The equivariant Dehn's lemma and loop theorem.

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The equivariant Dehn's lemma and loop theorem

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Introduction

In [4] the authors observed that the topological methods in the theory of three-dimensional manifolds can be modified to settle some old problems in the classical theory of minimal surfaces in euclidean space (see also [1], [12]). In [4] and [5] we found that we could use the theory of minimal surfaces to extend the theorems of Papakriakopoulous, Whitehead and Shapiro, Stalling and Epstein on the Dehn's lemma, loop theorem and sphere theorem. The key point to our approach to these topological theorems is the following: Given a certain family of maps of the disk or sphere into our three-dimensional manifold M, we minimize the area of the maps (with respect to the pulled back metric) in this family and prove the existence of the minimal map. Then by using the area minimizing property of the map and the tower construction in topology, we prove that any area minimizing map in the family is an embedding. In this way, we realize the solutions to the above topological theorems by minimal surfaces. In [4] and [5] we used the above area minimizing solutions to prove equivariant versions of the loop and the sphere theorem, and we applied these new theorems to the classification of compact group actions on $\mathbf{R}^3$ in [11].

In this paper we generalize some of the theorems in [4] and [5] to compact planar domains by proving the existence of embedded planar domains of least area of a given genus and by proving a certain disjointness property for planar domains of least area. We then use this disjointness property to prove the equivariant Dehn's lemma for planar domains.

On the other hand, we use a different variation approach to get a geodesic version of the loop theorem. More precisely, we prove the following: suppose that the induced map $i_*:\pi_1(\partial M)\to\pi_1(M)$ of the inclusion of the boundary has nontrivial kernel K. Then for any metric on ∂M, any nontrivial geodesic of least length in K is embedded and any two such geodesics are equal or disjoint. This geodesic loop theorem coupled with the above equivariant Dehn's lemma yields a new version of the equivariant loop theorem in [5]. As the placement of curves on a surface is easier to understand this new equivariant loop theorem is easier to apply to study group actions. Applications of this theorem to classification of group actions on $\mathbf{R}^3$ will appear in [11].

Throughout this paper we will be working with compact three-dimensional Riemannian manifolds M with convex boundary. For simplicity we sometimes refer to such an M as a *convex* manifold.

1. Dehn's lemma for planar domains

THEOREM 1 (Dehn's lemma for planar domains of a given genus). *Let $\Gamma=\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ be a collection of disjoint unoriented Jordan curves on the boundary of a three-dimensional orientable convex manifold M. Suppose that these Jordan curves bound a continuous mapping g from a smooth compact planar domain (possibly disconnected). Let F_k be the family of all piecewise smooth maps mapping from a compact planar domain with k components into M whose boundary consists of curves in Γ. Let A_k be the infimum of the areas of the maps in F_k. If A_k is strictly less than A_{k+1}, then there exists a branched minimal immersion which has least area among maps in F_k. Furthermore, any branched minimal immersion of least area in F_k is an embedding.*

Proof. The existence of a map $f:\Omega\to M$ of planar domain with k components and least area follows from the inequality $A_k<A_{k+1}$, from Morrey [7] and from Theorem 1 in [4]. From the approximation technique in the proof of Theorem 5 in [4], we may assume that the map f is a simplicial immersion with respect to some triangulations of Ω and M.

Since $f:\Omega\to M$ is a map of least area for a given genus, f restricted to each component Ω' of Ω is a map of least area from a planar domain with boundary curves $f(\partial\Omega')$. By Theorem 5 in [4], $f\,|\,\Omega'$ is an embedding. Suppose that there are two distinct components Ω_1, Ω_2 of Ω such that $f(\Omega_1)$ and $f(\Omega_2)$ intersect. In this case it is shown in [4] that there are Jordan curves $\gamma_1: S^1\to\mathring{\Omega}_1$ and $\gamma_2: S^1\to\mathring{\Omega}_2$ such that $f(\gamma_1(t))=f(\gamma_2(t))$. The standard cutting and gluing argument (see the end of the Proof of Theorem 5 in [4]) along the image curve $f(\gamma_1)=f(\gamma_2)$ produces a map of a planar surface with the same Euler characteristic as Ω and with the same area as f. However, the area of the new map can be decreased along the folding curve $f(\gamma_1)$. Since the Euler characteristic of a planar domain with n boundary curves determines the genus and the number of components, the existence of the new map contradicts the least area property for f. This contradiction proves Theorem 1.

In [4] the authors also proved a disjointness property for least area disks when Γ in the above theorem consists of one curve γ. In that paper we prove that any two geometrically distinct least area disks intersect only along their boundary. This disjointness property for least area disks is useful in proving equivariant group action theorems. For this reason we would like to generalize the disjointness property to the case of planar domains given in the above theorem. However, in the following example two Jordan curves in parallel planes in $\mathbf{R}^2$ are given which bound two distinct embedded annuli of least area that intersect their interiors.

EXAMPLE. Let δ_{-1000} be a circle of radius 10 in the xy plane centered at the point $(0, -1000, 0)$ and let δ_{1000} be a circle of radius 10 in the xy plane centered at the point $(0, 1000, 0)$. Let γ_1 be the connected sum of δ_{-1000} and δ_{1000} along part of the interval I joining $(0, -1000, 0)$ to $(0, 1000, 0)$ in such a way that γ_1 is the union of parts of δ_{-1000}, δ_{1000}, and the intervals $I+(-1, 0, 0)$ and $I+(1, 0, 0)$. Let $\gamma_2 = \gamma_1 + (0, 0, 1)$ be the curve on the plane of distance one from the xy plane. A least area annulus $f: \Omega \to \mathbf{R}^3$ connecting γ_1 and γ_2 appears as in Figure 1. Let $R: \mathbf{R}^3 \to \mathbf{R}^3$ be rotation by 180 degrees around the z-axis. Then the least area annuli $f(\Omega)$ and $R \circ f(\Omega)$ intersect in their interiors. (A rigorous proof of the existence of Ω produced in this example can be found by using the bridge theorem in [6].)

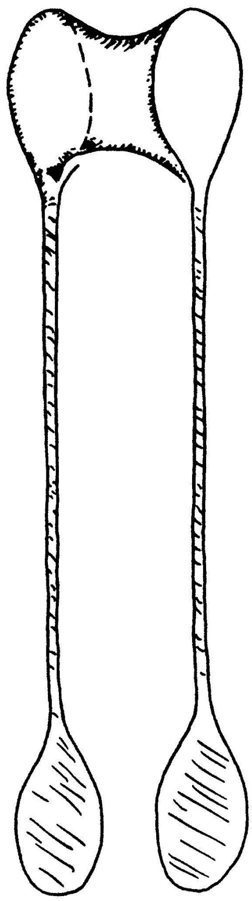


Figure 1.

In spite of this example, the disjointness property holds when the following assumptions on Γ hold.

THEOREM 2. *Let $\Gamma=\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ be a collection of disjoint unoriented Jordan curves on the boundary of a compact three-dimensional orientable convex manifold M. Suppose that γ_1 is homotopically nontrivial when n equals two or that Γ generates a rank $(n-1)$ subgroup of the first homology group of M. If there exists a continuous map g of a compact planar domain into M with boundary Γ, then*

(1) *there exists a branched minimal immersion of a compact planar domain which bounds Γ and has (finite) least area among all such maps.*
(2) *Every such map is an embedding of a connected planar domain.*
(3) *Any two such least area maps intersect only along their boundary Γ or else they differ by a conformal reparametrization.*

Proof. Part (1) is just the statement of Theorem 5 in [4]. Part (2) follows because the condition that the curves in Γ represent $n-1$ independent homology classes implies the connectedness of the surface. The proof of part (3) is based on the proof of Theorem 6 in [4]. The nontrivial approximation procedure in Theorem 6 in [4] reduces part (3) to the special case that the two least area maps $f:\Omega_1 \to M$ and $g:\Omega_2 \to M$ are simplicial with respect to some fixed triangulations of Ω_1, Ω_2 and M.

Suppose now that $X=f_1(\Omega_1)\cap f_2(\Omega_2)$ is not equal to the union of Γ. In [4] it is shown that X is a finite one-dimensional subcomplex of M with every vertex in X meeting at least two edges in X and the intersection of $f(\Omega_1)$ and $f(\Omega_2)$ is traverse except possibly at the vertices. A simple induction argument (see Lemma 10 in [4]) proves that X contains a closed Jordan curve α which is not contained in the union of Γ or for some i and $k>0$ there is a unique Jordan arc $\sigma:[0, 1]\to X$ with $\sigma([0, 1])\cap\Gamma=\{\sigma(0), \sigma(1)\}$ and $\sigma(0)\in\gamma_i$ and $\sigma(1)\in\gamma_{i+k}$.

Suppose that σ exists. By the classification of compact planar surfaces, there would be a smooth Jordan curve τ in the interior of Ω_1 such that $\tau\cap X=\tau\cap\sigma$ is one point which is not vertex and the intersection of τ and σ on Ω_1 is transverse at this point. As τ intersects Ω_2 transversely in one point, $[\tau]\cap[\Omega_2]$ is nonzero where $\cap$ denotes the intersection pairing on homology in M with Z_2-coefficients. However, as Ω_1 is a compact planar domain, τ is homologous with Z_2-coefficients to some sum of boundary curves of Ω_1. As the boundary curves of Ω_1 and Ω_2 are the same, some boundary curve γ_i of Ω_2 must intersect Ω_2 nontrivially in homology. However, M is orientable and therefore we can push γ_i off Ω_2 to create a curve γ_i' which is disjoint from Ω_2. This curve is homologous to γ_i but does not intersect Ω_2. This contradicts the intersection equation on homology and therefore σ can not exist. Hence there must be a Jordan curve α in X which is not contained in Γ.

Let $\alpha_1 : S^1 \to \Omega_1$ and $\alpha_2 : S^1 \to \Omega_2$ be the Jordan curves with $f(\alpha_1(t)) = g(\alpha_2(t))$ and $f(\alpha_1) = \alpha$. Suppose for the moment that α_1 and α_2 are contained in the interior of Ω_1 and Ω_2. The curve α_i disconnects Ω_i into two planar domains Ω_i', Ω_i'' where Ω_i' is the planar domain containing the Jordan curve γ_1.

Now consider the surface Σ obtained by gluing $f(\Omega_1'')$ and $f(\Omega_2'')$ along α. If Σ has a nonempty boundary, then for some i different from 1, an oriented boundary curve γ_i of Σ is homologous in Σ to a collection of curves in $\{\pm\gamma_2, \pm\gamma_3, \ldots, \pm\hat{\gamma}_i, \ldots, \pm\gamma_n\}$ where for the moment the curves Γ are oriented in an arbitrary manner. Therefore the curves $\Gamma - \{\gamma_1, \gamma_i\}$ generate a subgroup of $H_1(M, Z)$ with the same rank as Γ which is $n-1$. For $n \geq 3$, this contradicts our assumptions. If $n = 2$, then X is a disk and so γ_2 is homotopically trivial. This also contradicts our assumptions and so Σ must have no boundary.

As Σ has no boundary, the surfaces $f(\Omega_1')$ and $f(\Omega_2')$ have the same boundary curves. The usual cutting and gluing argument shows that f or g does not have least area and hence part (3) is valid if the Jordan curve α_1 lies in the interior of Ω_1. Actually the only reason that we chose the case "α_1 lies in the interior of Ω_1" was to make visualization of the intersection easier. The same argument still produces a contradiction when part of α intersects the union of the curves in Γ. This proves part (3) and completes the proof of the theorem.

Remark. Theorem 2 can be proved by assuming appropriate conditions about areas of planar domains which bound some subcollection of curves in Γ rather than topological conditions. For example, suppose that $\Gamma = \{\gamma_1, \gamma_2\}$ and that either γ_1 or γ_2 does not bound a disk with area less than twice the area of some annular region joining them. Then the planar domain of least area joining γ_1 and γ_2 will be an embedded annulus and any two such annular surfaces intersect only along their boundary curves. Note that this area condition fails for the example described before Theorem 2.

2. Embedding of the partially free boundary value problem

Another type of embedding theorem that can be proved using the topological tower construction is the partially free boundary value problem considered in Courant's book [2]. In its simplest topological form the partially free boundary value problem can be stated as follows. Let M be a compact three-dimensional Riemannian manifold and γ_1 be a Jordan curve on a boundary component ∂_1 of M which is freely homotopic to a closed curve γ_2 on another component ∂_2 of the boundary of M. Let F be the family of all maps from the annulus Ω to M which maps one boundary curve of Ω homeomorphically onto γ_1 and the other boundary curve of Ω into ∂_2. Then we say that a minimal immersion $f:\Omega\to M$ is a solution to the partially free boundary value problem for γ_1 and ∂_2 if $f\in F$ and f has least area in F.

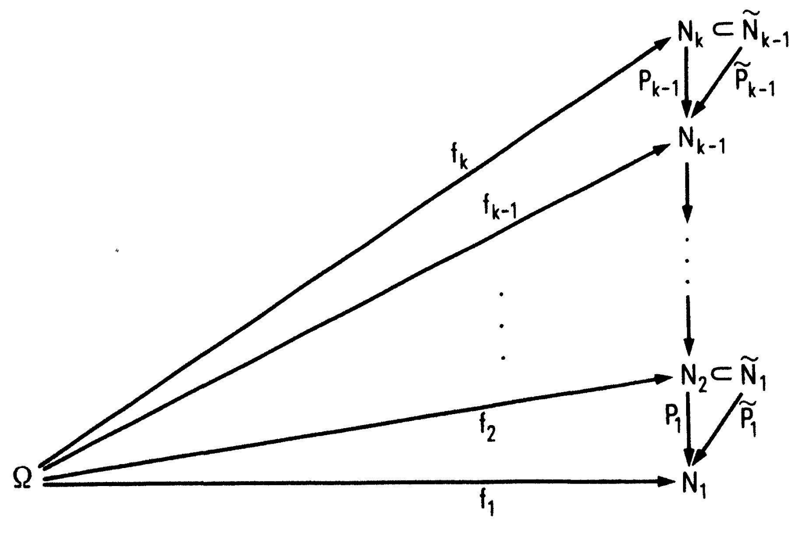
THEOREM 3. *Suppose M is a compact orientable Riemannian three-dimensional manifold with convex boundary, γ_1 is a Jordan curve on a component ∂_1 of the boundary of M and γ_1 is freely homotopic to a curve on a different component ∂_2 of the boundary. Then*

(1) *There exists a solution $f:\Omega\to M$ to the partially free boundary value problem for γ_1 and ∂_2 if the infimum of areas of maps in F is strictly less than the area of any map of a disk with boundary γ_1. Furthermore f is continuous in Ω and smooth in the interior of Ω.*

(2) *Any such solution f is one-to-one and everywhere orthogonal to ∂_2.*

Proof. The existence of a solution to the partially free boundary value problem can be proved using the methods in the proof of the free boundary value problem in [5].

After conformal reparametrization we may assume that Ω is a circular domain where the inner circle is the unit circle S^1 and $f(S^1)=\gamma_1$. From the approximation arguments in [4] and [5] we may assume that the map f is simplicial with respect to some triangulations of Ω and M. Therefore the image surface $f(\Omega)$ has a regular neighborhood N_1 in M. After restricting the range space of f to N_1, there is a new map $f_1:\Omega\to N_1$. Let H be the subgroup of $H_1(N_1, Z_2)$ generated by $f_1(S^1)$. If H is not all of $H_1(N_1, Z_2)$, then there exists a surjective homomorphism $\rho: H_1(N_1, Z_2)\to Z_2$ with $\rho([f_1(S^1)])=0$. This homomorphism induces a surjective



homomorphism $\bar{\rho}:\pi_1(N_1)\to Z_2$. Since the kernel of $\bar{\rho}$ has index two in $\pi_1(N_1)$, there is a 2-sheeted covering space $\tilde{P}_1:\tilde{N}_1\to N_1$ associated to this subgroup. Since the map $f:\Omega\to N_1$ satisfies $f_{1*}(\pi_1(\Omega))\subset\tilde{P}_{1*}(\pi_1(\tilde{N}_1))=\ker(\bar{\rho})$, the lifting theorem for covering spaces implies that f_1 lifts to a map $\tilde{f}_1:\Omega\to\tilde{N}_1$. After restricting the range of $\tilde{f}_1$ to a regular neighborhood N_2 of $\tilde{f}_2(\Omega)$, we get a new map $f_2:\Omega\to N_2$.

Repeating this construction k-times yields the tower below. As was discussed in [4] or [5], this construction terminates with a map $f_k:\Omega\to N_k$ with $f_k(S^1)$ generating $H_1(N_k, Z_2)$. Here P_i is the restriction of $\tilde{P}_i$ to N_{i+1} and each N_i is a Riemannian manifold with the pulled back Riemannian metric.

ASSERTION 1. *$f_k:\Omega\to N_k$ is one-to-one.*

Proof. As $H_1(N_k, Z_2)$ is generated by $f_k(S^1)$, $H_1(N_k, Z_2)$ is equal to the trivial group or the group Z_2. If $H_1(N_k, Z_2)$ is the trivial group, it is straightforward to check that the boundary of N_k consists entirely of spheres (see [4] for a proof). In this case $\gamma=f_k(S^1)$ lies on some sphere S in the boundary of N_k.

In [4] and [5] it is shown that there exists, after subdivision, a simplicial retraction $R:N_k\to f_k(\Omega)$ such that (1) $R\mid(\partial N_k-f_k(\partial\Omega))$ is locally one-to-one, and (2) $R\mid\partial N_k$ covers each 2-simplex of $f_x(\Omega)$ exactly two times.

The Jordan curve γ disconnects the sphere S into two disks D_1 and D_2. Computing areas, we have

$$\text{Area}\,(R\mid D_1)+\text{Area}\,(R\mid D_2)=\text{Area}\,(R\mid S)\le\text{Area}\,(R\mid\partial N_k)\le 2\,\text{Area}\,(f_k).$$

Hence either the area of $R\mid D_1$ or $R\mid D_2$ is not greater than Area $(f_k)=$ Area (f_1). Therefore we may assume that the area of, say, $g=P_1\circ P_2\circ\cdots\circ P_{k-1}\circ R\mid D_1$ is not greater than the area of f. Furthermore, the area of g can be decreased along a folding curve which is a self-intersection curve of $f_k(\Omega)$ in the case $f_k(D)$ is not embedded (see Theorem 4 in [4] for a rigorous proof of this fact). This contradicts the original assumption that f is a solution to the partially free boundary value problem.

Thus we may assume that $H_1(N_k, Z_2)$ is Z_2. In this case it is easy to show that $f_k(\partial\Omega)$ is contained in a torus component T of the boundary of N_k (see the proof of Theorem 5 in [4]). Furthermore, as $H_1(N_k, Z_2)$ is generated by $f(S_1)$, the boundary curves of $f_k(\Omega)$ are disjoint and are nontrivial homology classes on ∂N_k. From the simple topology of curves on a torus we may conclude that $f_k(\partial\Omega)$ disconnect T into a collection of closed planar domains, two of which are annular regions A_1 and A_2 where the boundary of the annular region A_i consists of $f_k(S^1)$ and part of the other boundary curve of $f_k(\Omega)$.

Let $R:N_k\to f_k(\Omega)$ be the retraction discussed above. Then, as before, Area $(R\mid A_1)+$ Area $(R\mid A_2)\le$ Area (f_k), and so we may assume that

Area $(R \mid A_1)$ is strictly less than Area (f_k). However, the boundary curves of $g = (P_1 \circ P_2 \circ \cdots \circ P_{k-1} \circ R) \mid A_1$ consists of γ_1 and a curve on the boundary component ∂_2. As Area $(g) <$ Area (f), we arrive at a contradiction which shows the map f_k is one-to-one and proves Assertion 1.

ASSERTION 2. *f_{k-1} is one-to-one.*

Proof. If f_{k-1} is not one-to-one, then the map f_{k-1} has singular points which are double points. As f_{k-1} is everywhere orthogonal to the boundary of N_{k-1}, the maximum principle or Lemma 5 in [4] implies that the image of the boundary component of Ω different from S^1 is not completely contained in the singular set $S(f_{k-1})$. The arguments in [4] show that there exists a Jordan curve $\alpha_1 : S^1 \to \Omega$ or a Jordan arc $\alpha_1 : [0, 1] \to \Omega$ with $\alpha(0), \alpha(1) \in \partial\Omega$ which bounds with some part of $\partial\Omega$ a closed connected domain Ω_1 in Ω with $\Omega_1 \cap S(f_{k-1}) = \alpha_1$. Let α_2 be the double curve corresponding to α_1. By our choice of α_1, the Jordan curve α_2 will bound, with some parts of $\partial\Omega$, a closed subdomain Ω_2 of Ω whose interior is disjoint from Ω_1.

A cutting and gluing argument shows that we can interchange the region Ω_1 and Ω_2 to get a new continuous piecewise smooth map $g : \Omega \to f_{k-1}(\Omega)$ with the same area as $f_{k-1}(\Omega)$ and such that $G = P_1 \circ P_2 \circ \cdots \circ P_{k-2} \circ g$ is a candidate for a solution to the partially free boundary value problem. However, the area of G can be decreased along the folding curve α_1 which contradicts the least area property for f. This contradiction proves the assertion which in turn implies part (2) of the theorem.

Remarks. The previous theorem can be generalized in a number of interesting ways. For example, one can replace γ_1 by a collection $\Gamma_1 = \{\gamma_1, \gamma_2, \ldots, \gamma_k\}$ of pairwise disjoint Jordan curves and γ_2 by $\Gamma_2 = \{\alpha_1, \ldots, \alpha_n\}$ a collection of curves which lies on distinct boundary components of ∂M different from the boundary components containing the Jordan curves in Γ_1. In this case we assume that there is a map of a planar domain into M whose boundary curves are $\Gamma_1 \cup \Gamma_2$. One can then pose a partially free boundary value problem and if there is a least area solution to this problem, one can prove that the solution is embedded. The proof of this fact can be shown using the techniques of proof given in Theorem 3 and in Theorem 5 of [4].

It is important to note that the existence of embedded solutions to other free boundary value problems can also be shown. For example, suppose we replace the condition that γ_1 and γ_2 lie on distinct components of the boundary of M, by the condition that γ_2 lies in the complement of some compact piece P of the boundary surface containing γ_1. Then if a solution to this free boundary value problem exists and the boundary of the map is disjoint from ∂P, then the solution is an embedding. Such free boundary value problems occur naturally for, say, certain convex subsets of euclidean three space.

The solution to the free boundary value problem in [5] can be generalized to annular or even planar domains. For example, suppose that γ_1 is a loop on a boundary component ∂_1 of a convex M, which is homotopically nontrivial in M. Suppose γ_1 is homotopic to a loop γ_2 on a different boundary component ∂_2 of M. Then there exists an immersion $f:\Omega \to M$ of an annulus of least area with one boundary curve on ∂_1 and the other boundary curve on ∂_2 and so that the induced map on fundamental groups is nontrivial. Furthermore, f is as regular as the metric of M and any such f is one-to-one.

3. The equivariant Dehn's lemma

In [5] we proved the equivariant loop theorem by using the disjointness property of least area disks. The disjointness property in Theorem 2 for least area planar domains can also be used to prove the following equivariant theorem.

THEOREM 4 (Equivariant Dehn's lemma for planar domains). *Suppose $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ is a collection of smooth disjoint unoriented Jordan curves on the boundary of an orientable three-dimensional manifold M. Suppose either the curves in Γ generate a rank $(n-1)$ subgroup of the first homology group of M or n equals to two and the curve γ_1 is homotopically nontrivial in M. Suppose also that the collection Γ is the image of the boundary of a map of a compact planar domain into M. If G is a compact subgroup of* Diff$^+(M)$ *which acts freely on the union of Γ, then Γ is the boundary of an embedded compact planar domain in M which is invariant under those elements of G that leave some γ_i in Γ invariant.*

Proof. As G is compact, we may assume that G acts on M as a group of isometries. Furthermore, it is elementary to construct an invariant metric on M with convex boundary by averaging the metric on ∂M and taking the product metric in a neighborhood of ∂M. We may also assume that M is compact by restricting the manifold to a regular neighborhood of the G orbits of the image of the map of the compact planar domain given in the hypothesis.

By Theorem 1, there exists a smooth embedded connected compact planar domain Ω of least area in M with boundary curves in Γ. By Theorem 2, any two such least area planar domains are either disjoint in the interior of M or equal.

Suppose now that $g:M \to M$ is an element of G which leaves invariant the Jordan curve γ_i and suppose $g(\Omega) \neq \Omega$. As g leaves γ_i invariant and has no fixed points on γ_i, g acts on a regular neighborhood of γ_i as a rotation. As $g(\Omega) \neq \Omega$ and $g(\Omega)$ is another planar domain of least area, $g(\Omega)$ is disjoint from Ω in the interior of M. This implies that $g(\Omega)$ lies locally on one side of Ω. By convexity (see for example [4] or [6]), the surfaces Ω and $g(\Omega)$ are immersed and transverse to the boundary of M.

Let $J(\gamma_i'(t))$ denote the vector obtained by rotating the tangent vector $\gamma_i'(t)$ clockwise by 90 degrees in the tangent space of $T_{\gamma_i(t)}\partial M$ with respect to the induced orientation. Define $\alpha_\Omega(\gamma_i(t))$ and $\alpha_{g(\Omega)}(\gamma_i(t))$ as the oriented angle between the vector $J(\gamma_i'(t))$ and the tangent planes of the corresponding surfaces. After integrating along γ_i, we have

$$\theta_\Omega = \int_0^1 \alpha_\Omega(\gamma(t))\,dt \quad \text{and} \quad \theta_{g(\Omega)} = \int_0^1 \alpha_{g(\Omega)}(\gamma(t))\,dt.$$

As g acts as rotation on the regular neighborhood of γ_i, $\alpha_\Omega(g(\gamma(t))) = \alpha_{g(\Omega)}(\gamma(t))$ and hence $\theta_\Omega = \theta_{g(\Omega)}$. On the other hand, as Ω lies locally on one side of $g(\Omega)$, either $\alpha_\Omega(\gamma_i(t)) \le \alpha_{g(\Omega)}(\gamma_i(t))$ for all t or else $\alpha_\Omega(\gamma_i(t)) \ge \alpha_{g(\Omega)}(\gamma_i(t))$ for all t. As the integrals are the same, $\alpha_\Omega(\gamma_i(t)) = \gamma_{g(\Omega)}(\gamma_i(t))$. This shows that $g(\Omega)$ and Ω are everywhere tangential to each other along γ_1. Therefore, the maximum principle (or Lemma 5 in [4]) implies that Ω and $g(\Omega)$ intersect in an open set. Hence the disjointness property of Ω implies that $\Omega = g(\Omega)$. This completes the proof of the theorem.

THEOREM 5 (Equivariant Dehn's lemma for disks). *Suppose $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ is a collection of disjoint Jordan curves on the boundary of an orientable three-dimensional manifold M. Suppose each γ_i is homotopically trivial in M. If G is a compact group acting on M as a group of orientation preserving diffeomorphism which acts freely on the union of Γ, then there exists a collection of embedded invariant disks $\{D_1, D_2, \ldots, D_n\}$ which are pairwise disjoint with $\partial D_i = \gamma_i$ and whose union is invariant under G.*

Proof. After picking an invariant metric, G acts as a group of isometries. As in the previous lemma, we can assume that this metric is convex and M is compact. Let D_1 be a disk of least area with boundary curve γ_1 and let $G \cdot D_1$ denote the union of the least area disks which are images of D_1 under G. By the argument given in the previous theorem, D_1 is the only disk in $G \cdot D_1$ whose boundary curve is γ_1. This implies that each of the curves in $G \cdot \gamma_1$ bound a unique disk in $G \cdot D_1$.

If $G \cdot \gamma_1$ is not all of Γ, then let D_2 be a disk of least area with a boundary curve in $\Gamma \backslash (G \cdot \gamma_1)$ and $G \cdot D_2$ be the union of the orbits of D_2 under the action of G. As before, these are embedded and disjoint. As the disks in $G_1 \cdot D_1$ and $G \cdot D_2$ can only intersect in their interiors and as they have least area, they do not intersect. This last fact is proved in [4] where we show that if two embedded minimal disks intersect only in their interiors, then there is a closed Jordan curve in their intersection which bounds two least area disks. Then by a cutting and gluing argument we can decrease the area of one of the $G(D_i)$ which is impossible.

If $G \cdot \gamma_1 \cup G \cdot \gamma_2$ does not exhaust the curves in Γ, then we can find a new least area disk D_3 with boundary curve in $\Gamma - (G \cdot \gamma_1 \cup G \cdot \gamma_2)$. Let $G \cdot D_3$ be the orbits of D_3. Continuing this process eventually, we can produce the required disks $G \cdot D_1, G \cdot D_2, \ldots, G \cdot D_k$.

COROLLARY. *Suppose $\tau : M \to M$ is an orientation preserving diffeomorphism of a three-dimensional manifold M which is an isometry with respect to some metric on M. If τ leaves invariant a Jordan curve γ on the boundary of M which is homotopically trivial in M, then τ has a fixed point on M.*

Proof. Let G be the closure in $\mathrm{Diff}^+(M)$ of the cyclic subgroup generated by τ. As γ lies on the boundary of an orientable three-dimensional manifold, G restricts to an effective action on γ. Here G is either a finite cyclic group or S^1. By the previous theorem there is either a fixed point of τ on γ or else there is a disk in M which is invariant under τ. If τ has no fixed points on γ, then the Brower fixed point theorem implies that τ has a fixed point on the invariant disk. This proves the corollary.

4. The equivariant loop theorem

In this section we are going to prove the equivariant loop theorem by first proving a disjointness property of a certain generating set of closed geodesics on the boundary of the three-dimensional manifold and then applying the equivariant Dehn's lemma of Section 3. We begin with the following

DEFINITION. Let M be an n-dimensional compact Riemannian manifold and let H be a normal subgroup of $\pi_1(M)$. Then a collection $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_n, \ldots\}$ of closed geodesics is said to be a *short generating set* for H if for each n, γ_n represents a closed curve in H of least length in the complement of the normal subgroup of H generated by the free homotopy classes $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_{n-1}\}$.

LEMMA 1. *Suppose γ_1 and γ_2 are embedded distinct closed geodesics on a boundary surface Σ of a three-dimensional Riemannian manifold M. If γ_1 and γ_2 intersect nontrivially and are homotopically trivial in M, then one of these geodesics, say γ_2, can be expressed as the product of two closed nongeodesic curves in $\gamma_1 \cup \gamma_2$,* *each with length less than or equal to length of γ_2, and these nongeodesic curves are homotopically trivial in M.*

Proof. Since an embedded geodesic is determined by its tangent vector at a single point and the exponential map is a local diffeomorphism, it is easily seen that γ_1 and γ_2 intersect transversally in a finite number of points. Hence we may consider γ_1 and γ_2 as simplicial curves on Σ with respect to some triangulation of M. By Dehn's lemma there exist embedded piecewise linear disks D_1 and D_2 with boundary curves γ_1 and γ_2 respectively, which are in general position.

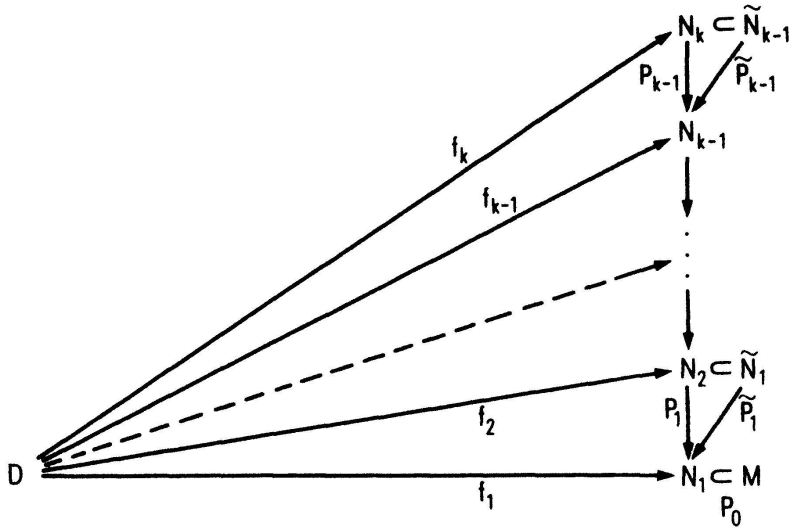
Since D_1 and D_2 are in general position, they intersect in a compact one-dimensional manifold with boundary. Let I be an interval component in $D_1 \cap D_2$. The interval I disconnects D_1 into two closed subdisks D_{11} and D_{12} and disconnects D_2 into two closed subdisks D_{21} and D_{22}. Let $\alpha_{ij} = D_{ij} \cap \Sigma$ and suppose that α_{11} is the shortest such arc. Then the length of the boundary of each of the disks $\tilde{D}_1 = D_{11} \cup_I D_{21}$ and $\tilde{D}_2 = D_{11} \cup_I D_{22}$ is less than or equal to the length of γ_2. Here $\cup_I$ means that we paste the disks along their common boundary arc I. On the other hand, γ_2 can be expressed as a product of $\partial\tilde{D}_1 \cdot \partial\tilde{D}_2 = (\alpha_{21}\alpha_{11}) \cdot (\alpha_{11}^{-1}\alpha_{22})$. Since $\partial\tilde{D}_1$ and $\partial\tilde{D}_2$ are not geodesics, $\partial\tilde{D}_1$ and $\partial\tilde{D}_2$ are the required closed curves. This completes the proof of the lemma.

THEOREM 6. *Suppose M is a compact orientable three-dimensional Riemannian manifold with a boundary component Σ. Let $K = \operatorname{Ker}(i_*)$ be the kernel of the map $i_*: \pi_1(\Sigma) \to \pi_1(M)$ induced by inclusion. Then with respect to any fixed metric on Σ, there exists a finite short generating set $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ for K. For any such generating set the geodesics in Γ are embedded. Furthermore, any two geodesics in the union of any two short generating sets are either equal or disjoint.*

Proof. We first show that there is a minimal generating set $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ for K consisting of embedded geodesics. Since there are only a finite number of free homotopy classes on a compact surface having length less than a given constant, we can choose a short generating set for K by sequentially picking the next free homotopy class of least length. To be precise, suppose by induction $\Gamma_{n-1} = \{\gamma_1, \gamma_2, \ldots, \gamma_{n-1}\}$ have been chosen. If Γ_{n-1} is not a short generating set, then we let γ_n be a closed geodesic of least length in the complement of the normal subgroup of K generated by Γ_{n-1}. We will now show that γ_n is embedded.

Since $\gamma_n: S^1 \to M$ is a closed geodesic, it is determined by its tangent vector at a point and its multiplicity which is the number of times it transverses the same path. As a geodesic of multiplicity one is always in general position with respect to itself, we may assume that $\gamma_n(S^1)$ is a simplicial curve with respect to some triangulation of M. Hence $\gamma_n: S^1 \to M$ is also simplicial with respect to the pulled back triangulation on S^1.

Let $f: D \to M$ be a branched simplicial immersion such that $f \mid \partial D = \gamma_n$. Then after restricting to a regular neighborhood N_1 of $f(D)$ in M, there is a new map $f_1: D \to N_1$. As in the proof of Dehn's lemma (see [4] or [1]), we can construct a tower where we may assume that the boundary of N_k consists entirely of spheres and each of the manifolds N_i are Riemannian with respect to the pulled back metric. Here $\tilde{P}_{i-1}: \tilde{N}_{i-1} \to N_{i-1}$ is the universal covering space of N_{i-1} and N_i is a regular neighborhood of the image of some lift $\tilde{f}_{i-1}$ to this universal covering space.



ASSERTION 1. *The lift f_k has an embedded boundary curve.*

Proof. Since $C = f_k(\partial D)$ lies on a sphere, every Jordan curve in the 1-complex C is homotopically trivial in N_{k-1}. As the fundamental group of C is generated as a $\pi_1(C, p)$ module by Jordan curves, there is a Jordan curve γ' in C such that $\gamma' = P_0 \circ P_1 \circ \cdots \circ P_{k-1}(\gamma)$ does not lie in the normal subgroup of K generated by Γ_{n-1}. If C is not a Jordan curve, then the length of γ_n is not minimal. This shows that C is a Jordan curve. Since C has less length than any nontrivial multiple of C and C is homotopically trivial, the lift $f_k \mid \partial D$ must be an embedding.

ASSERTION 2. *γ_n is embedded.*

Proof. If γ_n is not embedded, then there exists a smallest $m > 0$ such that $f_m \mid \partial D$ is not embedded. By the previous assertion $f_{m+1} \mid \partial D$ exists and is one-to-one. Let $\tilde{f}_m = i \circ f_{m+1}$ be the composition of f_{m+1} with the inclusion map into the total space of the universal covering space $\tilde{P}_m : \tilde{N}_m \to N_m$. By definition of $\tilde{f}_m$, $\tilde{f}_m$ is a lift of the map f_m to its universal covering space. Since f_m is not one-to-one, two points on $\tilde{f}_m(\partial D)$ must be identified under a nontrivial covering transformation τ: $\tilde{N}_m \to \tilde{N}_m$.

First, suppose $\tau(\tilde{f}_m(\partial D)) \neq \tilde{f}_m(\partial D)$. Then with respect to the pulled back metric on N_m, Lemma 1 implies that one of these geodesics, say $\tilde{f}_m(\partial D)$, can be expressed as a product of two closed nongeodesic curves α_1, α_2 with length $(\alpha_i) \leq$ length $(f_m(\partial D)) =$ length (γ_n). Hence either $P_0 \circ \cdots \circ P_{m-1} \circ \tilde{P}_m(\alpha_1)$ or $P_0 \circ \cdots \circ P_{m-1} \circ \tilde{P}_m(\alpha_2)$ does not lie in the normal subgroup of K generated by $\Gamma_{n-1} = \{\gamma_1, \gamma_2, \ldots, \gamma_{n-1}\}$ and has length less than the length of γ_n. This contradicts the least length property of γ_n and shows that γ_n is embedded in the case $\tau(f_m(\partial D)) \neq f_m(\partial D)$.

If $\tau(\tilde{f}_m(\partial D)) = \tilde{f}_m(\partial D)$, then by the Corollary to Theorem 5, τ has a fixed point in $\tilde{N}_m$ which implies that τ is the identity map contrary to our hypothesis about τ. This shows that this case can not occur and that γ_n is embedded. This ends the proof of Assertion 2.

By induction we can continue this process to find a short generating set Γ for K consisting of embedded geodesics. The argument given above also implies that any short generating set consists of embedded geodesics.

Let $\Gamma = \{\gamma_1, \ldots, \gamma_n, \ldots\}$ be a possibly infinite short generating set for K. We will now show that the embedded geodesics in Γ are disjoint and the number of elements in Γ are bounded by $3g$ where g is the genus of Σ. Suppose γ_i and γ_{i+k} are geodesics in Γ which intersect each other and where $k > 0$. Lemma 1 shows that the free homotopy class of one of these geodesics can be expressed as the sum of two homotopy classes of less length. This immediately contradicts the least length property for these geodesics and thereby proves the geodesics in Γ are disjoint. This argument also proves the last statement in the theorem.

If M^2 is a compact orientable surface of genus g and $\Gamma = \{\gamma_1, \ldots, \gamma_{3g+1}\}$ is a collection of $3g+1$ disjoint Jordan curves on the surface, then the classification theorem for compact surfaces can be used to show that two of these Jordan curves are isotopic. Hence there are at most $3g+1$ elements in a short generating set for K where g is the genus of Σ. This last observation completes the proof of the theorem.

THEOREM 7 (Equivariant loop theorem). *Suppose G is a finite group which acts on a compact orientable three-dimensional manifold M with boundary as a group of orientation preserving diffeomorphisms. Then there exists a collection $\Delta = \{D_1, D_2, \ldots, D_n\}$ of embedded pairwise disjoint disks in M which satisfy*

(1) $D_i \cap \partial M = \partial D_i$.
(2) *The normal subgroup generated by $\Gamma = \{\partial D_1, \partial D_2, \ldots, \partial D_n\}$ is the kernel K of the inclusion map of the fundamental group of each component Σ of the boundary of M into M.*
(3) *The union of Δ is G invariant.*

Proof. If we produce a collection of disjoint Jordan curves $\Gamma=\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ on ∂M such that the normal subgroup of $\pi_1(\partial M)$ generated by Γ is K and G acts freely on the union of Γ, then the theorem will follow from Theorem 5. To prove the existence of such a Γ, we first consider a short generating set $\Gamma'=\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ given by Theorem 6. If G acts freely on the union of Γ', then Γ' is the required collection of Jordan curves. If G has a fixed point on Γ', then we carry out the following procedure.

Let N_i be a regular neighborhood of the curve γ_i on ∂M that is small enough so that the collection of these neighborhoods is invariant under G and these neighborhoods are pairwise disjoint. Clearly, N_i is diffeomorphic to $S^1\times[0, 1]$. Let Γ be the collection of all the boundary circles of these regular neighborhoods. As G acts as a group of orientation preserving transformations of the boundary of M, and N_i is an annulus, any element $g\in G$ which has a fixed point on ∂N_i must be equal to the identity on N_i and hence the identity on M. Therefore G acts freely on the union of Γ. By the previous discussion this completes the proof of the theorem.

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