* $E_1 \to E'_1$ *is equal to the sum* $\sum_{\mu \in I+} \operatorname{rank}(f_\mu)$. Since each space E_{i+1} is the homology of (E_i, d_i) the rank of each homomorphism $E_i \to E'_i$ is bounded by the sum $\sum_\mu \operatorname{rank}(f_\mu)$.

(iii) For a sufficiently large i_0 the differentials d_i, $i > i_0$, vanish and so the sequence E_i stabilizes. The stable terms are denoted by E_∞. This space E_∞ is *functorially* isomorphic to the graded space associated to the filtered homology $\{F^i H_*(A)\}$. The word "functorially" means that the homomorphism $E_\infty \to E'_\infty$ corresponding to the inclusions $B_k \to B'_k$ is equal to the graded homomorphism associated to the (filtered) inclusion homomorphism $H_*(A) \to H_*(A')$.

(3) The following proposition generalizes the topological lemma of 2.3.

Let $B^i_k \subset V$, $k = 1, \dots, N$, $i = 0, 1, \dots, n+1$, $n = \dim(V)$, *be some open sets such that*

$$B^0_k \subset B^1_k \subset \cdots \subset B^{n+1}_k, \qquad k = 1, \dots, N.$$

Let A^i *denote the unions* $\bigcup_{k=1}^n B^i_k$ *and let* f^i_μ *denotes the inclusion homomorphisms*

$$H_*(B^i_{i_1} \cap \cdots \cap B^i_{i_l}) \to H_*(B^{i+1}_{i_1} \cap \cdots \cap B^{i+1}_{i_l}), \qquad (i_1, \dots, i_l) = \mu \in I_+.$$

Then the rank of the inclusion homomorphism $H_*(A^0) \to H_*(A^{n+1})$ *is bounded from above by the sum*

$$\sum_{\substack{i=0,\dots,n \\ \mu \in I_+}} \operatorname{rank}(f^i_\mu).$$

Proof. The properties of the spectral sequence imply that the rank of each homomorphism $E^i_\infty \to E^{i+1}_\infty$ is bounded by

$$\sum_{\mu \in I_+} \operatorname{rank}(f^i_\mu),$$

and the lemma in (1) applies.

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