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Autor(en): **McGehee, Richard**

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# Double collisions for a classical particle system with nongravitational interactions

Richard McGehee

## 1. Introduction

In 1907 K. F. Sundman proved that a solution of the Newtonian gravitational three body problem for which the angular momentum is nonzero can be written in terms of a power series convergent for all time [12]. The major obstacle overcome by Sundman was presented by collisions between the particles. The condition that the angular momentum be nonzero allowed him to eliminate the possibility of a collision between all three particles. Since he could not eliminate the possibility of a collision between two of the particles, he was forced to investigate the nature of solutions with double collisions.

Sundman discovered that a solution which has a double collision as $t \to t^*$ can be written as a convergent power series in $(t^*-t)^{1/3}$. This solution cannot be extended holomorphically to $t^*$. However, using analytic continuation in the complex plane around the branch point $t^*$, Sundman found a real analytic extension for $t > t^*$. Introducing a new variable, which locally has the form $(t^*-t)^{1/3}$ at each collision, he was able to extend the solution through each double collision and to write it as a convergent power series in the new variable. A complete description of Sundman's construction can be found in the book of Siegel and Moser [10].

From the vantage point provided us by the current theory of dynamical systems, we can question the significance of Sundman's power series expansion. Today an ordinary differential equation is viewed as a vector field on a manifold, and its solutions are viewed as determining a flow on that manifold. One attaches as much importance to the smoothness of the flow with respect to initial data as to its smoothness with respect to time. Sundman's extension of an orbit through a double collision would be considered of questionable significance if it were unrelated to nearby orbits. Since Sundman's extension accounts only for the dependence on time and not for the dependence on initial data, we do not know *a priori* whether his extension is related to nearby orbits.

Levi-Civita has given us a different method for extending orbits through double collisions [6]. He eliminated the singularities in the vector field by transforming the equations to ones without singularities. The extension through double collisions then is given by the transformed equations and is automatically a smooth function of initial data.

It happens that Sundman's extension and Levi-Civita's extension coincide. One is tempted to believe that somehow the power series technique carries with it the dependence on initial data. Since holomorphic functions are completely determined by their behavior on small sets, and since the solutions of the three body problem are holomorphic, one can easily imagine that an analytic extension in time will automatically give analytic dependence on initial data. It does so for the gravitational potential, but we shall see below that it fails to do so for other potentials.

To contrast these two methods of extension we study a particle system with nongravitational interactions. Instead of assuming that the pairwise potential energy is $r^{-1}$, where $r$ is the distance between the two particles, we assume that the potential energy is $r^{-\alpha}$, where $\alpha$ is some positive real number. We then study Sundman's technique versus Levi-Civita's. We shall see that solutions can be extended as a power series in time whenever $\alpha$ is in a certain dense subset of the rationals. However, only a few of these values of $\alpha$ produce double collisions which can be extended according to Levi-Civita. For the other values, Sundman's technique gives extensions which are not continuous with respect to initial data.

Thus we see that it is only a peculiar property of the gravitational interaction which gives us the same orbit extensions for the two different techniques. If one were to carry out Sundman's entire program for a three body problem with a nongravitational potential, one might obtain a power series solution, convergent for all time, but for which small changes in initial data could produce drastic changes in the orbits.

## 2. Equations of motion

We study a system of two particles. Since the two particles will always move in some fixed plane, we take $R^2$ for the position space. We fix the center of mass at the origin, so the system reduces to that of a single particle of unit mass in a central force field. We take the potential energy function to be

$$U(x) = -|x|^{-\alpha}, \qquad \alpha > 0,$$

where $x \in R^2$ is the position of the single particle. The motion is described by the differential equation

$$\ddot{x} = -\text{grad } U(x) = -\alpha\, |x|^{-\alpha-2}x, \tag{2.1}$$

where the double dot represents the second time derivative.

A double collision occurs if the two particles coincide and corresponds to $x = 0$ in the reduced problem. The origin is a "singularity" of equation (2.1), i.e. it is a point where the equation is undefined. Indeed, the potential energy approaches $-\infty$ as $x \to 0$.

A "singularity" of a solution of (2.1) is a time $t^*$ beyond which the solution cannot be extended as a smooth function of real $t$. A precise definition of this notion is given in Section 3 below. We shall show that, for equation (2.1), singularities of the solution correspond to singularities of the differential equation, i.e. $x \to 0$ as $t \to t^*$. The only singularities are therefore double collisions.

In Section 4 we develop some properties of the solutions of equation (2.1). We shall see that $\alpha = 2$ is a dividing line between two very different types of behavior. For example, equation (2.1) has many circular periodic orbits. For $\alpha < 2$, these orbits are all stable; for $\alpha > 2$, they are all unstable. As another example, the angular momentum must be zero for a collision orbit if $\alpha < 2$. However, if $\alpha > 2$, there are collision orbits with arbitrary angular momentum.

It is interesting to note that the case $\alpha = 2$ has another property not discussed here. A system of $n$ particles moving along a line with pairwise interaction given by an inverse square potential is completely integrable [9].

In Section 5 we discuss Sundman's technique of extending singular solutions. We call his technique "branch regularization." Roughly speaking, a singularity $t^*$ of a solution can be considered as a branch point, where time $t$ is regarded as complex. We then ask whether we can find a real analytic branch which extends the solution. We are able to find such a branch whenever $\alpha$ has the form $2((q/p)-1)$, where $q$ and $p$ are relatively prime integers with $q$ odd. If $p$ is even, as it is for the gravitational potential, then the extension can be described as a "reflection," i.e. the velocity vector reverses direction at collision, and the particles effectively bounce off each other. If $p$ is odd, then the extension can be described as a "transmission," i.e. the direction of the velocity vector is preserved at collision, and the particles effectively pass through each other.

In Section 6 we turn to Levi-Civita's idea of focusing attention on the singularities of the differential equation rather than on the singularities of the solution. It is difficult to describe his technique in general terms, since it involves an apparently *ad hoc* change of variable. However, Easton has given a general definition of regularization in the spirit of Levi-Civita [4], Easton's idea is to use an isolating block to examine whether orbits passing close to collision determine an extension for an orbit ending in collision. Easton calls his technique "regularization by surgery." However, this author feels that the use of surgery plays a secondary role to the use of isolating blocks and therefore prefers to call the technique "block regularization."

In Sections 6 and 7 we apply Easton's definition to equation (2.1). We find that the equation is block regularizable if and only if $\alpha = 2(1-n^{-1})$, where $n$ is a positive integer. We then discuss the similarities and differences between branch regularization and block regularization.

Finally in Section 8 we exhibit a transformation analogous to that given by Levi-Civita. Our transformation works for those values of $\alpha$ for which equation (2.1) is block regularizable. It reduces to Levi-Civita's transformation when $\alpha = 1$.

## 3. Singularities of solutions

In this section we prove that the only singularities of solutions of equation (2.1) are collisions. We begin with the general definition of a singularity of a solution of the differential equation

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}). \tag{3.1}$$

Here $\mathbf{F}$ is a real analytic vector field on some open set $U \subset R^n$, and the dot denotes differentiation with respect to time $t$. If $\mathbf{x}(0) \in U$, then the standard existence and uniqueness theorems give us a unique solution $\mathbf{x}(t)$ defined for $t^- < t < t^+$, where $-\infty \leqslant t^- < 0 < t^+ \leqslant +\infty$ and $(t^-, t^+)$ is the maximal interval over which the solution can be extended.

DEFINITION. If $t^+ < \infty$, then the solution $\mathbf{x}(t)$ is said to *end in a singularity at* $t^+$. If $t^- > -\infty$, then $\mathbf{x}(t)$ is said to *begin in a singularity* at $t^-$. In either case, $t^* = t^+$ or $t^-$ is said to be a *singularity* of the solution $\mathbf{x}(t)$.

We can write equation (2.1) as a first order system by introducing the momentum vector $y = \dot{x}$. The equation then becomes

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\alpha\,|x|^{-\alpha-2}x, \end{aligned} \tag{3.2}$$

which is a Hamiltonian system with Hamiltonian function

$$H(x, y) = \tfrac{1}{2}|y|^2 - |x|^{-\alpha}. \tag{3.3}$$

The function $H$ represents the total energy of the particle and is a conserved quantity; i.e. $H$ is constant along solutions of equations (3.2).

We now state the main result of this section. The proof is given after the proof of Lemma 3.2.

THEOREM 3.1. *Suppose that* $(x(t), y(t))$ *is a solution of* (3.2) *with a singularity at* $t^*$. *Then* $x(t) \to 0$ *as* $t \to t^*$.

LEMMA 3.2. *Suppose that* $(x(t), y(t))$ *is a solution of* (3.2) *with* $H(x, y) = h$, *and suppose that* $|x(t_0)| > 2\varepsilon > 0$. *Then there exist constants* $\delta$ *and* $M$, *depending on* $h$ *and* $\varepsilon$, *but independent of* $(x(t_0), y(t_0))$, *such that*

$$|y(t)| < M \quad \text{and} \quad |x(t)| > \varepsilon, \quad \text{whenever} \quad |t - t_0| < \delta.$$

*Proof.* We take

$$M = (2h + 2\varepsilon^{-\alpha})^{1/2} \quad \text{and} \quad \delta = \varepsilon / M.$$

It suffices to show that

$$|x(t)| > \varepsilon \quad \text{whenever} \quad |t - t_0| < \delta, \tag{3.4}$$

since $H(x(t), y(t)) = h$ implies that

$$|y(t)| = (2h + 2\,|x(t)|^{-\alpha})^{1/2},$$

and hence that

$$|y(t)| < M \quad \text{whenever} \quad |x(t)| > \varepsilon. \tag{3.5}$$

Suppose that (3.4) is false. Then there exists a $t_1$ such that $|t_1 - t_0| < \delta$ and $|x(t_1)| \leq \varepsilon$. By continuity one then can find a $t_2$ such that

$$\begin{aligned} &|t_2 - t_0| < \delta, \\ &|x(t_2)| = \varepsilon, \end{aligned} \tag{3.6}$$

and

$$|x(t)| > \varepsilon \quad \text{whenever} \quad |t - t_0| < |t_2 - t_0|.$$

Since $y = \dot{x}$, (3.5) now implies that

$$|\dot{x}(t)| < M \quad \text{for} \quad |t - t_0| < |t_2 - t_0|.$$

Hence $|x(t_2)| > |x(t_0)| - M\,|t_2 - t_0|$, i.e. $\varepsilon > 2\varepsilon - M\,|t_2 - t_0|$, or $|t_2 - t_0| > \varepsilon/M = \delta$. But this contradicts (3.6), establishing (3.4) and proving the lemma.

*Proof of Theorem* 3.1. Let $H(x, y) = h$. Assume that $x(t)$ does not approach 0 as $t \to t^*$. Then there exist an $\varepsilon > 0$ and a sequence $\{t_n\}$, with $t_n \to t^*$, such that

$$|x(t_n)| > 2\varepsilon \quad \text{for all} \quad n. \tag{3.7}$$

Let $\delta$ and $M$ be given by Lemma 3.2. Choose $n$ so that $|t^* - t_n| < \delta$. Then

$$|y(t)| < M \quad \text{and} \quad |x(t)| > \varepsilon \quad \text{whenever} \quad |t - t_n| < \delta.$$

Equations (3.2) then imply that

$$|\dot{x}(t)| < M \quad \text{and} \quad |\dot{y}(t)| < \alpha\varepsilon^{-\alpha-1} \quad \text{whenever} \quad |t - t_n| < \delta.$$

Hence $(x(t), y(t))$ actually approaches some limiting point $(x^*, y^*)$ as $t \to t^*$. By (3.7), $x^* \geq 2\varepsilon$. Therefore the solution can be extended beyond $t^*$, which contradicts the hypothesis that $t^*$ is a singularity. Hence $x(t) \to 0$ as $t \to t^*$ and the theorem is proved.

## 4. A geometric description of the flow

We now present a description of the orbit structure of system (3.2), with special emphasis on the orbits near collision. The transformations used here are essentially the same as those used by this author and others to study collisions in Newtonian gravitational systems. (See [8] for specific references.) Similar transformations have been used also by Devaney in his study of the anisotropic Kepler problem [3].

It is convenient to introduce the constants

$$\beta = \frac{\alpha}{2} \quad \text{and} \quad \gamma = \frac{1}{\beta + 1}.$$

We identify the real plane $R^2$ with the complex plane $C^1$ and consider $x$ as a complex number or as a vector in the Euclidean plane, depending on context. As usual, a bar denotes the complex conjugate. We shall use the angular momentum integral

$$\Omega(x, y) = \mathrm{Im}\,(\bar{x}y).$$

We introduce the new real coordinates $r > 0$, $\theta$, $w$, and $v$ by letting

$$\begin{aligned} x &= r^{\gamma} e^{i\theta}, \\ y &= r^{-\beta\gamma}(v + iw)e^{i\theta}. \end{aligned} \tag{4.1}$$

System (3.2) then becomes

$$\begin{aligned} \dot{r} &= (\beta + 1)v, \\ \dot{\theta} &= r^{-1}w, \\ \dot{w} &= r^{-1}(\beta - 1)wv, \\ \dot{v} &= r^{-1}(w^2 + \beta(v^2 - 2)). \end{aligned} \tag{4.2}$$

If we take the energy integral $H(x, y)$ given by (3.3) to have the constant value $h$ and the angular momentum integral $\Omega(x, y)$ to have the constant value $c$, then we can write

$$w^2 + v^2 - 2 = 2hr^{\alpha\gamma}, \tag{4.3}$$

$$r^{(1-\beta)\gamma} w = c. \tag{4.4}$$

We now define the constant energy manifold

$$\mathbf{M}(h) = \{(r, \theta, w, v) \in R^4 : r \geq 0 \text{ and (4.3) holds}\}. \tag{4.5}$$

System (4.2) determines a vector field on $\mathbf{M}(h)$ which is undefined when $r = 0$. We let

$$\mathbf{N} = \{(r, \theta, w, v) \in \mathbf{M}(h) : r = 0\}, \tag{4.6}$$

which is the manifold of states corresponding to collision for equation (2.1). From the definition of $\mathbf{M}(h)$, we see that

$$\mathbf{N} = \{(r, \theta, w, v) \in R^4 : r = 0 \quad \text{and} \quad w^2 + v^2 = 2\}$$

and hence that **N** is independent of $h$. Since $\theta$ is considered modulo $2\pi$, **N** is a two dimensional torus.

As mentioned above, the vector field given by (4.2) is not defined on **N**. A collision corresponds to an orbit which approaches **N** in finite time. However, we can scale the vector field in such a way that the new vector field can be extended to **N**. We accomplish this scaling by introducing a new time parameter $\tau$ given by

$$dt = r\,d\tau. \tag{4.7}$$

Equations (4.2) then become

$$\begin{aligned} r' &= (\beta+1)rv, \\ \theta' &= w, \\ w' &= (\beta-1)wv, \\ v' &= w^2+\beta(v^2-2), \end{aligned} \tag{4.8}$$

where the prime denotes differentiation with respect to $\tau$. For this new vector field, **N** is an invariant set. As we shall see in Lemma 4.1 below, collision orbits now approach **N** asymptotically as $\tau \to \pm\infty$.

The solutions of the entire system (4.8) are determined by the last two of those equations in the following sense. The equations for $w'$ and $v'$ do not involve $r$ and $\theta$ and therefore may be solved independently. Then $r$ and $\theta$ can be determined from the first two of equations (4.8). If $h \neq 0$, then $r$ could also be determined from equation (4.3).

We therefore consider the last two of equations (4.8) as a separate system:

$$\begin{aligned} w' &= (\beta-1)wv, \\ v' &= w^2+\beta(v^2-2). \end{aligned} \tag{4.9}$$

These equations admit the integral

$$\Lambda(w, v) = |w|^{\alpha}\, |v^2+w^2-2|^{1-\beta}. \tag{4.10}$$

This integral is derived from the angular momentum and energy integrals as follows.

$$\Lambda = |\Omega(x, y)|^{\alpha}\, |2H(x, y)|^{1-\beta}$$

is a function of integrals for system (3.2) and is therefore itself an integral. Written in terms of $r$, $\theta$, $w$, and $v$, this expression becomes equation (4.10).

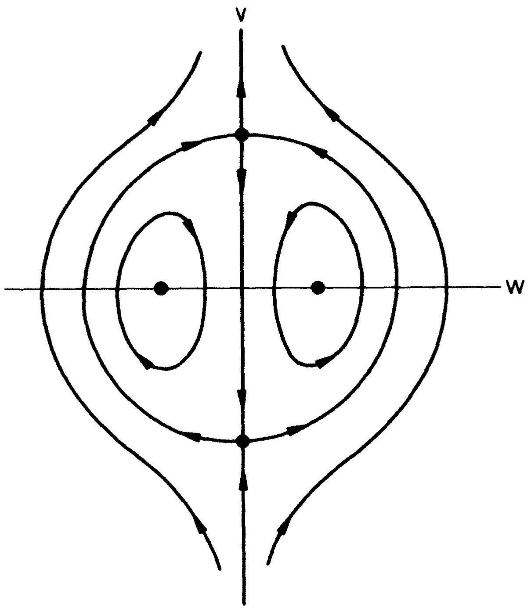


Figure 4.1.

The orbit structure of system (4.9) is completely determined by the integral $\Lambda$. There are three distinct cases depending on whether $\beta<1$, $\beta=1$, or $\beta>1$. These cases are shown in Figures 4.1, 4.2, and 4.3 respectively. In each case, the circle $\{w^2+v^2=2\}$ is invariant and corresponds to the collision manifold $\mathbf{N}$. The constant energy manifold $\mathbf{M}(h)$ projects to $\{w^2+v^2\leqslant 2\}$ when $h<0$, to $\{w^2+v^2\geqslant 2\}$ when $h>0$, and to $\{w^2+v^2=2\}$ when $h=0$.

In the remainder of this section we describe the relevant aspects of the solutions of system (4.8). Most of the statements can be derived from an examination of the integral $\Lambda$. Those that will be used in later sections are labeled as Lemmas and proved.

When $\beta<1$, the flow is similar to that of the Kepler problem, which is given by $\beta=\frac{1}{2}$. If we take $h\geqslant 0$, which corresponds to hyperbolic or parabolic orbits in the Kepler problem, and if we take $w\neq 0$, which means that the angular momentum is nonzero, then all solutions are unbounded in both directions. The

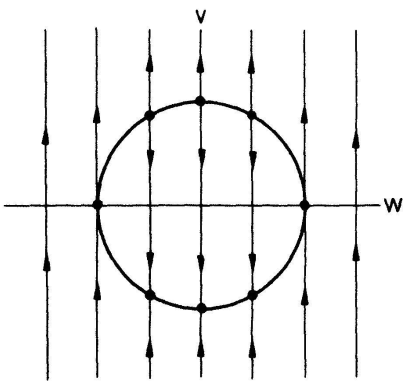


Figure 4.2.

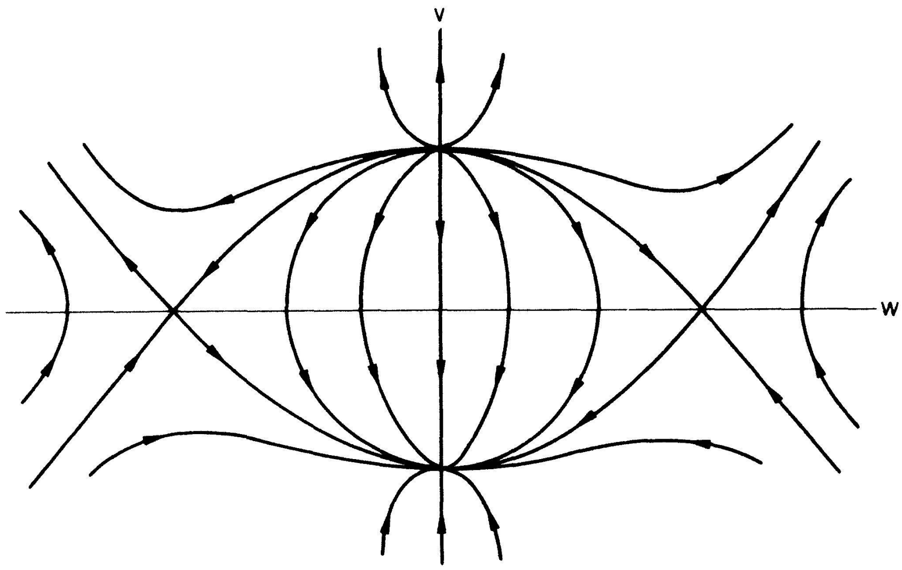


Figure 4.3.

orbits with zero angular momentum either begin or end in collision. If we take $h<0$, which corresponds to elliptic or circular orbits in the Kepler problem, then all solutions are bounded. There are two circular orbits, corresponding to the critical points $(w, v)=(\pm\sqrt{\alpha}, 0)$ in Figure 4.1. These circular orbits are stable. Taking the angular momentum to be zero distinguishes a one parameter family of orbits beginning and ending in collision. All other solutions move along invariant tori. One feature that distinguishes the Kepler problem is that each torus is foliated by periodic orbits. One does not expect this foliation for arbitrary $\alpha<2$. Instead one expects most of the tori to be filled with quasi-periodic orbits.

For $\beta=1$, the variable $w$ becomes the angular momentum integral, as can be seen from equation (4.4). In this case, the invariant circle $\{w^2+v^2=2\}$ becomes a circle of critical points for system (4.9). When $h<0$, every solution begins and ends in collision. When $h>0$ and $|w|\leq\sqrt{2}$, every orbit either begins or ends in collision and becomes unbounded in the other direction. When $h>0$ and $|w|>\sqrt{2}$, every orbit is unbounded in both time directions. When $h=0$, the variable $v$ also becomes an integral, as can be seen from equation (4.3). If $v\neq 0$, then the first of equations (4.8) immediately implies that $r=r_0e^{(\beta+1)v\tau}$ and hence that every solution begins or ends in collision and becomes unbounded in the other direction. Setting $v=0$ distinguishes a one parameter family of circular periodic orbits $r=r_0$, where $r_0$ is constant.

When $\beta>1$, the flow is drastically different from the Kepler problem. The circular periodic orbits, corresponding to the critical points $(w, v)=(\pm\sqrt{\alpha}, 0)$ in Figure 4.3, now occur for positive energy and are unstable. Among the orbits asysmptotic to one of the circular orbits, some are collision orbits while the others are unbounded. When $h>0$ there are unbounded collision orbits as well as orbits beginning and ending in collision and orbits unbounded in both directions. When $h \leq 0$ all solutions begin and end in collision except those with $h = 0$ and $c = 0$. These begin or end in collision and are unbounded in the other direction.

In the following lemmas, we shall consider $r$ as a function of $t$ or of $\tau$, depending on whether we are considering a solution of system (4.2) or of system (4.8). The orbits of both systems are the same; only the rate at which solutions move along the orbits is different. Using equation (4.7), we can write $\tau$ as a function of $t$ along a particular orbit:

$$\tau(t) = \tau_0 + \int_{t_0}^{t} \frac{ds}{r(s)}. \tag{4.11}$$

Here $s$ is a dummy variable for $t$, so the $r(s)$ under the integral is a solution of (4.2). Note that $\tau_0 = \tau(t_0)$ and that $r(\tau_0) = r(t_0)$, where the first $r$ now denotes a solution of (4.8). Note also that any solution of (4.2) must have $r(t) > 0$ for all $t$.

We now show that orbits with singularities in the original time variable $t$ become orbits for which $r$ approaches 0 asymptotically in the new time variable $\tau$.

LEMMA 4.1. *Suppose that* $(r, \theta, w, v)(t)$ *satisfies* (4.2) *and that* $r \to 0$ *as* $t \to t^* \pm$. *Then* $\tau(t) \to \mp\infty$.

*Proof.* We prove that $t \to t^*-$ implies that $\tau(t) \to +\infty$. The other case is similar. Equation (4.11) implies that $\tau(t)$ is defined for all $t_0 < t < t^*$ and is increasing. Hence $\tau(t) \to \tau^*$ as $t \to t^*-$, where $\tau^*$ may be infinite. Then $r(\tau) \to 0$ as $\tau \to \tau^*$. Since equations (4.8) are defined and smooth on $\mathbf{M}(h)$, and since $\mathbf{N}$ is a compact invariant set, it is impossible for a solution to approach $\mathbf{N}$ in finite time. Therefore $\tau^* = \infty$, and the lemma is proved.

For the following two lemmas, we use the notation

$$\mathbf{S}^{\pm} = \{(r, \theta, w, v) \in \mathbf{N} : v = \pm\sqrt{2}\}. \tag{4.12}$$

By the definition of $\mathbf{N}$,

$$\mathbf{S}^{\pm} = \{(r, \theta, w, v) \in R^4 : r = 0,\ w = 0,\ v = \pm\sqrt{2}\},$$

and hence $\mathbf{S}^+$ and $\mathbf{S}^-$ are both circles. We denote by $\omega(\mathbf{p})$ the omega limit set of the point $\mathbf{p} = (r, \theta, w, v)$ under the flow on $\mathbf{M}(h)$ defined by equations (4.8). Similarly, $\alpha(\mathbf{p})$ denotes the alpha limit set.

LEMMA 4.2. *Assume* $\beta \neq 1$. *Let* $\mathbf{p}(\tau) = (r, \theta, w, v)(\tau)$ *be a solution of* (4.8), *and write* $\mathbf{p}_0=\mathbf{p}(0)$. *Then*

(a) $r\to 0$ *as* $\tau\to+\infty$ *if and only if* $\omega(\mathbf{p}_0)\subset\mathbf{S}^-$, *and*

(b) $r\to 0$ *as* $\tau\to-\infty$ *if and only if* $\alpha(\mathbf{p}_0)\subset\mathbf{S}^+$.

*Proof.* As above, we prove only (a), since (b) is similar. Since $\mathbf{N}$ is exactly the set on $\mathbf{M}(h)$ where $r=0$, $r\to 0$ as $\tau\to+\infty$ if and only if $\omega(\mathbf{p}_0)\subset\mathbf{N}$. Since $\mathbf{S}^-\subset\mathbf{N}$, it therefore suffices to show that $\omega(\mathbf{p}_0)\subset\mathbf{N}$ implies that $\omega(\mathbf{p}_0)\subset\mathbf{S}^-$. Using the energy relation (4.3) with $r=0$, we can rewrite the equation for $v'$ in system (4.8) restricted to $\mathbf{N}$ as

$$v'=(1-\beta)w^2.$$

Therefore $v$ defines a Liapunov function on $\mathbf{N}$. Since a Liapunov function must be constant on an omega limit set [1], we must have $v'=0$ on $\omega(\mathbf{p}_0)$. But $v'=0$ on $\mathbf{N}$ exactly on $\mathbf{S}^+$ and $\mathbf{S}^-$. We can rule out $\mathbf{S}^+$ since $v$ is positive in a neighborhood of $\mathbf{S}^+$ and the first of equations (4.8) implies that $r$ is increasing whenever $r$ is positive. Therefore $\omega(\mathbf{p}_0)\subset\mathbf{S}^-$ and the proof is complete.

LEMMA 4.3. *Assume* $\beta\neq 1$. *Let* $(r,\theta,w,v)(t)$ *be a solution of* (4.2) *such that* $r\to 0$ *as* $t\to t^*\pm$. *Then, as* $t\to t^*\pm$,

$$w(t)\to 0,\qquad v(t)\to\pm\sqrt{2},\quad \textit{and}\quad \theta(t)\to\theta^*,$$

*for some constant* $\theta^*$. *Furthermore, if* $\beta<1$, *then* $w(t)\equiv 0$ *and* $\theta(t)\equiv\theta^*$.

*Proof.* As before, we prove only the case $t\to t^*-$. By Lemma 4.1 it suffices to prove the same limits for system (4.8) as $\tau\to+\infty$. By Lemma 4.2 it suffices to prove these limits for all points in $\mathbf{M}(h)$ on the stable manifold of $\mathbf{S}^-$. On $\mathbf{S}^-$, $\theta'=0$, so $\mathbf{S}^-$ consists entirely of rest points. A computation shows that the eigenvalues at each of these rest points are 0, $-\sqrt{2}(\beta-1)$, and $-2\sqrt{2}\beta$. The zero eigenvalue corresponds to the tangential direction along $\mathbf{S}^-$, while the two others correspond to the normal directions. For $\beta\neq 1$, $\mathbf{S}^-$ has hyperbolic normal structure, and hence the stable manifold of $\mathbf{S}^-$ is the union of the stable manifolds of each of the points on $\mathbf{S}^-$ [5]. Thus the solution $(r,\theta,w,v)(t)$ approaches some point on $\mathbf{S}^-$, i.e. $\theta\to\theta^*$, $w\to 0$, and $v\to-\sqrt{2}$.

When $\beta<1$, one normal eigenvalue is positive and the other is negative. In this case the stable manifold is two dimensional and, in fact, consists only of points with $w=0$. (See Figure 4.1) Since $\theta'=w=0$, $\theta$ must be constant. Therefore $w\equiv 0$ and $\theta\equiv\theta^*$, and the proof is complete.

Equation (4.4) tells us that the angular momentum is zero if $w$ is zero and $r$ is positive. Therefore the final statement of Lemma 4.3 implies that, for $\beta < 1$, the angular momentum must be zero on a collision orbit. Furthermore, all the motion takes place along a fixed line. These properties are not true for $\beta \geq 1$, as is seen in Figures 4.2 and 4.3.

## 5. Branch regularization of solutions

In this section we define the classical notion of regularization of solutions. We then determine for which $\alpha$ equation (2.1) is regularizable in this sense. We begin with a general definition concerning the solutions of the equation

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}). \tag{5.1}$$

DEFINITION. Let $\phi_1(t)$ and $\phi_2(t)$ be solutions of (5.1). Suppose that $\phi_1$ ends in a singularity at time $t^*$ and that $\phi_2$ begins in a singularity at the same time. Suppose that there is a multivalued analytic function having a branch at $t^*$ and extending both $\phi_1$ and $\phi_2$, Then $\phi_1$ is said to be a *branch extension* of $\phi_2$ at $t^*$, and $\phi_2$ is said to be a *branch extension* of $\phi_1$ at $t^*$,

DEFINITION. A solution $\phi$ of equation (5.1) which either begins or ends in a singularity at $t^*$ is said to be *branch regularizable* at $t^*$ if it has a unique branch extension at $t^*$. Equation (5.1) is said to be *branch regularizable* if every solution is branch regularizable at every singularity.

It is important to note that time $t$ is considered to be complex in the above definitions. The solutions $\phi_1$ and $\phi_2$ are real analytic, i.e. they are real for real values of $t$. However, they are extensions of one another through complex values of $t$.

It is also important not to be confused by our use of complex numbers to represent points in the Euclidean plane. A solution $x(t)$ of equation (2.1) is not a a real analytic function of $t$. Instead, $x_1(t)$ and $x_2(t)$ are real analytic functions of $t$, where $x = x_1 + ix_2$.

We are interested in branch extensions for solutions of equation (2.1) or, equivalently, of system (3.2). Since $y = \dot{x}$, $y(t)$ will automatically have a branch extension whenever $x(t)$ has one. It is therefore sufficient to consider only $x(t)$.

We denote the rational numbers by $Q$. Whenever we write $p/q \in Q$, we mean that $p$ and $q$ are relatively prime integers. A special subset of $Q$ is important here:

$$Q_0 = \{p/q \in Q : q \text{ is odd and } 0 < p < q\}.$$

The following theorem tells us for which values of $\alpha$ equation (2.1) is branch regularizable. Recall that $\alpha = 2\beta$ and that $\gamma = (1+\beta)^{-1}$. Since $\alpha > 0$, we must have $0 < \gamma < 1$.

THEOREM 5.1. *Equation* (2.1) *is branch regularizable if and only if* $\gamma \in Q_0$.

We break this theorem into two parts, one for each direction of the implication. Theorem 5.1 is an immediate consequence of the following two theorems. Note that Theorem 5.2 is actually stronger than the implication stated in Theorem 5.1.

THEOREM 5.2. *If* $\gamma \notin Q_0$, *then no singular solution of equation* (2.1) *is branch regularizable.*

THEOREM 5.3. *If* $\gamma \in Q_0$, *then every singular solution of equation* (2.1) *is branch regularizable.*

To prove these theorems we must know the exact analytical dependence on time of the solutions near a singularity. This information is provided by the following three lemmas, which will be proved after they are first used to establish Theorems 5.2 and 5.3.

LEMMA 5.4. *Suppose* $\beta < 1$. *Let* $x(t)$ *be a solution of equation* (2.1) *such that* $x(t) \to 0$ *as* $t \to 0+$. *Then*

$$x(t) = \kappa t^{\gamma} \Phi(t^{2\beta\gamma}),$$

*where* $\kappa$ *is a complex constant with* $|\kappa| = 1$ *and where* $\Phi$ *is a real analytic function defined on a neighborhood of* 0, *with* $\Phi(0) = (\sqrt{2}(\beta+1))^{\gamma}$.

LEMMA 5.5. *Suppose* $\beta > 1$. *Let* $x(t)$ *be a solution of equation* (2.1) *such that* $x(t) \to 0$ *as* $t \to 0+$. *Then*

$$x(t) = \kappa[t^{\gamma}\Psi_1(t^{2\beta\gamma}, t^{2(\beta-1)\gamma}) + it^{\beta\gamma}\Psi_2(t^{2\beta\gamma}, t^{2(\beta-1)\gamma})],$$

*where* $\kappa$ *is a complex constant with* $|\kappa| = 1$ *and where* $\Psi_1$ *and* $\Psi_2$ *are real analytic functions defined on a neighborhood of* $(0, 0)$, *with* $\Psi_1(0, 0) = (\sqrt{2}(\beta+1))^{\gamma}$.

LEMMA 5.6. *Suppose* $\beta = 1$. *Let* $x(t)$ *be a solution of equation* (2.1) *such that* $x(t) \to 0$ *as* $t \to 0+$. *Let* $c$ *be the constant value of the angular momentum* $\Omega$ *and* $h$

*be the constant value of the energy $H$ along the solution. Then $c^2 \leq 2$, and $h>0$ if $c^2=2$. Also,*

$$x(t)=\kappa\sqrt{2}t^{1/2}t^{ic/2a}(a+ht)^{1/2}(a+ht)^{-ic/2a} \quad \text{if} \quad c^2<2, \text{ and}$$

$$x(t)=\kappa\sqrt{2ht}e^{-ic/2ht} \quad \text{if} \quad c^2=2,$$

*where $a=(2-c^2)^{1/2}$ and $\kappa$ is a complex constant with $|\kappa|=1$.*

*Proof of Theorem* 5.2. First note that, since the equations of motion are autonomous, we need consider only solutions $x(t)$ with a singularity at $t^*=0$. Theorem 3.1 implies that $x(t)\to 0$ as $t\to 0$. Now note that, if $x=\phi(t)$ is a solution of equation (2.1), then $x=\phi(-t)$ is also a solution. Therefore it suffices to show that no solution satisfying $x(t)\to 0$ as $t\to 0+$ has a real analytic branch for $t<0$.

If $\gamma$ is irrational, or if $\gamma=p/q$, with $q$ even, then $t^\gamma$ has no real branch for $t<0$. The formulas of Lemmas 5.4 and 5.5 then immediately imply that $x(t)$ has no real branch for $t<0$, if $\beta\neq 1$. If $\beta=1$, which corresponds to $\gamma=p/q=\frac{1}{2}$, then the formulas of Lemma 5.6 immediately imply that $x(t)$ has no real branch for $t<0$. In any case, no singular solution is branch regularizable when $\gamma\notin Q_0$.

*Proof of Theorem* 5.3. By the considerations at the beginning of the proof of Theorem 5.2, it suffices to show that every solution $x(t)$ satisfying $x(t)\to 0$ as $t\to 0+$ has a unique branch extension for $t<0$. We first show that every such solution has a unique real analytic continuation to $t<0$. We then show that this continuation is a solution.

Since $\gamma=p/q$, we have that $\beta\gamma=(q-p)/q$ and $(\beta-1)\gamma=(q-2p)/q$. Using Lemmas 5.4 and 5.5, we can then write

$$x=\phi(t)=\kappa[u^pX_1(u^2)+iu^{q-2p}X_2(u^2)], \quad t>0,$$

where

$$u=t^{1/q},$$

$$X_1(z)=\begin{cases}\Phi(z^{q-p}), & \text{when } \beta<1,\\ \Psi_1(z^{q-p}, z^{q-2p}), & \text{when } \beta>1,\end{cases}$$

and

$$X_2(z)=\begin{cases}0, & \text{when } \beta<1,\\ \Psi_2(z^{q-p}, z^{q-2p}), & \text{when } \beta>1.\end{cases}$$

Since $q$ is odd, $t^{1/q}$ has a unique real analytic continuation to $t<0$, namely $-(-t)^{1/q}$. Therefore $\phi(t)$ has a unique real analytic continuation to $t<0$, which can be written

$$\phi_b(t)=\kappa[(-v)^p X_1(v^2)+i(-v)^{q-2p}X_2(v^2)],\quad t<0,$$

where

$$v=(-t)^{1/q}.$$

We have only left to show that $x=\phi_b(t)$ is also a solution of equation (2.1). For $t<0$ we can write

$$\phi_b(t)=\begin{cases}-\phi(-t), & \text{if } p \text{ is odd},\\ \overline{\phi(-t)}, & \text{if } p \text{ is even}.\end{cases}$$

One can check easily that $x=-\phi(-t)$ and $x=\overline{\phi(-t)}$ are both solutions of equation (2.1) for $t<0$, since $x=\phi(t)$ is a solution for $t>0$. Therefore every singular solution has a unique branch extension, and the proof is complete.

Before proceeding to the proofs of the lemmas, we digress briefly to discuss the regularization provided by the proof of Theorem 5.3. If $\beta<1$, then $X_2$ is identically zero, and hence we can write

$$\phi_b(t)=\begin{cases}-\phi(-t), & \text{if } p \text{ is odd},\\ \phi(-t), & \text{if } p \text{ is even}.\end{cases}$$

Since the motion of a singular orbit takes place along a fixed line when $\beta<1$, we can interpret this extension as a transmission if $p$ is odd and a reflection if $p$ is even. Note that the Newtonian gravitational potential corresponds to $\alpha=1$, $\beta=\frac{1}{2}$, and $\gamma=p/q=\frac{2}{3}\in Q_0$. Hence collisions are branch regularizable and the regularization is a reflection.

Four other examples are shown in Figure 5.1. In each case $q=5$, so the real analytic continuation is given by a rotation through an angle of $5\pi$ in the complex $t$-plane. When $\beta=\frac{1}{4}$, $p$ is even, so the extension is a reflection. When $\beta=\frac{2}{3}$, $p$ is odd, so the extension is a transmission. When $\beta=\frac{3}{2}$, we have $\beta\geq 1$, so there are collision orbits with nonzero angular momentum. Such an orbit and its branch extension are shown in the figure.

We note in passing that equation (2.1) is explicitly solvable in terms of circular or elliptic functions for the following values of $\alpha$: $\frac{1}{2}$, $\frac{2}{3}$, 1, $\frac{4}{3}$, $\frac{3}{2}$, 2, 3, 4, and 6 [13].

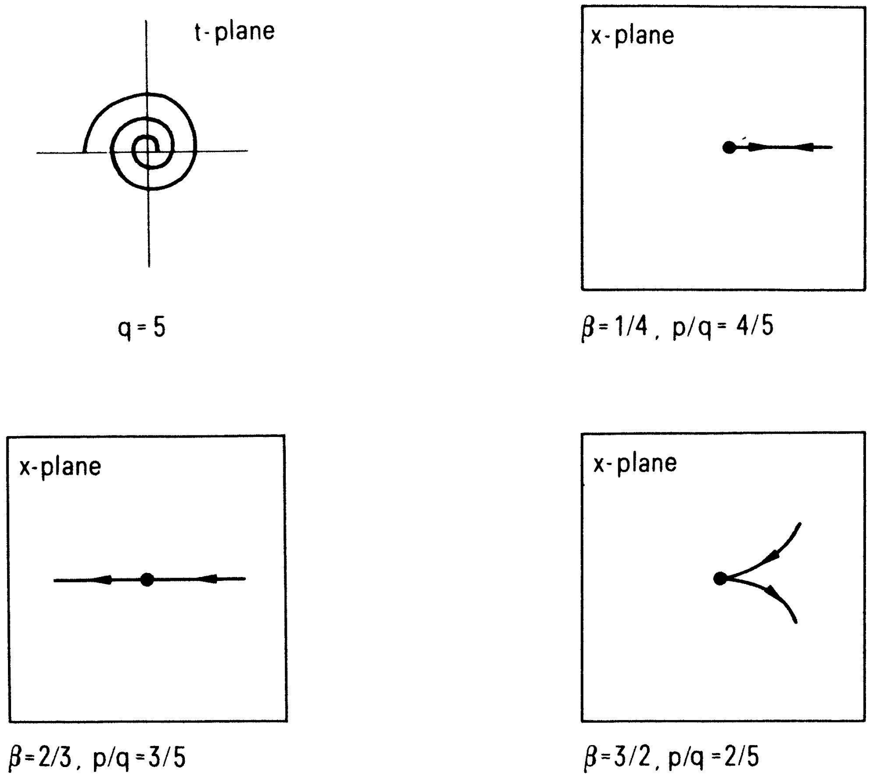


Figure 5.1.

The equation is branch regularizable for all of these values except $\frac{2}{3}$, 2, and 6. Thus there appears to be no relationship between regularizability and explicit solvability.

The remainder of this section is devoted to the proofs of Lemmas 5.4, 5.5, and 5.6. We start by stating and proving two more lemmas. The first could be proved using formal power series expansions and convergence arguments. However, in keeping with the geometric spirit of this paper, we prove it using the stable manifold theorem.

LEMMA 5.7. *Let $k$, $\alpha_1$, and $\alpha_2$ be positive, and let $f: C^2 \to C^1$ be real analytic near* $(0,0)$, *with* $f(0,0)=0$. *Assume that the function $\phi$ satisfies $\phi(x)>0$ for $x>0$ and $\phi(x)\to 0$ as $x\to 0+$, and suppose that $y=\phi(x)$ is a solution of*

$$\frac{dy}{dx}=k+f(y^{\alpha_1}, y^{\alpha_2}). \tag{5.2}$$

*Then there exists a function $g: C^2\to C^1$, real analytic near* $(0,0)$, *with* $g(0,0)=0$, *and such that*

$$\phi(x)=x(k+g(x^{\alpha_1}, x^{\alpha_2})).$$

*Proof.* We consider the differential equation (5.2) as a vector field in the plane by introducing a time variable $t$ which satisfies $x\,dt = dx$ and writing

$$\begin{aligned} \dot{x} &= x, \\ \dot{y} &= x(k+f(y^{\alpha_1}, y^{\alpha_2})), \end{aligned} \tag{5.3}$$

where the dot denotes differentiation with respect to $t$. This system is a real analytic vector field on

$$\mathbf{Q} \equiv \{(x, y) \in R^2 : x>0 \quad \text{and} \quad y>0\}.$$

We now introduce the new variables

$$\xi_1 = x^{\alpha_1}, \qquad \xi_2 = x^{\alpha_2}, \quad \text{and} \quad \eta = y/x. \tag{5.4}$$

Writing system (5.3) in these variables we obtain

$$\begin{aligned} \dot{\xi}_1 &= \alpha_1\xi_1, \\ \dot{\xi}_2 &= \alpha_2\xi_2, \\ \dot{\eta} &= -\eta + k + f(\xi_1\eta^{\alpha_1}, \xi_2\eta^{\alpha_2}). \end{aligned} \tag{5.5}$$

Note that the point $(\xi_1, \xi_2, \eta) = (0, 0, k) \equiv \mathbf{p}$ is a rest point for this system. Since $k>0$, system (5.5) is real analytic in a neighborhood of $\mathbf{p}$. Applying the real analytic version of the stable manifold theorem [11] to $\mathbf{p}$, we find that the local unstable manifold $\mathbf{W}$ of $\mathbf{p}$ can be written

$$\mathbf{W} = \{(\xi_1, \xi_2, \eta) \in R^3 : \eta = k + g(\xi_1, \xi_2) \text{ for } (\xi_1, \xi_2) \text{ near } (0, 0)\},$$

where $g: C^2 \to C^1$ is real analytic near $(0, 0)$ with $g(0, 0) = 0$.

Now observe that system (5.5) has a two dimensional invariant manifold

$$\mathbf{M} \equiv \{(\xi_1, \xi_2, \eta) \in R^3 : \xi_1>0, \xi_2>0, \eta>0, \quad \text{and} \quad \xi_1^{\alpha_2} = \xi_2^{\alpha_1}\},$$

and that equations (5.4) define a real analytic diffeomorphism from $\mathbf{Q}$ to $\mathbf{M}$ which carries the vector field (5.3) to the vector field (5.5). Furthermore, since the function $\phi$ can be extended to $x=0$ as a $C^1$ function with $\phi(0)=0$ and $\phi'(0)=k$, the graph of $\phi$ is mapped to the unique orbit in $\mathbf{W} \cap \mathbf{M}$. Hence $\phi$ can be written

$$\phi(x) = y = x\eta = x(k + g(x^{\alpha_1}, x^{\alpha_2})),$$

which completes the proof.

LEMMA 5.8. *Let $\lambda$, $\alpha_1$, and $\alpha_2$ be positive, and let $f: C^2 \to C^1$ be real analytic near* $(0,0)$. *Then the function*

$$\psi(x) \equiv \int_0^x \zeta^{\lambda-1} f(\zeta^{\alpha_1}, \zeta^{\alpha_2})\, d\zeta$$

*can be written*

$$\psi(x) = x^\lambda g(x^{\alpha_1}, x^{\alpha_2}),$$

*where $g: C^2 \to C^1$ is real analytic near* $(0,0)$.

*Proof.* Substituting $\zeta = xt$ into the above integral gives us

$$\psi(x) = x^\lambda \int_0^1 t^{\lambda-1} f((xt)^{\alpha_1}, (xt)^{\alpha_2})\, dt.$$

The result follows immediately upon writing

$$g(\xi_1, \xi_2) \equiv \int_0^1 t^{\lambda-1} f(\xi_1 t^{\alpha_1}, \xi_2 t^{\alpha_2})\, dt.$$

*Proof of Lemma* 5.4. Transformation (4.1) gives us a solution $(r, \theta, w, v)(t)$ of system (4.2) such that $r \to 0$ as $t \to 0+$. Lemma 4.3 implies that $v(t) \to +\sqrt{2}$ as $t \to 0+$ and that $w(t) \equiv 0$ and $\theta(t) \equiv \theta^*$. Using equation (4.3) we find that

$$v(t) = (2 + 2hr(t)^{2\beta\gamma})^{1/2}$$

for small positive $t$. The first of equations (4.2) then implies that

$$\frac{dr}{dt} = (\beta+1)(2 + 2hr^{2\beta\gamma})^{1/2}$$

for small positive $t$. Applying Lemma 5.7, we obtain a function $R$ which is real analytic in a neighborhood of 0 with $R(0) = \sqrt{2}(\beta+1)$ such that

$$r(t) = tR(t^{2\beta\gamma}).$$

Transformation (4.1) then gives us

$$x(t) = \kappa t^\gamma \Phi(t^{2\beta\gamma}),$$

where $\kappa = e^{i\theta^*}$ and $\Phi(z) = R(z)^\gamma$.

*Proof of Lemma* 5.5. Transformation (4.1) gives us a solution $(r, \theta, w, v)(t)$ of system (4.2) such that $r \to 0$ as $t \to 0+$. Lemma 4.3 implies that $v(t) \to +\sqrt{2}$, $w(t) \to 0$, and $\theta(t) \to \theta^*$ as $t \to 0+$. Using equations (4.4) and (4.3) we compute that

$$w(t) = cr^{(\beta-1)\gamma}$$

and that

$$v(t) = (2 + 2hr(t)^{2\beta\gamma} - c^2 r(t)^{2(\beta-1)\gamma})^{1/2}$$

for small positive $t$. The first of equations (4.2) then implies that

$$\frac{dr}{dt} = (\beta+1)(2 + 2hr^{2\beta\gamma} - c^2 r^{2(\beta-1)\gamma})^{1/2}$$

for small positive $t$. Applying Lemma 5.7, we obtain a function $R$ which is real analytic in a neighborhood of $(0, 0)$ with $R(0, 0) = \sqrt{2}(\beta+1)$ such that

$$r(t) = tR(t^{2\beta\gamma}, t^{2(\beta-1)\gamma}).$$

The second of equations (4.2) combined with equation (4.4) now implies that

$$\frac{d\theta}{dt} = ct^{(\beta-1)\gamma-1} R(t^{2\beta\gamma}, t^{2(\beta-1)\gamma})^{(\beta-1)\gamma-1}.$$

Applying Lemma 5.8 we find that

$$\theta(t) = \theta^* + t^{(\beta-1)\gamma} \Theta(t^{2\beta\gamma}, t^{2(\beta-1)\gamma}),$$

where $\Theta$ is real analytic near $(0, 0)$. Transformation (4.1) then gives us that

$$\begin{aligned} x &= r^\gamma e^{i\theta^*}[\cos(\theta-\theta^*) + i \sin(\theta-\theta^*)] \\ &= \kappa t^\gamma [\Psi_1(\xi_1, \xi_2) + i\sqrt{\xi_2}\Psi_2(\xi_1, \xi_2)], \end{aligned} \tag{5.6}$$

where $\xi_1 = t^{2\beta\gamma}$, $\xi_2 = t^{2(\beta-1)\gamma}$,

$$\Psi_1(\xi_1, \xi_2) = R(\xi_1, \xi_2)^\gamma \cos(\sqrt{\xi_2}\Theta(\xi_1, \xi_2)),$$

and

$$\Psi_2(\xi_1, \xi_2) = R(\xi_1, \xi_2)^\gamma \frac{\sin(\sqrt{\xi_2}\,\Theta(\xi_1, \xi_2))}{\sqrt{\xi_2}}.$$

Since $R$ and $\Theta$ are real analytic in a neighborhood of $(0, 0)$, with $R(0, 0) > 0$, and since cos is an even real analytic function, $\Psi_1$ is real analytic in a neighborhood of $(0, 0)$. Note that $\Psi_1(0, 0) = R(0, 0)^\gamma$. Since sin is an odd real analytic function, $\Psi_2$ is real analytic in a neighborhood of $(0, 0)$. Writing (5.6) in terms of $t$ finishes the proof.

*Proof of Lemma* 5.6. Since $\beta = 1$, we have that $\alpha = 2$ and $\gamma = \frac{1}{2}$. Equation (4.4) then becomes

$$w = c,$$

while equation (4.3) becomes

$$v^2 + c^2 - 2 = 2hr, \tag{5.7}$$

from which follows the inequality

$$c^2 - 2 \leq 2hr.$$

Since $r \to 0$ as $t \to 0$, we must have that $c^2 \leq 2$.

We now wish to find the explicit solutions of system (3.2) or, equivalently, of system (4.2). Rewriting (4.2) with $\beta = 1$, we have

$$\begin{aligned} \dot{r} &= 2v, \\ \dot{\theta} &= c/r, \\ \dot{w} &= 0, \\ \dot{v} &= 2h. \end{aligned} \tag{5.8}$$

Using equation (5.7), we see that

$$v^2 \to 2 - c^2 = a^2 \quad \text{as} \quad t \to 0+.$$

The possibility that $v$ would approach a negative value is eliminated by the first of equations (5.8). Hence we have that

$$v \to a \quad \text{as} \quad t \to 0+.$$

We can now solve system (5.8) explicitly to obtain

$$v(t) = a + 2ht,$$

and

$$r(t) = 2at + 2ht^2, \tag{5.9}$$

Substituting the transformation (4.1) into system (5.8), we obtain the following differential equation for $x$:

$$\dot{x} = \frac{v + ic}{r} x = \frac{a + ic + 2ht}{2t(a + ht)} x. \tag{5.10}$$

First consider the case $c^2 < 2$, so that $a > 0$. The general solution of equation (5.10) can be written

$$x(t) = Kt^{1/2} t^{ic/2a} (a + ht)^{1/2} (a + ht)^{-ic/2a},$$

where $K$ is a complex constant. Equation (5.9) implies that $|K|^2 = 2$, which gives the desired formula for the case $c^2 < 2$.

Now consider the case $c^2 = 2$, so that $a = 0$. Equation (5.7) becomes

$$v^2 = 2hr.$$

If $h$ were zero, then $v$ would also be zero and $r$ would be constant, which would contradict the hypothesis that $r \to 0$. Thus we must have $h > 0$. Equation (5.10) now becomes

$$\dot{x} = \frac{ic + 2ht}{2ht^2} x,$$

and the general solution can be written

$$x(t) = Kte^{-ic/2ht}.$$

Equation (5.9) now implies that $|K|^2 = 2h$, which gives the desired formula for the case $c^2 = 2$ and completes the proof.

## 6. An isolating block about collision

Our next goal is to describe the geometric notion of regularization defined by Easton. We proceed by first developing the notation of isolating blocks, following Conley and Easton [2]. We next find an isolating block for our central force problem. We then complete the business of the present section by using equations (4.8) to establish some properties of the block. In Section 7 we give Easton's definition of regularization [4] and use the results of the present section to apply his definition to the central force problem. Finally, in Section 8, we finish our discussion of regularization by describing a transformation similar to that of Levi-Civita [6].

We begin with isolating blocks. Let $\mathbf{M}$ be a smooth manifold and let $\psi:\mathbf{M}\times R^1\to\mathbf{M}$ be a flow on $\mathbf{M}$. A subset $\mathbf{N}\subset\mathbf{M}$ is called *invariant* if $\psi(\mathbf{N}, R^1)=\mathbf{N}$.

DEFINITION. A compact invariant set $\mathbf{N}\subset\mathbf{M}$ is called *isolated* if there exists an open set $\mathbf{U}$ containing $\mathbf{N}$ such that $\Psi(\mathbf{x}, R^1)\subset\mathbf{U}$ implies $\mathbf{x}\in\mathbf{N}$. The set $\mathbf{U}$ is called an isolating neighborhood for $\mathbf{N}$.

Now let $\mathbf{B}$ be a compact subset of $\mathbf{M}$ with non-empty interior and suppose that $\mathbf{b}=\partial\mathbf{B}$ is a smooth submanifold of $\mathbf{M}$. Define

$$\begin{aligned}
&\mathbf{b}^+\equiv\{\mathbf{x}\in\mathbf{b}: \psi(\mathbf{x}, (-\varepsilon, 0))\cap\mathbf{B}=\phi \text{ for some } \varepsilon>0\},\\
&\mathbf{b}^-\equiv\{\mathbf{x}\in\mathbf{b}: \psi(\mathbf{x}, (0, \varepsilon))\cap\mathbf{B}=\phi \text{ for some } \varepsilon>0\},\\
&\mathbf{t}\equiv\{\mathbf{x}\in\mathbf{b}: \dot{\psi}(\mathbf{x}, 0) \text{ is tangent to } \mathbf{b}\}.
\end{aligned}\tag{6.1}$$

DEFINITION. $\mathbf{B}$ is called an *isolating block* if $\mathbf{t}=\mathbf{b}^+\cap\mathbf{b}^-$.

DEFINITION. Let $\mathbf{N}$ be an isolated invariant set, and let $\mathbf{B}$ be an isolating block. Then $\mathbf{B}$ is said to *isolate* $\mathbf{N}$ if int $(\mathbf{B})$ is an isolating neighborhood for $\mathbf{N}$.

The following theorem was proved by Conley and Easton [2].

THEOREM 6.1. *If* $\mathbf{N}$ *is an isolated invariant set, then there exists an isolating block which isolates* $\mathbf{N}$. *If* $\mathbf{B}$ *is an isolating block, then there exists an isolated invariant set (possibly empty) which is isolated by* $\mathbf{B}$.

Since it is convenient in our application to define $\mathbf{B}$ in terms of a real-valued function on $\mathbf{M}$, we introduce some more notation. Let $I:\mathbf{M}\to R^1$ be smooth. Write

$$I^*(\mathbf{x}, t)\equiv I(\psi(\mathbf{x}, t)),$$

and define

$$\dot{I}(x) \equiv \dot{I}^*(\mathbf{x}, 0) \quad \text{and} \quad \ddot{I}(x) \equiv \ddot{I}^*(\mathbf{x}, 0),$$

where $\dot{I}^*$ and $\ddot{I}^*$ as usual denote derivatives with respect to time $t$. The following lemma is proved by Wilson and Yorke [14]. The symbol "$D$" denotes derivative.

LEMMA 6.2. *Let* $I: \mathbf{M} \to [0, \infty)$, *and let* $\delta_0 > 0$. *Suppose that* $DI(\mathbf{x}) \neq 0$ *whenever* $0 < I(\mathbf{x}) \leq \delta_0$. *Suppose also that* $\ddot{I}(\mathbf{x}) > 0$ *whenever* $0 < I(\mathbf{x}) \leq \delta_0$ *and* $\dot{I}(\mathbf{x}) = 0$. *Then* $\mathbf{N} \equiv I^{-1}(0)$ *is an isolated invariant set and* $I^{-1}([0, \delta])$ *is an isolating block for* $\mathbf{N}$ *for each* $\delta \in (0, \delta_0]$.

We now define the subsets of $\mathbf{b}$ which are asymptotic to $\mathbf{N}$.

$$\mathbf{a}^+ = \{\mathbf{x} \in \mathbf{b}^+ : \psi(\mathbf{x}, [0, \infty)) \subset \mathbf{B}\},$$

$$\mathbf{a}^- = \{\mathbf{x} \in \mathbf{b}^- : \psi(\mathbf{x}, (-\infty, 0]) \subset \mathbf{B}\}.$$

By definition, if $\mathbf{x} \in \mathbf{b}^+ - \mathbf{a}^+$, then there exists a $t > 0$ such that $\psi(\mathbf{x}, t) \notin \mathbf{B}$. Thus we may define the time spent in the block for a point $\mathbf{x} \in \mathbf{b}^+ - \mathbf{a}^+$ by

$$T(\mathbf{x}) \equiv \inf\{t > 0 : \psi(\mathbf{x}, t) \notin \mathbf{B}\}.$$

Note that $\psi(\mathbf{x}, [0, T(\mathbf{x})]) \in \mathbf{B}$ and that $\psi(\mathbf{x}, T(\mathbf{x})) \in \mathbf{b}^-$. Now define the map across the block

$$\Psi : \mathbf{b}^+ - \mathbf{a}^+ \to \mathbf{b}^- : \mathbf{x} \to \psi(\mathbf{x}, T(\mathbf{x})).$$

The following theorem was also proved by Conley and Easton [2].

THEOREM 6.3. *If* $\mathbf{B}$ *is an isolating block, then* $\Psi : \mathbf{b}^+ - \mathbf{a}^+ \to \mathbf{b}^- - \mathbf{a}^-$ *is a diffeomorphism.*

DEFINITION. An isolating block $\mathbf{B}$ is said to be *trivializable* if $\Psi$ extends uniquely to a differmorphism $\mathbf{b}^+ \to \mathbf{b}^-$.

The following lemma shows that trivializability is actually a property of an isolated invariant set. The proof follows from techniques in Conley's notes [1] and will be omitted here.

LEMMA 6.4. *Suppose that* $\mathbf{N}$ *is an isolated invariant set and that* $\mathbf{B}_1$ *and* $\mathbf{B}_2$ *isolate* $\mathbf{N}$. *Then* $\mathbf{B}_1$ *is trivializable if and only if* $\mathbf{B}_2$ *is trivializable.*

DEFINITION. Let $\mathbf{B}$ isolate $\mathbf{N}$. Then $\mathbf{N}$ is said to be *trivializable* if $\mathbf{B}$ is trivializable.

We now return to the central force problem. For the remainder of this section we take $\psi$ to be the flow on $\mathbf{M}(h)$ determined by equations (4.8). The manifold $\mathbf{M}(h)$ is given by (4.5), while the invariant set $\mathbf{N}$ is given by (4.6). Dots now denote differentiation with respect to $\tau$. Define

$$I:\mathbf{M}(h)\to R^1:(r,\theta,w,v)\to r,$$

$$\mathbf{B}(h,\delta)\equiv\{x\in\mathbf{M}(h):I(\mathbf{x})\leq\delta\}.$$

LEMMA 6.5. *Let* $\beta<1$. *Given any* $h$, *there exists a* $\delta_0>0$ *such that* $\mathbf{B}(h,\delta)$ *is an isolating block for* $\mathbf{N}$ *whenever* $0<\delta\leq\delta_0$.

*Proof.* We use Lemma 6.2. The tangent space to $\mathbf{M}(h)$ at the point $\mathbf{x}=(r,\theta,w,v)$ is given by

$$\{(\dot r,\dot\theta,\dot w,\dot v):-h\alpha\gamma r^{\alpha\gamma-1}\dot r+w\dot w+v\dot v=0\}.$$

Since $DI(\mathbf{x})(\dot r,\dot\theta,\dot w,\dot v)=\dot r$, we have that $DI(\mathbf{x})\neq 0$ if $(w,v)\neq(0,0)$. Equation (4.3) implies that

$$|w^2+v^2-2|=2\,|h|\,\delta^{\alpha\gamma},$$

where $I(\mathbf{x})=\delta$. Therefore, if $\delta_0$ is small enough, then $(w,v)\neq(0,0)$ whenever $\delta\leq\delta_0$. Hence $DI(\mathbf{x})\neq 0$ when $0<I(\mathbf{x})\leq\delta_0$. Now, using equations (4.8), we see that

$$\dot I=(\beta+1)rv\quad\text{and}\quad\ddot I=(\beta+1)r(w^2+(2\beta+1)v^2-2\beta).$$

If $I(\mathbf{x})=\delta$ and if $\dot I(\mathbf{x})=0$, then $r=\delta$ and $v=0$. Using (4.3), we then have

$$\ddot I=2(\beta+1)\delta(h\delta^{\alpha\gamma}+1-\beta).$$

Since $1-\beta>0$, we may choose $\delta_0$ small enough so that $\ddot I>0$ whenever $0<\delta\leq\delta_0$. Finally, note that $\mathbf{N}=I^{-1}(0)$. Hence, by Lemma 6.2, $\mathbf{B}(h,\delta)$ is an isolating block for $\mathbf{N}$ and the proof is complete.

By examining Figures 4.2 and 4.3, one can observe that **N** is not isolated if $\beta \geq 1$. Theorem 6.1 then tells us that it is impossible to construct an isolating block about **N** when $\beta \geq 1$.

We now exhibit for the block $\mathbf{B}(h, \delta)$ the various subsets defined above. We fix $h$ and assume that $0 < \delta \leq \delta_0$, where $\delta_0$ is given by Lemma 6.5. We abbreviate $\mathbf{B} = \mathbf{B}(h, \delta)$ and $\mathbf{x} = (r, \theta, w, v)$.

$$\mathbf{b} = \{\mathbf{x} \in \mathbf{M}(h) : r = \delta\},$$

$$\mathbf{b}^+ = \{\mathbf{x} \in \mathbf{b} : v \leq 0\},$$

$$\mathbf{b}^- = \{\mathbf{x} \in \mathbf{b} : v \geq 0\},$$

$$\mathbf{t} = \{\mathbf{x} \in \mathbf{b} : v = 0\},$$

$$\mathbf{a}^+ = \{\mathbf{x} \in \mathbf{b}^+ : w = 0\},$$

$$\mathbf{a}^- = \{\mathbf{x} \in \mathbf{b}^- : w = 0\}.$$

The projections of some of these sets to the $(w, v)$-plane are shown in Figure 6.1 for the case $h > 0$.

The following theorem gives us the values of $\beta$ for which **N** is trivializable. The proof will be broken into two parts. The first part is proved in Lemma 6.7 below, while the second part is postponed until Section 8. The relevance of this theorem to the question of whether collisions can be regularized will be discussed in the next section.

THEOREM 6.6. *The set* **N** *given by equation* (4.6) *is a trivializable isolated invariant set for equations* (4.8) *if and only if* $\beta = 1 - n^{-1}$, *where n is a positive integer.*

LEMMA 6.7. *Suppose that* **N** *is a trivializable isolated invariant set for equations* (4.8). *Then* $\beta = 1 - n^{-1}$, *where n is a positive integer.*

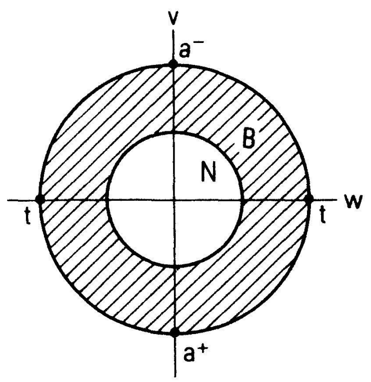


Figure 6.1.

*Proof.* Since $\mathbf{N}$ is not an isolated invariant set when $\beta \geq 1$, we must have $\beta < 1$. Using the definition of $\mathbf{B}$ and equation (4.3), we write

$$\mathbf{b} = \{(r, \theta, w, v) \in R^4 : r = \delta \text{ and } w^2 + v^2 = 2 + 2h\delta^{\alpha\gamma}\}.$$

Using the integral $\Lambda$ (4.10) for system (4.9), we can write the map across the block as

$$\Psi : \mathbf{b}^+ - \mathbf{a}^+ \to \mathbf{b}^- - \mathbf{a}^- : (r, \theta, w, v) \to (r, \Psi_\theta(\theta, w), w, -v),$$

where $\Psi_\theta$ is the second component of $\Psi$. Here we are using $(\theta, w)$ as coordinates on $\mathbf{b}^+$, so $r = \delta$ and $v = -(2 + 2h\delta^{\alpha\gamma} - w^2)^{1/2}$. Since equations (4.8) are independent of $\theta$,

$$\Psi_\theta(\theta, w) = \theta + \Gamma(w).$$

The function $\Gamma$ is defined for all $w$ such that $0 < w^2 \leq 2 + 2h\delta^{\alpha\gamma}$. By symmetry,

$$\Gamma(-w) = -\Gamma(w). \tag{6.2}$$

By hypothesis, $\mathbf{B}$ is trivializable. Therefore $\Psi$ extends to a continuous map $\mathbf{b}^+ \to \mathbf{b}^-$. Thus

$$\theta + \Gamma(0+) = \theta + \Gamma(0-) + 2\pi n,$$

where $n$ is an integer. By (6.2), $\Gamma(0-) = -\Gamma(0+)$. Hence we must have

$$\Gamma(0+) = \pi n. \tag{6.3}$$

The number $\Gamma(0+)$ can be computed using geometric methods. Consider a point $\mathbf{p}_0 \in \mathbf{a}^+$. Recall the definition (4.12) of $\mathbf{S}^\pm$. By Lemma 4.3, $\omega(\mathbf{p}_0)$ is a point $\mathbf{s}^-$ in $\mathbf{S}^-$. The orbit through $\mathbf{p}_0$ is the stable manifold of $\mathbf{s}^-$. Now let $\mathbf{p} \in \mathbf{b}^+$ be close to $\mathbf{p}_0$. The orbit through $\mathbf{p}$ follows closely the stable manifold of $\mathbf{s}^-$, passes close to $\mathbf{s}^-$, and then follows closely the unstable manifold of $\mathbf{s}^-$. We therefore must determine the unstable manifold of $\mathbf{s}^-$.

The unstable manifold of $\mathbf{s}^-$ is a subset of $\mathbf{N}$, so we study the flow on $\mathbf{N}$ determined by equations (4.8). On $\mathbf{N}$, $w^2 + v^2 = 2$, so we introduce the angular variable $\chi$ by

$$w + iv = \sqrt{2}e^{i\chi}.$$

Equations (4.8), restricted to **N**, then become

$$\theta' = w,$$

$$\chi' = (1-\beta)w.$$

Hence, in the $(\theta, \chi)$ variables, the unstable manifolds of points on $\mathbf{S}^-$ are just straight lines with slope $1-\beta$. (See Figure 6.2.) We are interested in the branch of the unstable manifold of $\mathbf{s}^-$ for which $w \geqslant 0$. Therefore we take $-\pi/2 \leqslant \chi \leqslant \pi/2$. Write

$$\mathbf{s}^- = (\delta, \theta_0, 0, -\sqrt{2}).$$

Then the unstable manifold of the point $\mathbf{s}^-$ is exactly the stable manifold of the point

$$\mathbf{s}^+ \equiv (\delta, \theta_0 + \pi(1-\beta)^{-1}, 0, \sqrt{2}).$$

We now can determine $\Gamma(0+)$. The orbit through $\mathbf{p}$ first follows the stable manifold of $\mathbf{s}^-$, then follows the unstable manifold of $\mathbf{s}^-$, which coincides with the stable manifold of $\mathbf{s}^+$, and finally follows the unstable manifold of $\mathbf{s}^+$. Note that $\theta$ does not change along the stable manifold of $\mathbf{s}^-$ or along the unstable manifold of $\mathbf{s}^+$. Therefore, as $\mathbf{p} \to \mathbf{p}_0$, the change in $\theta$ along the orbit approaches the difference in $\theta$ between $\mathbf{s}^-$ and $\mathbf{s}^+$. This difference is $\pi(1-\beta)^{-1}$. Hence $\Gamma(0+) = \pi(1-\beta)^{-1}$. Combining this result with (6.3), we have $\beta = 1 - n^{-1}$, where $n$ is an integer. Since $0 \leqslant \beta < 1$, $n$ is positive and the proof is complete.

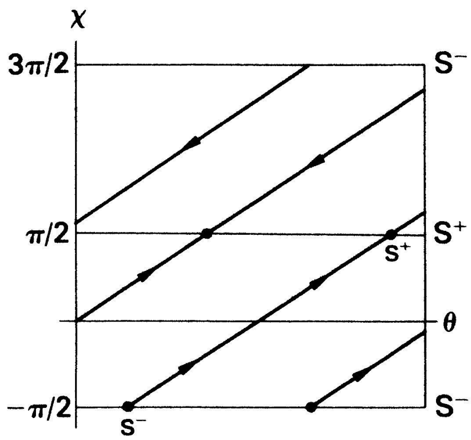


Figure 6.2.

## 7. Block regularization of the vector field

We now turn to Easton's definition of regularization. We are interested in whether equations (3.2) can be regularized. We consider the equivalent system (4.2). The set $\mathbf{N}$ given by (4.6) is the set of singularities of equations (4.2), i.e. it is the set where the vector field fails to be defined. In the previous section we worked with system (4.8), for which $\mathbf{N}$ is an invariant set. On $\mathbf{M}(h)-\mathbf{N}$ the orbits for the two systems are identical; only the parameterization is different. Later in this section we shall make use of this relationship between the two systems, but first we describe Easton's definition.

Let $\mathbf{M}$ be a smooth manifold, let $\mathbf{N}$ be a compact subset of $\mathbf{M}$, and let $\mathbf{F}$ be a vector field on $\mathbf{M}-\mathbf{N}$. In the previous section $\mathbf{N}$ was an invariant set. In this section, $\mathbf{N}$ is the set of singularities of the vector field $\mathbf{F}$. Let $\phi$ be the flow on $\mathbf{M}-\mathbf{N}$ given by $\mathbf{F}$. We use the term "flow" loosely here, since we do not require that $\phi(\mathbf{x}, t)$ be defined for all $t$.

Again let $\mathbf{B}$ be a compact subset of $\mathbf{M}$ with non-empty interior, and suppose that $\mathbf{b}=\partial\mathbf{B}$ is a smooth manifold which does not intersect $\mathbf{N}$. As before, $\mathbf{b}^+$, $\mathbf{b}^-$, and $\mathbf{t}$ are defined by (6.1). The definition of isolating block is also the same. Let $\mathcal{O}(\mathbf{x})$ denote the orbit through $\mathbf{x}$, i.e.

$$\mathcal{O}(\mathbf{x})=\{\phi(\mathbf{x}, t): \phi(\mathbf{x}, t) \text{ is defined}\}.$$

DEFINITION. An isolating block $\mathbf{B}$ is said to isolate the singularity set $\mathbf{N}$ if $\mathbf{N}\subset \operatorname{int}(\mathbf{B})$ and if $\mathcal{O}(\mathbf{x})\not\subset\mathbf{B}$ for all $\mathbf{x}\in\mathbf{B}-\mathbf{N}$.

The subsets $\mathbf{a}^+$ and $\mathbf{a}^-$ are the same as before, except that now we must allow for solutions which are not defined for all $t$. Thus

$$\mathbf{a}^+=\{\mathbf{x}\in\mathbf{b}^+: \phi(\mathbf{x}, t)\in\mathbf{B} \text{ for all } t\geq 0 \text{ for which } \phi(\mathbf{x}, t) \text{ is defined}\},$$

$$\mathbf{a}^-=\{\mathbf{x}\in\mathbf{b}^-: \phi(\mathbf{x}, \mathrm{t})\in\mathbf{B} \text{ for all } t\leq 0 \text{ for which } \phi(\mathbf{x}, t) \text{ is defined}\}.$$

The map $\Phi:\mathbf{b}^+-\mathbf{a}^+\to\mathbf{b}^-$ is defined in exactly the same way as the map $\Psi$ in Section 6. Again Theorem 6.3 holds [4], and again we have the same definition of a trivializable block $\mathbf{B}$.

DEFINITION. The singularity set $\mathbf{N}$ is said to be *block regularizable* if there exists a trivializable block $\mathbf{B}$ which isolates $\mathbf{N}$.

Easton gives a general procedure, which he calls "regularization by surgery," whereby one can replace the given vector field by a vector field without singularities [4]. For our purposes we give the following interpretation of the above definition. Suppose that one can isolate the singularities of a given system with an isolating block, and suppose that the map across the block can be extended. Solutions passing close to the singularities then will determine uniquely an extension for a solution ending in a singularity. Thus one can construct an extended flow with the property of differentiability with respect to initial data. If, on the other hand, the map across the block does not extend, then such an extended flow does not exist.

As a side remark we note that it may be useful in some applications to relax the definition of regularization and to require only that the map across the block extends to a homeomorphism. If the differentiability of the flow were of little interest, then such a relaxation would probably be more appropriate.

We now return again to our central force problem. Note that whether a certain set is or is not an isolating block is independent of the parameterization of the flow. The map across the block is also independent of the parameterization. Therefore, $\mathbf{B}(h, \delta)$ is an isolating block for system (4.2) if and only if it is an isolating block for system (4.8), and $\Phi = \Psi$. Hence $\mathbf{B}(h, \delta)$ is trivializable for (4.2) if and only if it is trivializable for (4.8). We then have, as an immediate consequence of Theorem 6.6, our main result about block regularization of system (3.2).

THEOREM 7.1. *The singularity set* $\mathbf{N}$ *for system* (4.3) *is block regularizable if and only if* $\beta = 1 - n^{-1}$, *where* $n$ *is a positive integer.*

The extension provided by this theorem was computed in the proof of Lemma 6.7. There we saw that orbits passing close to collision are deflected through an angle of $\Gamma(0+) = \pi(1-\beta)^{-1}$ in the $x$-plane. If this angle is a multiple of $\pi$, then the particle emerges in the same direction regardless of whether it passes the singularity on the right or on the left. Hence, if $\beta = 1 - n^{-1}$, then the extension of a singular orbit is a rotation through $n\pi$. This extension is a reflection for even $n$ and a transmission for odd $n$. The proof of Lemma 6.7 also shows that the map across the block extends to a homeomorphism when $\beta = 1 - n^{-1}$. To complete the proof of Theorem 6.6, we have only to prove that this homeomorphism is in fact a diffeomorphism. This proof is given in the next section.

In Figure 7.1 we illustrate four examples of the regularization provided by Theorem 7.1. In case $\beta = \frac{1}{4}$, $\Gamma(0+) = 4\pi/3$, so no unique extension is determined. In case $\beta = \frac{1}{2}$, $\Gamma(0+) = 2\pi$, so the unique extension is a reflection. In case $\beta = \frac{2}{3}$, $\Gamma(0+) = 3\pi$, so the unique extension is a transmission. In case $\beta = \frac{3}{2}$, $\beta \geq 1$, so there are no nearby orbits from which to determine an extension.

It is interesting to compare Figure 7.1 with Figure 5.1. In the cases $\beta = \frac{1}{4}$ and $\beta = \frac{3}{2}$, branch regularization determines an extension while block regularization

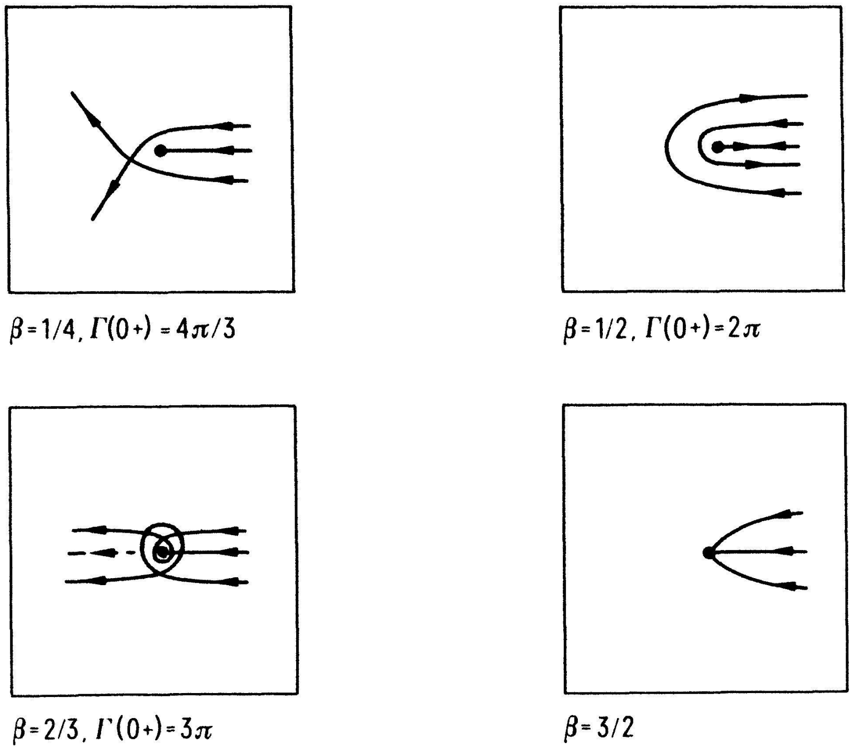


Figure 7.1.

does not. In the cases $\beta=\frac{1}{2}$ and $\beta=\frac{2}{3}$, both regularizations provide the same extensions.

One easily sees that system (3.2) is branch regularizable whenever it is block regularizable and that the two techniques give the same extension. If it is block regularizable, then $\beta=1-n^{-1}$, so $p/q=(1+\beta)^{-1}=n/(2n-1)$. Therefore $q=2n-1$ is odd, and the system is branch regularizable. The extension is a reflection when $p=n$ is even and a transmission when $p=n$ is odd.

In general, block regularization does not imply branch regularization. In a previous paper [7] this author gave an example of a block regularizable system which is not branch regularizable. It seems reasonable to suppose that there are systems which are both branch and block regularizable, but for which the two extensions are different. However, this author does not know such an example.

## 8. Levi-Civita regularization

We now return to the original Hamiltonian system (3.2) and introduce a change of variables analogous to the Levi-Civita transformation of the Kepler problem [6]. Consider the Hamiltonian function $H(x, y)$ given by equation (3.3) and assume that

$$\alpha=2\beta=2(1-n^{-1}),$$

where $n$ is a positive integer. As usual, we fix the total energy $H(x, y) = h$. Define new complex variables $z$ and $w$ by

$$x = z^n$$
$$y = w\bar{z}^{1-n}. \tag{8.1}$$

Note that these equations define a canonical transformation with multiplier $n$, i.e.

$$\operatorname{Re}(d\bar{x}\, dy) = n \operatorname{Re}(d\bar{z}\, dw).$$

Note also that the transformation is an $n$-to-1 mapping. Now consider the new Hamiltonian function

$$K(z, w) \equiv |z|^{2(n-1)}(H(z^n, w\bar{z}^{1-n}) - h)$$
$$= \tfrac{1}{2}|w|^2 - h\,|z|^{2(n-1)} - 1.$$

We consider (8.1) as an isoenergetic transformation from the manifold $\{H = h\}$ to the manifold $\{K = 0\}$. The vector field (3.2) on $\mathbf{M}(h)$ transforms to

$$\frac{dz}{d\sigma} = w,$$
$$\frac{dw}{d\sigma} = 2(n-1)h\,|z|^{2(n-2)}z, \tag{8.2}$$

on the manifold $\{K(z, w) = 0\}$. The new time variable $\sigma$ is given by

$$dt = n\,|z|^{2(n-1)}\,d\sigma.$$

Equations (8.2) extend to $z = 0$ and hence the singularity at collision has been "regularized" in the sense of Levi-Civita. Since $|w|^2 = 2$ when $z = 0$, solutions of (8.2) pass right through $z = 0$. To recover the behavior of system (3.2) we apply transformation (8.1). The case when $n = 3$ ($\alpha = \frac{4}{3}$) is shown in Figure 8.1.

We now state and prove our final lemma. This lemma, together with Lemma 6.7, establishes Theorem 6.6 and hence Theorem 7.1.

LEMMA 8.1. *Let $\beta = 1 - n^{-1}$, where $n$ is a positive integer. Then the invariant set* $\mathbf{N}$ *for equations* (4.8) *is trivializable.*

*Proof.* We use transformation (8.1). The block $\mathbf{B}(h, \delta)$ written in the $(z, w)$

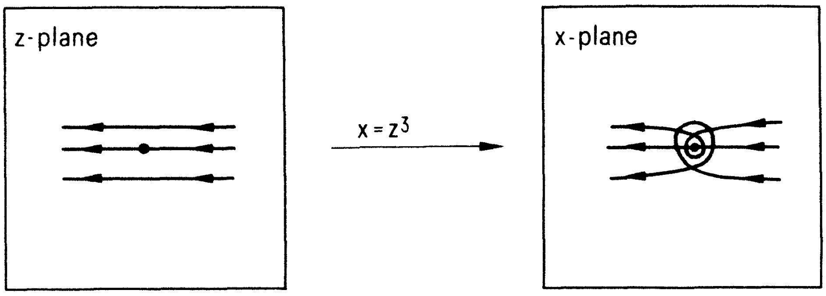


Figure 8.1.

coordinates becomes

$$\mathbf{B}' = \{(z, w) \in C^2 : K(z, w) = 0, |z|^n \leq \delta\}.$$

Vector field (8.2) has no invariant set inside $\mathbf{B}'$. Hence the asymptotic sets $\mathbf{a}^+$ and $\mathbf{a}^-$ for $\mathbf{B}'$ are empty. By Theorem 6.3, the map across the block is a diffeomorphism from the entire incoming set to the entire outgoing set. Transforming back to the coordinates of system (4.8), we see that $\Psi$ extends to a diffeomorphism.

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*School of Mathematics*
*University of Minnesota*
*Minneapolis, MN 55455 USA*

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