Heat kernel estimates and lower bound of eigenvalues.

Autor(en): **Cheng, Shiu-Yuen / Li, Peter**

Objekttyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **56 (1981)**

PDF erstellt am: **22.07.2024**

Persistenter Link: https://doi.org/10.5169/seals-43247

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.
Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.
Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

Ein Dienst der *ETH-Bibliothek*
ETH Zürich, Rämistrasse 101, 8092 Zürich, Schweiz, www.library.ethz.ch

http://www.e-periodica.ch

Comment. Math. Helvetici 56 (1981) 327–338 0010–2571/81/003327-12$01.50+0.20/0
© 1981 Birkhäuser Verlag, Basel

Heat kernel estimates and lower bound of eigenvalues

SHIU-YUEN CHENG* and PETER LI

§0. Introduction

Let M^n be a n-dimensional compact Riemannian manifold with possibly empty boundary, ∂M. We consider the eigenvalues of the Laplace operator on M with either Dirichlet or Neumann boundary condition. The famous asymptotic formulas of H. Weyl give

$$\lambda_k^{n/2} \sim A(n)k/V \tag{0.1}$$

and

$$\mu_k^{n/2} \sim A(n)k/V \tag{0.2}$$

where λ_k is the kth non-zero eigenvalue on M (with Neumann boundary condition if $\partial M \neq \emptyset$) and μ_k is the kth eigenvalue on M with Dirichlet boundary condition. In view of this, it is interesting to investigate if the eigenvalues λ_k and μ_k possess lower estimates which are compatible with the Weyl formulas.

When M is assumed to be a domain in a simply connected non-positively curved manifold, the answer was obtained in [7, 3], where they showed

$$\mu_k^{n/2} \geq B(n)k/V$$

for some constant $B(n)$ which only depend on n. Lower estimates for λ_k on general Riemannian manifold were derived in [5]. However the order of λ_k was not optimal, instead he proved

$$\lambda_k^{n-1} \geq C(M)kV^{-2(n-1)/n}$$

where $C(M)$ is a constant depending on M. More precisely, $C(M)$ depends on the Sobolev constant of M which in term can be bounded from below by the

* Research partially supported by an NSF grant and a Sloan Fellowship

diameter, d; lower bound of the Ricci curvature, $(n-1)K$; and the volume of M, V.

Later in [6], using a completely different method Li and Yau showed that if M is non-negatively curved then

$$\lambda_k^{n/2} \geq B'(n)k/V$$

for k not less than a constant that depends on the geometry of M. Also if the Ricci curvature of M is non-negative, then

$$\lambda_k^{n/2} \leq B''(n)(k+1)/V$$

for all k, $B'(n)$ and $B''(n)$ are constants depending only on n. For general manifolds, lower bounds for small eigenvalues are more complicated. They proved that

$$\lambda_k^{n/2} \geq C'(M)k/V, \text{ for all } k,$$

where $C'(M)$ depends on d, V, n, lower bound of the sectional curvature, and radius of convexity of M. Upper bound in the form

$$\lambda_k^{n/2} \leq C(K, d, n)[(k+1)/V] + C(K, n)$$

was also derived. The constants in this case depends on the quantities described.

In view of the upper estimates for λ_k and previous results, it is not surprising to speculate that

$$\lambda_k^{n/2} \geq C(K, d, n, V) \cdot k$$

for all k. In fact, the purpose of the work is to show the above inequality. More precisely, we will prove that

$$\lambda_k^{n/2} \geq \alpha(n) C_1^{n/2} k/V$$

where $\alpha(n)$ is a constant depending only on n, and C_1 is the Sobolev constant from Lemma 1. Together with the estimates of the isoperimetric constant obtained in [2], we conclude that C_1 has a lower bound depending on n, K, d, and V.

In a case where M is a manifold with boundary, we showed that

$$\mu_k^{n/2} \geq \alpha(n) C_2^{n/2} k/V \quad \text{and} \quad \lambda_k^{n/2} \geq \alpha(n) C_1^{n/2} k/V$$

where μ_k and λ_k are eigenvalues of the Dirichlet and Neumann boundary problems respectively. Since C_2 is equivalent to the isoperimetric constant, when M is a domain in a simply connected non-positively curved manifold, we recovered the results in [3].

It is interesting to consider the special case where M is a compact manifold without boundary of constant -1 curvature. If $n \geq 4$, we showed that

$$\lambda_k^{n/2} \geq \alpha(n)k/V^{n^2+n+1}$$

where $\alpha(n)$ is a constant depending only on n. This inequality is known to be false for the case $n=2$ and $k \leq 2g-3$ (see [8]).

§1. Sobolev inequalities

The standard Sobolev inequality for compact manifold M with or without boundary asserts that there exists a constant C_0 depending only on M, such that

$$\int_M |\nabla f| \geq C_0 \left(\inf_{a \in \mathbf{R}} \int_M |f-a|^{n/(n-1)} \right)^{(n-1)/n} \tag{1.1}$$

for any $f \in H_{1,1}(M)$.

In this section we will derive a weaker form of (1.1) which is more suitable for our application.

LEMMA 1. *Let M^n be a compact Riemannian manifold with possibly empty boundary. Then there exists a constant C_1 depending on M such that*

$$\int_M |\nabla f|^2 \geq C_1 \left(\int_M f^2 \right)^{(2+n)/n} \left(\int_M |f| \right)^{-4/n}$$

for any $f \in H_{1,2}(M)$ which satisfies

$$\int_M f = 0.$$

Proof. The proof is divided into two cases.

Case 1. When $n \geq 3$, in [5] we derived the following weaker form of (1.1), namely

$$\int_M |\nabla f|^2 \geq C_0^2 \left[\alpha(n) \left(\int_M |f|^{2n/(n-2)} \right)^{(n-2)/n} - \beta(n) V^{-2/n} \int_M f^2 \right] \tag{1.2}$$

for $f \in H_{1,2}(M)$, where $\alpha(n)$ and $\beta(n)$ are constants depending only on n and V = volume of M. However if $\int_M f = 0$, then the Poincaré inequality gives

$$\int_M |\nabla f|^2 \geq \lambda_1 \int f^2 \tag{1.3}$$

On the other hand, it is known that (see [5])

$$\lambda_1 \geq \gamma(n) C_0^2 V^{-2/n} \tag{1.4}$$

for $\gamma(n)$ = constant depending on n alone. Combining (1.2), (1.3), and (1.4) we conclude that

$$\int_M |\nabla f|^2 \geq C_0^2 \alpha(n) \left(\int_M |f|^{2n/(n-2)} \right)^{(n-2)/n} \tag{1.5}$$

for all $f \in H_{1,2}(M)$ satisfying $\int_M f = 0$. The lemma for $n \geq 3$ then follows by setting $C_1 = C_0^2 \cdot \alpha(n)$ and using Hölder inequality.

Case 2. When $n = 2$, we will prove the lemma directly from (1.1). The condition for the constant $k \in \mathbf{R}$ such that

$$\int_M (g-k)^2 = \inf_{a \in \mathbf{R}} \int_M |g-a|^2 \tag{1.6}$$

is equivalent to

$$\int_M (g-k) = 0 \tag{1.7}$$

Hence (1.1) can be written in the form

$$\left(\int_M |\nabla g| \right)^2 \geq C_0^2 \int_M (g-k)^2 \tag{1.8}$$

for any $g \in H_{1,1}(M)$ with $kV = \int_M g$. If we let $g = (\operatorname{sgn} f)\, |f|^{3/2}$, then we have

$$\left(\frac{3}{2} \int_M |f|^{1/2} |\nabla f| \right)^2 \geq C_0^2 \left[\frac{1}{2} \int |f|^3 - Vk^2 \right]. \tag{1.9}$$

However, since

$$\int_M |f| \int_M |\nabla f|^2 \geq \int_M |f|^{1/2} |\nabla f| \tag{1.10}$$

and

$$Vk^2 = \frac{(\int_M g)^2}{V} \leq \frac{(\int_M |g|)^2}{V} = V^{-1}\left(\int_M |f|^{3/2}\right)^2 \leq V^{-1} \int_M |f| \int_M f^2 \tag{1.11}$$

we have

$$\frac{9}{4}\int_M |f| \int_M |\nabla f|^2 + \frac{C_0^2}{V}\int_M |f| \int_M f^2 \geq \frac{C_0^2}{2}\int_M |f|^3 \geq \frac{C_0^2}{2}\frac{(\int_M f^2)^2}{\int_M |f|}$$

On the other hand, Poincaré inequality and (1.4) yield

$$\left(\frac{9}{4} + \frac{1}{\gamma(2)}\right)\int_M |f| \int_M |\nabla f|^2 \geq \frac{C_0^2}{2}\left(\int_M f^2\right)^2 \left(\int_M |f|\right)^{-1}.$$

Hence $C_1 = C_0^2\ (\frac{9}{2} + 2/\gamma(2))^{-1}$ for the case $n = 2$.

LEMMA 2. *Let M^n be a compact Riemannian manifold with boundary ∂M. Then there exists a constant C_2 depending only on M such that*

$$\int_M |\nabla f|^2 \geq C_2 \left(\int_M f^2\right)^{(2+n)/n} \left(\int_M |f|\right)^{-4/n}$$

for all $f \in H_{1,2}(M)$ with $f/\partial M \equiv 0$.

Proof. The lemma follows from the appropriate Sobolev inequality for Dirichlet boundary condition, namely

$$\int_M |\nabla f| \geq \tilde{C}_0 \left(\int_M |f|^{n/(n-1)}\right)^{(n-1)/n} \tag{1.12}$$

for $f \in H_{1,1}(M)$ satisfying $f/\partial M \equiv 0$. Here we use Lemma 6 of [5] instead, which gives for the case when $n \geq 3$,

$$\int_M |\nabla f|^2 \geq \tilde{C}_0 \alpha(n) \left(\int_M |f|^{2n/(n-2)}\right) \tag{1.13}$$

By setting $C_2 = \alpha(n)\tilde{C}_0^2$ and using the Hölder inequality, the lemma follows for $n \geq 3$.

When $n=2$, we substitute the function $(\operatorname{sgn} f)\,|f|^{3/2}$ instead of f into (1.12), and the rest of the proof follows similarly to that of Lemma 1.

Remarks. It is known that [1], the constants C_0 and $\tilde{C}_0$ are equivalent to the following isoperimetric inequalities.

(i) Let M be a compact manifold with or without boundary. We define the constant

$$I(M) = \inf_N \frac{A(N)^{n/(n-1)}}{\operatorname{Min}\{V(M_1), V(M_2)\}}, \tag{1.14}$$

where the infemum is taken over all codimension-1 submanifolds N which divide M into two parts, M_1 and M_2. Then

$$I(M) \leq C_0 \leq 2^{1/(n-1)} I(M). \tag{1.15}$$

(ii) Let M be a compact manifold with boundary ∂M. We define another isoperimeter constant

$$\tilde{I}(M) = \inf_N \frac{A(N)^{n(n-1)}}{V(\tilde{M})} \tag{1.16}$$

where infemum is taken over all codimension-1 submanifolds N which divide M into two parts, and $\tilde{M}$ is the part such that $\partial\tilde{M} \cap \partial M = \varnothing$. Then

$$\tilde{C}_0 = \tilde{I}(M) \tag{1.17}$$

Together with the proofs of Lemmas 1 and 2, we have

$$C_1 \geq \alpha(n) I^2(M) \tag{1.18}$$

and

$$C_2 \geq \alpha(n) \tilde{I}^2(M).$$

When $\partial M = \varnothing$, it was shown in [2] that $I(M)$ can be estimated from below by a constant depending only on the diameter $= d$, volume of $M = V$, and the lower bound of the Ricci curvature of M, $(n-1)K$. In fact,

$$I(M) \geq \alpha(n)\left[\frac{V}{\int_0^d F_k(r)\,dr}\right]^{(n+1)/(n-1)} \tag{1.19}$$

where

$$F_K(r)=\begin{cases}(-K)^{-1/2}(\sinh\sqrt{-K}r)^{n-1} & \text{if} \quad K<0\\ r^{n-1} & \text{if} \quad K=0\\ (K)^{-1/2}(\sinh\sqrt{K}r)^{n-1} & \text{if} \quad K>0\end{cases}$$

§2. Eigenvalue estimates

We are now ready to derive lower estimates for the eigenvalues. In view of eigenfunction expansion it is natural to consider the fundamental solutions of the heat equations with either Dirichlet or Neumann boundary condition. In fact, we utilize the Sobolev inequalities, i.e. Lemmas 1 and 2, to obtain an upper bound for the heat kernel and hence lower bounds for the eigenvalues.

THEOREM 1. *Let M be a compact manifold with or without boundary. Suppose λ_k is the* kth *non-zero eigenvalue (with Neumann boundary condition if $\partial M \neq \varnothing$) on M. Then there exists a constant $\alpha(n)$ depending only on n, such that*

$$\lambda_k^{n/2} \geq \alpha(n) C_1^{n/2} k/V$$

In particular if $\partial M=\varnothing$, then $C_1^{n/2}$ has a lower bound depending only on d, V, and K. In which case

$$\lambda_k^{n/2} \geq \alpha(n)\frac{k}{V}\left[\frac{V}{\int_0^d F_k(r)\,dr}\right]^{n(n+1)/(n-1)}$$

where $F_K(r)$ has been defined by (1.19).

Proof. Let $H(x, y, t)$ be the fundamental solution of the heat equation

$$\square_y H(x,y,t)=\left(\Delta_y-\frac{\partial}{\partial t}\right)H(x,y,t)=0$$

When $\partial M \neq \varnothing$, $H(x, y, t)$ is taken to be the heat kernel with Neumann boundary condition.

We consider the function $G(x, y, t)$ defined on $M\times M\times[0,\infty]$ by

$$G(x, y, t)=H(x, y, t)-\frac{1}{V}$$

Since $H(x, y, t)$ has the eigenfunction expansion

$$H(x, y, t)=\sum_{i=0}^{\infty} e^{-\lambda_i t}\phi_i(x)\phi_i(y) \tag{2.1}$$

and also because $\lambda_0=0$ with $\phi_0(x)=V^{-1/2}$,

$$G(x, y, t)=\sum_{i=1}^{\infty} e^{-\lambda_i t}\phi_i(x)\phi_i(y). \tag{2.2}$$

It is easy to check that G satisfies the semi-group property

$$G(x, y, t+s)=\int_M G(x, z, s)G(z, y, t)\, dz \tag{2.3}$$

for x, $y\in M$, and t, $s\in[0,\infty]$. Moreover

$$\int_M G(x, y, t)\, dy=0 \tag{2.4}$$

because $\int_M \phi_i(y)\, dy=0$, for $i\geq 1$. Also

$$\int_M |G(x, y, t)|\, dy\leq\int_M |H(x, y, t)|\, dy+1=2 \tag{2.5}$$

because $H(x, y, t)\geq 0$ and $\int_M H(x, y, t)=1$.

To prove the theorem, we first estimate $G(x, x, t)$ from above. By (2.3)

$$G'(x, x, t)=\int_M G'(x, y, t/2)G(x, y, t/2)\, dy=\int_M \Delta_y G(x, y, t/2)G(x, y, t/2)\, dy \tag{2.6}$$

because $0=\Box_y H(x, y, t/2)=\Box_y G(x, y, t/2)$. Integration by parts and the boundary condition of G yields

$$-G'(x,x,t)=\int_M |\nabla_y G(x,y,t/2)|^2\,dy \geq 2^{-4/n}C_1\left(\int_M G^2(x,y,t/2)\right)^{(2+n)/n}, \tag{2.7}$$

by Lemma 1 and (2.5). However the semi-group property (2.3) of G implies

$$-G'(x,x,t)(G(x,x,t))^{-(2+n)/n} \geq 2^{-4/n}C_1. \tag{2.8}$$

Integrating both sides of (2.8) with respect to t, and using the fact that

$$G(x,x,t)=(H(x,x,t)-1/V) \to \infty$$

as $t \to 0$, we have

$$\frac{n}{2}G^{-2/n}(x,x,t) \geq 2^{-4/n}C_1 t \tag{2.9}$$

Hence

$$G(x,x,t) \leq 4\left(\frac{2C_1}{n}\right)^{-n/2} t^{-n/2}$$

Integrating both sides with respect to x and using the expansion (2.2), we have

$$\sum_{i=1}^{\infty} e^{-\lambda_i t} \leq \alpha(n)(C_1 t)^{-n/2} V.$$

Let $t=1/\lambda_k$, and observe that $\lambda_1/\lambda_k \leq 1$ for $i \leq k$, (2.10) becomes

$$\alpha(n)\left(\frac{\lambda_k}{C_1}\right)^{n/2} V \geq \sum_{i=1}^{\infty} e^{-\lambda_i/\lambda_k} \geq \sum_{i=1}^{k} e^{-\lambda_i/\lambda_k} \geq ke^{-1}$$

This proves the theorem.

THEOREM 2. *Let M be a compact manifold with boundary ∂M. Then the kth eigenvalue μ_k for Dirichlet boundary condition satisfies*

$$\mu_k^{n/2} \geq \alpha(n) C_2^{n/2} k/V$$

Proof. The proof here is similar to the above theorem. Instead, we use the heat kernel for Dirichlet boundary condition and observe the fact that it satisfies the semi-group property, its L^1 norm is identically 1, and it has the eigenfunction expansion. Also Lemma 2 will be used in terms of Lemma 1 for the Sobolev inequality.

COROLLARY 1. *Let M be a compact manifold. Suppose ϕ satisfies $\Delta\phi = -\lambda\phi$, and*

(i) *If $\partial M \neq \emptyset$, and $\partial\phi/\partial\nu \equiv 0$ on ∂M, then*

$$\|\phi\|_\infty^2 \leq \alpha(n)\left(\frac{\lambda}{C_1}\right)^{n/2}$$

(ii) *If $\partial M = \emptyset$, then*

$$\|\phi\|_\infty^2 \leq \alpha(n)\left(\frac{\lambda}{C_1}\right)^{n/2}$$

(iii) *If $\partial M \neq \emptyset$, and if $\phi/\partial M \equiv 0$, then*

$$\|\phi\|_\infty^2 \leq \alpha(n)\left(\frac{\lambda}{C_2}\right)^{n/2}$$

Proof. Clearly the proof follows from (2.9) by letting $t = 1/\lambda$, we have

$$e^{-1}\phi^2(x) \leq \sum_{i=1}^{\infty} e^{-\lambda_i/\lambda}\phi_i^2(x) \leq \alpha(n)\left(\frac{\lambda}{C_1}\right)^{n/2} \tag{2.12}$$

for all $x \in M$.

COROLLARY 2. *Let M be a compact manifold without boundary. Suppose $n \geq 4$ and the curvature of M is constant -1, then*

$$\lambda_k^{n/2} \geq \alpha'(n)\left[\frac{V^{(n^2+1)/(n-1)}}{(\sinh \beta(n) V)^{n(n+1)}}\right]k,$$

where $\alpha'(n)$, $\beta(n)$ are positive constants depending only on n.

Proof. By Theorem 1, (1.18), and (1.19), we have

$$\lambda_k^{n/2} \geq \alpha(n)\left[\frac{V^{n(n+1)/(n-1)-1}}{(\int_0^d F_{-1}(r)\,dr)^{n(n+1)/(n-1)}}\right]k$$

However, since

$$\sinh r \leq \cosh r$$

this gives

$$\lambda_k^{n/2} \geq \alpha'(n)\left[\frac{V^{(n^2+1)/(n-1)}}{(\sinh d)^{n(n+1)}}\right]k \tag{2.14}$$

On the other hand, a theorem of Gromov [4] showed that

$$d \leq \beta(n)V - 1 < \beta(n)V \tag{2.15}$$

Hence

$$\lambda_k^{n/2} \geq \alpha'(n)\left[\frac{V^{(n^2+1)/(n-1)}}{(\sinh \beta(n)V)^{n(n+1)}}\right]k$$

has to be proved.

COROLLARY 3 (Lieb, Donnelly–Li). *Let M be a compact domain on a simply connected non-positively curved manifold. Then there exists a constant $\alpha(n)$ depending only on n, such that*

$$\mu_k^{n/2} \geq \alpha(n)k/V$$

for all $k \geq 1$.

Proof. This follows trivially from Theorem 2, (1.16) and (1.18). We also observe that by a standard comparison theorem

$$\tilde{I}(M) \geq \frac{A(S^{n-1})^{n/(n-1)}}{V(D^n)} = \text{constant}$$

where $A(S^{n-1})$ and $V(D^n)$ stand for the area of the unit $(n-1)$-sphere in $\mathbf{R}^n$ and the volume of the unit n-disk in $\mathbf{R}^n$, respectively.

REFERENCES

[1] BOMBIERI, E., *Theory of minimal surfaces and a counter-example to the Bernstein conjecture in high dimension*, Lecture Notes, Courant Institute (1970).

[2] CROKE, C., *Some isoperimetric inequalities and eigenvalue estimates*, preprint.

[3] DONNELLY H. and LI, P., *Lower bounds for the eigenvalues of negatively curved manifolds*, to appear in Math. Z.

[4] GROMOV, M., *Manifolds of negative curvature*, preprint.

[5] LI, P., *On the Sobolev constant and the p-spectrum of a compact Riemannian manifold*, preprint.

[6] LI, P. and YAU, S. T., *Estimates of eigenvalues of a compact Riemannian manifold*, Proc. Symp. Pure Math. *36* (1980), 205–240.

[7] LIEB, E., *The number of bound states of one body Schroedinger operators and the Weyl problem*, Proc. Symp. Pure Math. *36* (1980), 241–252.

[8] SCHOEN, R., WOLPERT, S. and YAU, S. T., *Geometric bounds on the low eigenvalues of a compact surface*, Proc. Symp. Pure Math. *36* (1980), 279–286.

Princeton University
Dept. of Mathematics
Fine Hall - Box 37
Princeton, N.J. 08544

Received May 14, 1980